A simple generalization of El-Gamal cryptosystem to non-abelian groups

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Abstract. In this paper we propose the group of unitriangular matrices over a finite field as a non-abelian group and composition of inner, diagonal and central automorphisms as a group of automorphisms for the MOR cryptosystem.

1 Introduction

Most of the cryptosystems popular today are built on abelian groups. It is natural to try to generalize them to non-abelian groups, not only because the current systems are getting old with time, but also because it is an interesting academic adventure in trying to do so. The cryptosystem that we have in mind is the *El-Gamal* cryptosystem [3, Section 2] which is built on the *Discrete Logarithm Problem* [3, Section 2]. Discrete logarithm problem can be generalized in different ways, to mention just two of them – one was done in [7] and the other is the MOR cryptosystem [12].

The MOR cryptosystem attracted a lot of attention and some well written papers [4, 11, 14]. We in this article propose a new group and a set of automorphisms for the MOR cryptosystem. Our group is the *group of unitriangular matrices over a finite field* and the automorphism is the composition of *diagonal, inner and the central automorphism*. We show that for this group and the set of automorphisms, MOR is at most as secure as the discrete logarithm problem in finite fields.

There is still a lot of interest with cryptosystems using the discrete logarithm problem in finite fields, like the El-Gamal cryptosystem. From [4, 11] and our study of the MOR cryptosystem it seems reasonable to expect that the proposed MOR cryptosystem is as secure as the El-Gamal cryptosystem over finite fields. So, we claim that we had a reasonable amount of success with these groups and automorphisms. Though the most desirable consequences of this research would be no *sub-exponential* attack on the cryptosystem.

There is one other shift in our proposed MOR cryptosystem. We are using *polycyclic* groups [13, Chapter 9] for the cryptosystem, computation with this class of groups is done differently than with the multiplicative group of finite fields. We are yet to understand the consequence of this shift, from arithmetic in finite fields to arithmetic in a polycyclic group and the use of automorphisms instead of exponentiation.

It is often expected of a proposer of a new cryptosystem to provide parameters and to show that the cryptosystem is *semantically secure*. We are not yet in a position to provide parameters, because the discrete logarithm problem in the automorphisms, on which the security of our cryptosystem depends is not well studied. Moreover since the best known attack is the discrete logarithm problem in finite fields, hence one can pick parameters out of any cryptosystem using the discrete logarithm problem, *e.g.* the El-Gamal cryptosystem. MOR cryptosystem is a straightforward generalization of the El-Gamal cryptosystem, so it is easy to see that MOR is not semantically secure (indistinguishability) [3], however it can be made secure against *indistinguishability against chosen ciphertext attack* using ideas similar to Cramer-Shoup [1].

2 The MOR cryptosystem

In this section we discuss the MOR cryptosystem [12] and critique some of the points discussed by the authors there. There are two different concepts used in this paper [12] for the security of their cryptosystem.

i The discrete logarithm problem in the group of inner automorphisms.
ii Membership problem in a finite cyclic group.

Let us describe the MOR cryptosystem in details. Let $G = \langle \gamma_1, \gamma_2, \ldots, \gamma_s \rangle$ be a finite non-abelian group. Let ϕ_g be an inner automorphism of G defined by $\phi_g(x) = g^{-1}xg$ for all $x \in G$. Then $\phi_g^m(x) = g^{-m}xg^m$ for all $x \in G$ and m a positive integer. Now suppose Eve wants to set up a public key for herself. Then she chooses g and publishes ϕ_g and ϕ_g^m. She however doesn't publish g and g^m, instead she publishes $\{\phi_g(\gamma_i)\}_{i=1}^s$ and $\{\phi_g^m(\gamma_i)\}_{i=1}^s$. Then to send a message (plaintext) $a \in G$, Bob computes the ϕ_g^r and ϕ_g^{mr} from the public information and for a random $r \in \mathbb{N}$ and computes $\phi_g^{mr}(a)$ and sends Eve $\left(\phi_g^r, \phi_g^{mr}(a)\right)$. Just like El-Gamal cryptosystem Alice knowing m can compute ϕ_g^{mr} from ϕ_g^r and hence the inverse ϕ_g^{-mr} and can find out a the plaintext.

What does the security of this protocol depends on? Firstly if one can solve the discrete logarithm problem in ϕ_g and ϕ_g^m then the protocol is

broken. On the other hand since the inner automorphisms are presented as the action on generators it might be difficult to find g from the public information of $\{\phi_g(\gamma_i)\}_{i=1}^s$. Moreover $\phi_g = \phi_{gz}$ for some $z \in Z(G)$. Hence even if there is an algorithm to find g, that g might not be unique. The authors of the MOR cryptosystem uses this fact for security as follows, suppose one knows the g from ϕ_g and then tries to determine the g^m in ϕ_{g^m} then by solving the conjugacy problem they will come up with $g^m z$ and then they will have to solve the membership problem in the cyclic group $\langle g \rangle$ before they can even try to solve the discrete logarithm problem. Of course this attack on the system doesn't include that some one might be able to solve for m from the public informations $\{\phi_g(\gamma_i)\}_{i=1}^s$ and $\{\phi_{g^m}(\gamma_i)\}_{i=1}^s$. Moreover as shown in [4, Theorem 1] there is an effective way using only *black box group* operations to get around this membership problem by switching to discrete logarithm problem in $G/Z(G)$.

The idea behind this scheme seems to be novel and the idea of using membership problem in Public Key Cryptography might have interesting applications. However, the biggest test for an idea to develop a public key protocol is the ability to find groups that produces fast encryption, fast decryption and is secure.

The idea of using automorphisms, where the public information about these automorphisms is its action on generators puts severe restrictions to the groups useful in this scheme.

> The group used should have a fast algorithm to express an element of the group as a word in generators. Unless every group element is presented as words in generators, *e.g.* polycyclic groups where fast collection algorithms are available, this is hard to achieve.

What concerns us the most is the use of two different cryptographic primitives simultaneously! It can be argued that two insecure locks doesn't make one secure lock, just get two guys to work on them simultaneously or use a *meet in the middle* attack. The converse of the idea is that one secure lock is enough to guard a secret. Stating plainly, the idea of using membership problem and the discrete logarithm problem simultaneously in a protocol is probably not wise. On top of that since MOR is a generalization of the El-Gamal cryptosystem whose security depends on discrete logarithm problem, computational Diffie-Hellman problem and Decision Diffie-Hellman problem [7, Section 2.3]or [3, Section 2], this cryptosystem is not ideally suited to exploit the membership problem. This was echoed in [11]. In the definition of the MOR cryptosystem in [11] the whole automorphism group was considered instead of the group of inner

automorphisms as in [12] and the requirement that the automorphisms be presented as action on generators was dropped.

The basic scheme for a MOR cryptosystem is as follows and is an adaptation of [11, Section 2].

Let G be a group and $\phi : G \to G$ is an automorphism. Alice's keys are as follows:

Public Key ϕ and ϕ^m.
Private Key m.

Encryption

a To send a message $a \in G$ Bob computes ϕ^r and ϕ^{mr} for a random $r \in \mathbb{N}$.
b The ciphertext is $(\phi^r, \phi^{mr}(a))$.

Decryption

a Alice knows m, so if she receives the ciphertext $(\phi^r, \phi^{mr}(a))$, she computes ϕ^{mr} from ϕ^r and then ϕ^{-mr} and then from $\phi^{mr}(a)$ computes a.

Alice can compute ϕ^{-mr} two ways, if she has the information necessary to find out the order of the automorphism ϕ then she can use the identity $\phi^{t-1} = \phi^{-1}$ whenever $\phi^t = 1$. Or, she can find out the order of some subgroup in which ϕ belongs and use the same identity. However the smaller the subgroup more efficient is the decryption algorithm.

3 Proposed group for the MOR cryptosystem

The non-abelian group we are proposing for the MOR cryptosystem is the group of unitriangular matrices over a finite field $\mathbb{F}_q$ of characteristic p, where p is a prime number. The group of unitriangular matrices is often denoted by $UT(n, \mathbb{F}_q)$. This group consists of all square matrices of dimension n, the diagonal elements are 1 (the multiplicative identity of the field) and all entries below the diagonal are 0 (the additive identity of the field). The entries above the diagonal can be any element of the finite field $\mathbb{F}_q$. The group operation is matrix multiplication. An arbitrary element $g \in UT(4, \mathbb{F}_q)$ looks like,

$$g = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $*$ denotes a field element. From a simple counting argument it follows that $UT(n, \mathbb{F}_q)$ is a Sylow p-subgroup of the general linear group $GL(n, \mathbb{F}_q)$ where p is the characteristic of the finite field $\mathbb{F}_q$.

Let e_{ij} for $i < j$ represent the matrix with 1 in the (i, j) position and 0 elsewhere. It is customary to represent $g \in UT(n, \mathbb{F}_q)$ as $1 + \sum_{i<j} a_{ij} e_{ij}$, where $a_{ij} \in \mathbb{F}_q$. Notice that 1 above is the identity matrix. We will abuse the notation a little bit and will use 1 as the identity of $UT(n, \mathbb{F}_q)$ and $\mathbb{F}_q$ simultaneously. It should be clear from the context which 1 we are referring to.

There are two fundamental set of relations in $UT(n, \mathbb{F}_q)$ along with the relations in the field $\mathbb{F}_q$. For $(1 + ae_{ij})$, $(1 + be_{kj}) \in UT(n, \mathbb{F}_q)$ where $a, b \in \mathbb{F}_q$ they are as follows:

$$(1 + ae_{ij})(1 + be_{ij}) = 1 + (a + b)e_{ij} \tag{1}$$

$$[1 + ae_{ij}, 1 + be_{kl}] = \begin{cases} 1 + abe_{il} & \text{if} \quad j = k, \ \ i \neq l \\ 1 - abe_{kj} & \text{if} \quad i = l, \ \ j \neq k \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

Here $[x, y] = x^{-1}y^{-1}xy$ is the commutator of elements $x, y \in G$ for any group G. It is well known that the additive group of $\mathbb{F}_q$, often written as $\mathbb{F}_q^+$ is a γ dimensional vector space over $\mathbb{Z}_p$, where $p^\gamma = q$. It follows [15, Page 455] that the minimal set of generators of $UT(n, \mathbb{F}_q)$ are $1+\delta_k e_{i,i+1}$, $k = 1, 2, \ldots, \gamma$ and $i = 1, 2, \ldots, n-1$. The set $\{\delta_1, \delta_2, \ldots, \delta_\gamma\}$ is a basis of $\mathbb{F}_q^+$ over $\mathbb{Z}_p$. The center of $UT(n, \mathbb{F}_q)$ is $1 + ke_{1,n}$ where $k \in \mathbb{F}_q$.

Since $UT(n, \mathbb{F}_q)$ is a finite p-group hence it is a finite *nilpotent* group and a *polycyclic* group [13, Proposition 3.4].

Definition 1 (Polycyclic Group). *A group G is a polycyclic group if there is a finite chain of subgroups $G = G_1 \supset G_2 \supset \ldots \supset G_k \supset G_{k+1} = 1$ such that G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is cyclic.*

Since in a polycyclic group G, G_i/G_{i+1} is cyclic hence there is a a_i in G_i such that the image of a_i in G_i/G_{i+1} generates G_i/G_{i+1}. It is easy to see that $\{a_1, a_2, \ldots, a_k\}$ generates the group G and is known as the *polycyclic generating set*. Since we are dealing with finite groups hence $|G_{i+1} : G_i| = m_i$ is finite. It follows that (see [13, Section 9.4]) every word in G can be expressed uniquely as $a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_k^{\alpha_k}$ where $0 \leq \alpha_j < m_j$ for $j = 1, 2, \ldots, k$. These words are called *collected words*. Using collection algorithm [13, Section 9.4] any word in $\{a_1, \ldots, a_k\}$ can be expressed as a collected word. So in this group computing the inverse and the product

is fast and easy, i.e., there is a fast implementation of polycyclic groups and their arithmetic [2, Polycyclic Package].

Now let us talk about the polycyclic generating set of $UT(n, \mathbb{Z}_p)$, for an arbitrary finite field $\mathbb{F}_q$ this can be similarly done. For sake of simplicity we take $n = 4$. Let $a_1 = 1 + e_{12}$, $a_2 = 1 + e_{23}$, $a_3 = 1 + e_{34}$, $a_4 = 1 + e_{13}$, $a_5 = 1 + e_{24}$ and $a_6 = 1 + e_{14}$. Then it is shown in [13, Section 9.4, Example 4.1] that $\{a_1, a_2, \ldots, a_5\}$ form a polycyclic generating set for $UT(4, \mathbb{Z})$. It is easy to see that this is also a polycyclic generating set for $UT(4, \mathbb{Z}_p)$ for an arbitrary prime p.

3.1 The diagonal automorphism

Let D be an diagonal matrix, i.e., a matrix of dimension n over the field $\mathbb{F}_q$, and the only non-zero elements are in the diagonals. We will represent a diagonal matrix D as $[w_1, w_2, w_3, \ldots, w_n]$, where w_i are non-zero elements of the field K and are the diagonal elements of the matrix D. It is easy to see that if $w_1 = w_2 = \ldots = w_n$ then the diagonal matrix is a scalar matrix. Weir[15, Section 4] introduced the diagonal automorphisms on $UT(n, \mathbb{F}_q)$. Let D be a diagonal matrix given by $[w_1, w_2, \ldots, w_n]$, then from matrix multiplication it follows that $D^{-1}xD$ for an $x \in UT(n, \mathbb{F}_q)$ where $x = 1 + \sum_{i<j} a_{ij}e_{ij}$ is given by $1 + \sum_{i<j}(w_i^{-1}a_{ij}w_j)e_{ij}$. Since the scalar matrices have the same diagonal elements, hence the group of diagonal automorphisms has order $(q-1)^{n-1}$.

These diagonal automorphisms are clearly not inner automorphism because the diagonal matrices are not unitriangular. We will now study the MOR cryptosystem using these diagonal automorphisms. It is easy to see that if $D = [w_1, w_2, \ldots, w_n]$ and $\phi(x) = D^{-1}xD$ for $x \in UT(n, \mathbb{F}_q)$ then $\phi^m(x) = D^{-m}xD^m$ where $D^m = [w_1^m, w_2^m, \ldots, w_n^m]$ where $m \in \mathbb{N}$. So if Alice makes D and D^m public then finding the m is solving the discrete logarithm problem in the multiplicative group $\mathbb{F}_q^*$ of the finite field $\mathbb{F}_q$.

If the plaintext is $a \in UT(n, \mathbb{F}_q)$ then computing $\phi^m(a)$ is easy and can be done easily from the formula above. So, using this diagonal automorphism one can have a secure protocol similar to that of El-Gamal cryptosystem. Clearly, there is no advantage for using this protocol over El-Gamal, the security depends on the discrete logarithm problem in the multiplicative group of the finite fields but one has to do more work than the El-Gamal cryptosystem for encryption and decryption. However this protocol seems to be computationally secure and can be made semantically secure using ideas similar to Cramer-Shoup. Notice that it is essential for the above mentioned use that the w_i are all different from

one another, otherwise valuable information about the plaintext will be leaked.

3.2 The inner automorphism

Inner automorphisms are the easiest of the automorphisms to study, they are defined as $I_g(x) = g^{-1}xg$ for all $x \in UT(n, \mathbb{F}_q)$ and $g \in UT(n, \mathbb{F}_q)$. It is well known that the group of inner automorphisms $I(G)$ for a arbitrary group G is a normal subgroup of the automorphism group of G. It is also known that $I(G)$ is isomorphic to $G/Z(G)$. From which it follows that the order of the group of inner automorphisms is $q^{\frac{n^2-n-2}{2}}$. We now see what happens when we use the inner automorphisms for the MOR cryptosystem.

Let $\phi = I_g$ as defined before. Since the conjugacy problem is easy and we are not using the membership problem, we can safely assume that g and g^n is public. If

$$g = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then

$$g^n = \begin{pmatrix} 1 & na_{12} & * & * \\ 0 & 1 & na_{23} & * \\ 0 & 0 & 1 & na_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $*$ represents a field element.

Now the discrete logarithm problem to find n essentially becomes discrete logarithm problem in $\mathbb{F}_q^+$. Since the discrete logarithm problem in the additive group of a finite field is known to be easy we don't believe that using only inner automorphisms one can build a secure MOR cryptosystem.

3.3 The central automorphism

The group of central automorphisms are the group most widely studied after the group of inner automorphism. The reason of their popularity is that the group of central automorphisms are the centralizers of the group of inner automorphisms, i.e., the central automorphisms commute with the inner automorphisms and fixes the derived subgroup elementwise. It can be shown that if ψ is a central automorphism of a group G then

$\psi(g) = gz_g$ where $z_g \in Z(G)$ and depends on g. It follows [5] that a description of the central automorphism $\zeta_i(\lambda)$ is

$$\zeta_r(\lambda) : 1 + a_{r,r+1}e_{r,r+1} \mapsto 1 + a_{r,r+1}e_{r,r+1} + \lambda\left(a_{r,r+1}\right)e_{1,n}$$

where λ is an endomorphism of $\mathbb{F}_q^+$ and $r = 1, 2, \ldots, n-1$. Now since λ is an endomorphism and $\mathbb{F}_q^+$ is a γ-dimensional vector space over $\mathbb{Z}_p$, then if $\lambda(\delta_i) = b_i$ then we arrive at [15, Page 463] where a description of the central automorphisms for the $UT(n, \mathbb{F}_q)$ is given as $1 + \delta_i e_{r,r+1} \mapsto 1 + \delta_i e_{r,r+1} + b_i e_{1,n}$ where $r = 1, 2, \ldots, n-1$, b_i is an arbitrary element of $\mathbb{F}_q$. This can also be represented as $1+\delta_i e_{r,r+1} \mapsto (1+\delta_i e_{r,r+1})(1+b_i e_{1,n})$. So composing this map n times gives us $1 + \delta_i e_{r,r+1} \mapsto (1 + \delta_i e_{r,r+1})(1 + nb_i e_{1,n})$. Notice that if $r = 1, n-1$ then the central automorphisms are inner automorphisms and from this it follows that the order of the group of central automorphisms is $q^{\gamma(n-3)}$ where $p^\gamma = q$. Since the description of the central automorphisms depend on λ hence unlike the inner or the diagonal automorphisms the only possible description of a central automorphism is by action on generators of the group G.

So if we take a central automorphism to use in MOR cryptosystem then from the public information the discrete logarithm problem is the same as the discrete logarithm problem in $\mathbb{F}_q^+$. Since the discrete logarithm problem in the additive group of the field is easy so central automorphism alone doesn't provide us with a secure MOR cryptosystem.

4 A proposed automorphism for the MOR cryptosystem

Currently the proposed group for the MOR cryptosystem [12] is $SL(2, \mathbb{Z}_p) \rtimes \mathbb{Z}_p$. This is a split extension of $SL(2, \mathbb{Z}_p)$ by $\mathbb{Z}_p$. The automorphisms proposed are the inner automorphisms. It is shown in [11, Theorem 2] that the discrete logarithm problem in the group of inner automorphisms of $SL(2, \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ is the same as the discrete logarithm problem in $SL(2, \mathbb{Z}_p)$. In [9] the authors show that the discrete logarithm problem in $GL(n, q)$, the general linear group over the finite field $\mathbb{F}_q$, is at most as hard as discrete logarithm problem in some finite extension field of $\mathbb{F}_q$. Since there are sub-exponential attacks on discrete logarithm problem on finite fields like the index calculus attack, there is every reason (practical as well as academic) to look for other groups and automorphisms in these groups.

In [4] the authors came up with *central commutator attack*, they showed that inner automorphisms are not well suited for MOR cryptosystem, especially when the group is nilpotent.

So it is clear that if we are using nilpotent groups, ($UT(n, \mathbb{F}_q)$ is a finite p-group and hence nilpotent) then we have to look for outer automorphisms. The diagonal and the central automorphisms are outer automorphisms. On the other hand as we saw in the last section, diagonal automorphism does provide us with a secure MOR cryptosystem and the only way to represent a central automorphism is its action on generators. On the other hand the security with diagonal automorphisms turns out to be discrete logarithm problem in the multiplicative group of the finite fields and the central and the inner automorphisms, from their presentation reveals valuable information.

Now we are in a position to describe the automorphism that we are going to use for the MOR cryptosystem, it is

central *composed* inner *composed* diagonal automorphism.

Let us denote by $\mathcal{I}$, $\mathcal{D}$ and $\mathcal{L}$ the group of inner, diagonal and the central automorphisms respectively. From the definition of $\mathcal{L}$ it is clear that $\mathcal{I} \times \mathcal{L}$ is a subgroup of the automorphism group of $UT(n, \mathbb{F}_q)$. Since the diagonal automorphisms don't commute with the inner automorphisms and from a simple argument it follows that the specific automorphisms we plan on using belongs to the subgroup $(\mathcal{I} \times \mathcal{L}) \rtimes \mathcal{D}$. From this it follows that the smallest subgroup containing the above automorphisms has order

$$q^{\frac{n^2-n-2}{2}} \times (q-1)^{n-1} \times q^{\gamma(n-3)}.$$

We now show by means of a small example that with the best of our efforts we were not able to beat the sub-exponential attack on finite fields. It is shown that the with the proposed group and the automorphism the MOR cryptosystem is as secure as discrete logarithm problem in a finite field.

4.1 A small example

We now explain the MOR cryptosystem with a small example. We used [2, Polycyclic Package] for this example, notations are from Section 3. Let $n = 4$ and $q = 1297$ where 1297 is a prime. We pick three random integers 984, 807 and 452. Then we define a central automorphisms (see Section 3.3) $map1$ as

$$map1 = \begin{cases} a_1 \longrightarrow a_1 a_6^{984} \\ a_2 \longrightarrow a_2 a_6^{807} \\ a_3 \longrightarrow a_3 a_6^{452} \end{cases}$$

all other generators remain fixed. Next we pick a random element $h := a_1^{83} a_2^{462} a_3^{1202} a_4^{1209} a_5^{793} a_6^{152}$ and compute the inner automorphism (see Sec-

tion 3.2), $map2: \ \ x \mapsto h^{-1}xh$ corresponding to h.

$$map2 := \begin{cases} a_1 \longrightarrow & a_1 a_4^{462} a_6^{1001} \\ a_2 \longrightarrow & a_2 a_4^{1214} a_5^{1202} a_6^{103} \\ a_3 \longrightarrow & a_3 a_5^{835} a_6^{88} \\ a_4 \longrightarrow & a_4 a_6^{1202} \\ a_5 \longrightarrow & a_5 g^6 1214 \\ a_6 \longrightarrow & a_6 \end{cases}$$

Then we take the diagonal automorphism (see Section 3.1) corresponding to $[624, 155, 538, 126]$, the diagonal automorphism $map3$ is

$$map3 = \begin{cases} a_1 \longrightarrow a_1^{576} \\ a_2 \longrightarrow a_2^{1267} \\ a_3 \longrightarrow a_3^{574} \\ a_4 \longrightarrow a_4^{878} \\ a_5 \longrightarrow a_5^{938} \\ a_6 \longrightarrow a_6^{768} \end{cases}$$

Then the automorphism Alice will make public is $\phi = map1 \cdot map2 \cdot map3$ and that is given by

$$\phi = \begin{cases} a_1 \longrightarrow & a_1^{576} a_4^{972} a_6^{538} \\ a_2 \longrightarrow & a_2^{1267} a_4^{1055} a_5^{383} a_6^{508} \\ a_3 \longrightarrow & a_3^{574} a_5^{1139} a_6^{558} \\ a_4 \longrightarrow & a_4^{878} a_6^{118} \\ a_5 \longrightarrow & a_5^{938} a_6^{1168} \\ a_6 \longrightarrow & a_6^{736} \end{cases}$$

and if Alice chooses her private key to be 65 then

$$\phi^{65} = \begin{cases} a_1 \longrightarrow & a_1^{450} a_4^{1145} a_6^{618} \\ a_2 \longrightarrow & a_2^{1263} a_4^{1269} a_5^{1242} a_6^{1093} \\ a_3 \longrightarrow & a_3^{526} a_5^{708} a_6^{279} \\ a_4 \longrightarrow & a_4^{264} a_6^{1190} \\ a_5 \longrightarrow & a_5^{274} a_6^{836} \\ a_6 \longrightarrow & a_6^{85} \end{cases}$$

The automorphisms ϕ and ϕ^{65} is public, (see description of the MOR cryptosystem in Section 2). Notice that $(576)^{65} \mod 1297 = 450$. An avid reader will further notice that from the public information of ϕ and ϕ^{65} it is clear that if k_j' is the exponent of a_j in $\phi^{65}(a_j)$ and if k_j is the exponent of a_j in $\phi(a_j)$ for $j = 1, 2, 3$ and $j = 6$. Then k_j' is k_j^{65}. The reason for

this is that the inner and the central automorphisms leave the exponent of a_1, a_2, a_3, a_6 unchanged. The only thing that changes $\{a_1, a_2, a_3, a_6\}$ is the diagonal automorphism and then the change is like $a_j \mapsto a_1^{w_j^{-1}w_{j+1}}$ for $j = 1, 2, 3$ and $a_6 \mapsto a_6^{w_1^{-1}w_4}$. Then composing the map m many times gives us $a_j \mapsto a_j^{\left(w_j^{-1}w_{j+1}\right)^m}$ for $j = 1, 2, 3$ and $a_6 \mapsto a_6^{\left(w_1^{-1}w_4\right)^m}$.

This leads us to the best known attack against this cryptosystem. If one can solve the discrete logarithm problem in a finite field then he can figure out the m from the public information of ϕ and ϕ^m as demonstrated above. There are sub-exponential algorithms, like index calculus methods, in finite fields to solve the discrete logarithm problem.

5 Conclusion

In this paper we studied a new group and a group of outer automorphisms for the MOR cryptosystem. The security of any proposed cryptosystem is always an open question. Since, this is the first time the group of unitriangular matrices and automorphisms over it is proposed for public key cryptography more work needs to be done to assure one of the security of the said system. There are two important question that comes out of this cryptosystem.

1. We saw that if one can solve the discrete logarithm problem in finite fields then he can break the MOR cryptosystem. Is the security of MOR cryptosystem equivalent to the El-Gamal cryptosystem?

2. Is it computationally more expensive to use MOR cryptosystem than the traditional El-Gamal cryptosystem? This is a relevant question because the MOR cryptosystem uses the algorithms for polycyclic groups.

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