The simplest method for constructing APN polynomials EA-inequivalent to power functions

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Abstract

The first APN polynomials EA-inequivalent to power functions have been constructed in [7, 8] by applying CCZ-equivalence to the Gold APN functions. It is a natural question whether it is possible to construct APN polynomials EA-inequivalent to power functions by using only EA-equivalence and inverse transformation on a power APN function: this would be the simplest method to construct APN polynomials EA-inequivalent to power functions. In the present paper we prove that the answer to this question is positive. By this method we construct a class of APN polynomials EA-inequivalent to power functions. On the other hand it is shown that the APN polynomials from [7, 8] cannot be obtained by the introduced method.

Keywords: Affine equivalence, Almost bent, Almost perfect nonlinear, CCZ-equivalence, Differential uniformity, Nonlinearity, S-box, Vectorial Boolean function

1 Introduction

A function $F : \mathbb{F}_2^m \to \mathbb{F}_2^m$ is called *almost perfect nonlinear* (APN) if, for every $a \neq 0$ and every b in $\mathbb{F}_2^m$, the equation $F(x)+F(x+a) = b$ admits at most two solutions (it is also called *differentially* 2*-uniform*). Vectorial Boolean functions used as S-boxes in block ciphers must have low differential uniformity to allow high resistance to the differential cryptanalysis (see [2, 28]). In this sense APN functions are optimal. The notion of APN function is closely connected to the notion of almost bent (AB) function. A function $F : \mathbb{F}_2^m \to \mathbb{F}_2^m$ is called AB if the minimum Hamming distance between all the Boolean functions $v \cdot F$, $v \in \mathbb{F}_2^m \setminus \{0\}$, and all affine Boolean functions on $\mathbb{F}_2^m$ is maximal. AB functions exist for m odd only and oppose an optimum resistance to the linear cryptanalysis (see [26, 14]). Besides, every AB function is APN [14], and in the m odd case, any quadratic function is APN if and only if it is AB [13].

The APN and AB properties are preserved by some transformations of functions [13, 28]. If F is an APN (resp. AB) function, A_1, A_2 are affine permutations and A is affine then the

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function $F' = A_1 \circ F \circ A_2 + A$ is also APN (resp. AB); the functions F and F' are called extended affine equivalent (*EA-equivalent*). Another case is the *inverse transformation*, that is, the inverse of any APN (resp. AB) permutation is APN (resp. AB). Until recently, the only known constructions of APN and AB functions were EA-equivalent to power functions $F(x) = x^d$ over finite fields ($\mathbb{F}_{2^m}$ being identified with $\mathbb{F}_2^m$). Table 1 gives all known values of exponents d (up to multiplication by a power of 2 modulo $2^m - 1$, and up to taking the inverse when a function is a permutation) such that the power function x^d over $\mathbb{F}_{2^m}$ is APN. For m odd the Gold, Kasami, Welch and Niho APN functions from Table 1 are also AB (for the proofs of AB property see [10, 11, 21, 22, 24, 28]).

Table 1
Known APN power functions x^d on $\mathbb{F}_{2^m}$.

Functions	Exponents d	Conditions	Proven in
Gold	$2^i + 1$	$\gcd(i, m) = 1$	[21, 28]
Kasami	$2^{2i} - 2^i + 1$	$\gcd(i, m) = 1$	[23, 24]
Welch	$2^t + 3$	$m = 2t + 1$	[18]
Niho	$2^t + 2^{\frac{t}{2}} - 1$, t even $2^t + 2^{\frac{3t+1}{2}} - 1$, t odd	$m = 2t + 1$	[17]
Inverse	$2^{2t} - 1$	$m = 2t + 1$	[1, 28]
Dobbertin	$2^{4t} + 2^{3t} + 2^{2t} + 2^t - 1$	$m = 5t$	[19]

In [13], Carlet, Charpin and Zinoviev introduced an equivalence relation of functions, more recently called CCZ-equivalence, which corresponds to the affine equivalence of the graphs of functions and preserves APN and AB properties. EA-equivalence is a particular case of CCZ-equivalence and any permutation is CCZ-equivalent to its inverse [13]. In [7, 8], it is proven that CCZ-equivalence is more general, and applying CCZ-equivalence to the Gold mappings classes of APN functions EA-inequivalent to power functions are constructed. These classes are presented in Table 2. When m is odd, these functions are also AB.

Table 2
Known APN functions EA-inequivalent to power functions on $\mathbb{F}_{2^m}$.

Functions	Conditions	Alg. degree
$x^{2^i+1} + (x^{2^i} + x + \mathrm{tr}(1) + 1)\,\mathrm{tr}(x^{2^i+1} + x\,\mathrm{tr}(1))$	$m \geq 4$ $\gcd(i, m) = 1$	3
$[x + \mathrm{tr}_{m/3}(x^{2(2^i+1)} + x^{4(2^i+1)}) + \mathrm{tr}(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2i}(2^i+1)})]^{2^i+1}$	m divisible by 6 $\gcd(i, m) = 1$	4
$x^{2^i+1} + \mathrm{tr}_{m/n}(x^{2^i+1}) + x^{2^i}\,\mathrm{tr}_{m/n}(x) + x\ \mathrm{tr}_{m/n}(x)^{2^i}$ $+[\mathrm{tr}_{m/n}(x)^{2^i+1} + \mathrm{tr}_{m/n}(x^{2^i+1}) + \mathrm{tr}_{m/n}(x)]^{\frac{1}{2^i+1}}(x^{2^i} + \mathrm{tr}_{m/n}(x)^{2^i} + 1)$ $+[\mathrm{tr}_{m/n}(x)^{2^i+1} + \mathrm{tr}_{m/n}(x^{2^i+1}) + \mathrm{tr}_{m/n}(x)]^{\frac{2^i}{2^i+1}}(x + \mathrm{tr}_{m/n}(x))$	$m \neq n$ m odd m divisible by n $\gcd(i, m) = 1$	$n + 2$

These new results on CCZ-equivalence have solved several problems (see [7, 8]) and have also raised some interesting questions. One of these questions is whether the known classes of APN power functions are CCZ-inequivalent. Partly the answer is given in [5]: it is proven that in general the Gold functions are CCZ-inequivalent to the Kasami and Welch functions, and that for different parameters $1 \leq i, j \leq \frac{m-1}{2}$ the Gold functions x^{2^i+1} and x^{2^j+1} are CCZ-inequivalent. Another interesting question is the existence of APN polynomials CCZ-inequivalent to power functions. Examples of APN polynomials CCZ-inequivalent to power functions have been constructed in [20, 16, 27] and infinite classes of such functions in [3, 4, 5, 6]. In the present paper we consider the natural question whether it is possible to construct APN polynomials EA-inequivalent to power functions by applying only EA-equivalence and the inverse transformation on a power APN function. We prove that the answer is positive and construct a class of AB functions EA-inequivalent to power functions by applying this method to the Gold AB functions. It should be mentioned that the functions from Table 2 cannot be obtained by this method. It can be illustrated, for instance, by the fact that for $m = 5$ the functions from Table 2 and for m even the Gold functions are EA-inequivalent to permutations [7, 8, 29], therefore, the inverse transformation cannot be applied in these cases and the method fails.

2 Preliminaries

Let $\mathbb{F}_2^m$ be the m-dimensional vector space over the field $\mathbb{F}_2$. Any function F from $\mathbb{F}_2^m$ to itself can be uniquely represented as a polynomial on m variables with coefficients in $\mathbb{F}_2^m$, whose degree with respect to each coordinate is at most 1:

$$F(x_1, ..., x_m) = \sum_{u \in \mathbb{F}_2^m} c(u)\Big(\prod_{i=1}^{m} x_i^{u_i}\Big), \qquad c(u) \in \mathbb{F}_2^m.$$

This representation is called the *algebraic normal form* of F and its degree $d^\circ(F)$ the *algebraic degree* of the function F.

Besides, the field $\mathbb{F}_{2^m}$ can be identified with $\mathbb{F}_2^m$ as a vector space. Then, viewed as a function from this field to itself, F has a unique representation as a univariate polynomial over $\mathbb{F}_{2^m}$ of degree smaller than 2^m:

$$F(x) = \sum_{i=0}^{2^m-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^m}.$$

For any k, $0 \leq k \leq 2^m - 1$, the number $w_2(k)$ of the nonzero coefficients $k_s \in \{0, 1\}$ in the binary expansion $\sum_{s=0}^{m-1} 2^s k_s$ of k is called the 2-*weight* of k. The algebraic degree of F is equal to the maximum 2-weight of the exponents i of the polynomial $F(x)$ such that $c_i \neq 0$, that is, $d^\circ(F) = \max_{0 \leq i \leq m-1, c_i \neq 0} w_2(i)$ (see [13]).

A function $F : \mathbb{F}_2^m \to \mathbb{F}_2^m$ is *linear* if and only if $F(x)$ is a linearized polynomial over $\mathbb{F}_{2^m}$, that is,

$$\sum_{i=0}^{m-1} c_i x^{2^i}, \quad c_i \in \mathbb{F}_{2^m}.$$

The sum of a linear function and a constant is called an *affine function.*

Let F be a function from $\mathbb{F}_{2^m}$ to itself and A_1, $A_2 : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$ be affine permutations. The functions F and $A_1 \circ F \circ A_2$ are then called *affine equivalent.* Affine equivalent functions have the same algebraic degree (i.e. the algebraic degree is *affine invariant*).

As recalled in Introduction, we say that the functions F and F' are *extended affine equivalent* if $F' = A_1 \circ F \circ A_2 + A$ for some affine permutations A_1, A_2 and an affine function A. If F is not affine, then F and F' have again the same algebraic degree.

Two mappings F and F' from $\mathbb{F}_{2^m}$ to itself are called Carlet-Charpin-Zinoviev equivalent (*CCZ-equivalent*) if the graphs of F and F', that is, the subsets $G_F = \{(x, F(x)) \mid x \in \mathbb{F}_{2^m}\}$ and $G_{F'} = \{(x, F'(x)) \mid x \in \mathbb{F}_{2^m}\}$ of $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$, are affine equivalent. Hence, F and F' are CCZ-equivalent if and only if there exists an affine automorphism $\mathcal{L} = (L_1, L_2)$ of $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ such that

$$y = F(x) \Leftrightarrow L_2(x, y) = F'(L_1(x, y)).$$

Note that since $\mathcal{L}$ is a permutation then the function $L_1(x, F(x))$ has to be a permutation too (see [5]). As shown in [13], EA-equivalence is a particular case of CCZ-equivalence and any permutation is CCZ-equivalent to its inverse.

For a function $F : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$ and any elements $a, b \in \mathbb{F}_{2^m}$ we denote

$$\delta_F(a, b) = |\{x \in \mathbb{F}_2^m : F(x + a) + F(x) = b\}|.$$

F is called a *differentially δ-uniform* function if $\max_{a \in \mathbb{F}_{2^m}^*, b \in \mathbb{F}_{2^m}} \delta_F(a, b) \leq \delta$. Note that $\delta \geq 2$ for any function over $\mathbb{F}_{2^m}$. Differentially 2-uniform mappings are called *almost perfect nonlinear.*

For any function $F : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$ we denote

$$\lambda_F(a, b) = \sum_{x \in \mathbb{F}_{2^m}} (-1)^{\mathrm{tr}(bF(x) + ax)}, \qquad a, b \in \mathbb{F}_{2^m},$$

where $\mathrm{tr}(x) = x + x^2 + x^4 + ... + x^{2^{m-1}}$ is the trace function from $\mathbb{F}_{2^m}$ into $\mathbb{F}_2$. The set $\Lambda_F = \{\lambda_F(a, b) : a, b \in \mathbb{F}_{2^m}, b \neq 0\}$ is called the *Walsh spectrum* of the function F and the multiset $\{|\lambda_F(a, b)| : a, b \in \mathbb{F}_{2^n}, b \neq 0\}$ is called the *extended Walsh spectrum* of F. The value

$$\mathcal{NL}(F) = 2^{m-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^m}, b \in \mathbb{F}_{2^m}^*} |\lambda_F(a, b)|$$

equals the *nonlinearity* of the function F. The nonlinearity of any function F satisfies the inequality

$$\mathcal{NL}(F) \leq 2^{m-1} - 2^{\frac{m-1}{2}}$$

([14, 30]) and in case of equality F is called *almost bent* or *maximum nonlinear.*

Obviously, AB functions exist only for n odd. It is proven in [14] that every AB function is APN and its Walsh spectrum equals $\{0, \pm 2^{\frac{m+1}{2}}\}$. If m is odd, every APN mapping which is quadratic (that is, whose algebraic degree equals 2) is AB [13], but this is not true for nonquadratic cases: the Dobbertin and the inverse APN functions are not AB (see [11, 13]).

When m is even, the inverse function x^{2^m-2} is a differentially 4-uniform permutation [28] and has the best known nonlinearity [25], that is $2^{m-1}-2^{\frac{m}{2}}$ (see [11, ?]). This function has been chosen as the basic S-box, with $m=8$, in the Advanced Encryption Standard (AES), see [15]. A comprehensive survey on APN and AB functions can be found in [12].

It is shown in [13] that, if F and G are CCZ-equivalent, then F is APN (resp. AB) if and only if G is APN (resp. AB). More generally, CCZ-equivalent functions have the same differential uniformity and the same extended Walsh spectrum (see [7]). Further invariants for CCZ-equivalence can be found in [20] (see also [16]) in terms of group algebras.

3 The new construction

In this section we show that it is possible to construct APN polynomials EA-inequivalent to power functions by applying only EA-equivalence and the inverse transformation on a power APN function. We shall illustrate it on the Gold AB functions and in order to do it we need the following result from [7, 8].

Proposition 1 ([7, 8]) *Let $F:\mathbb{F}_{2^m}\to\mathbb{F}_{2^m}$, $F(x)=L(x^{2^i+1})+L'(x)$, where $\gcd(i,m)=1$ and L, L' are linear. Then F is a permutation if and only if, for every $u\neq 0$ in $\mathbb{F}_{2^m}$ and every v such that $\mathrm{tr}(v)=\mathrm{tr}(1)$, the condition $L(u^{2^i+1}v)\neq L'(u)$ holds.* $\square$

Further we use the following notations for any divisor n of m

$$\begin{aligned}\mathrm{tr}_{m/n}(x) &= x+x^{2^n}+x^{2^{2n}}...+x^{2^{n(m/n-1)}},\\ \mathrm{tr}_n(x) &= x+x^2+...+x^{2^{n-1}}.\end{aligned}$$

Theorem 1 *Let $m\geq 9$ be odd and divisible by 3. Then the function*

$$F'(x)=\left(x^{\frac{1}{2^i+1}}+\mathrm{tr}_{m/3}(x+x^{2^{2i}})\right)^{-1},$$

with $1\leq i\leq m$, $\gcd(i,m)=1$, is an AB permutation over $\mathbb{F}_{2^m}$. The function F' is EA-inequivalent to the Gold functions and to their inverses, that is, to x^{2^j+1} and $x^{\frac{1}{2^j+1}}$ for any $1\leq j\leq m$.

Proof. To prove that the function F' is an AB permutation we only need to show that the function $F_1(x)=x^{\frac{1}{2^i+1}}+\mathrm{tr}_{m/3}(x+x^{2^{2i}})$ is a permutation. Since the function x^{2^i+1} is a permutation when m is odd and $\gcd(i,m)=1$ then F_1 is a permutation if and only if the function $F(x)=F_1(x^{2^i+1})=x+\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)})$, with $s=i \mod 3$, is a permutation.

By Proposition 1 the function F is a permutation if for every $v\in\mathbb{F}_{2^m}$ such that $\mathrm{tr}(v)=1$ and every $u\in\mathbb{F}_{2^m}^*$ the condition $\mathrm{tr}_{m/3}(u^{2^i+1}v+(u^{2^i+1}v)^{2^{2s}})\neq u$ holds. Obviously, if $u\notin\mathbb{F}_{2^3}^*$ then $\mathrm{tr}_{m/3}(u^{2^i+1}v+(u^{2^i+1}v)^{2^{2s}})\neq u$. For any $u\in\mathbb{F}_{2^3}^*$ the condition $\mathrm{tr}_{m/3}(u^{2^i+1}v+(u^{2^i+1}v)^{2^{2s}})\neq u$ is equivalent to $u^{2^i+1}\,\mathrm{tr}_{m/3}(v)+(u^{2^i+1}\,\mathrm{tr}_{m/3}(v))^{2^{2s}}\neq u$. Therefore, F is a permutation if for every $u, w \in \mathbb{F}_{2^3}^*$, $\mathrm{tr}_3(w) = 1$ the condition $u^{2^i+1}w + (u^{2^i+1}w)^{2^{2s}} \neq u$ is satisfied. Then F is a permutation if $x + x^{2^i+1} + x^{2^{2s}(2^i+1)}$ is a permutation on $\mathbb{F}_{2^3}$ and that was easily checked by a computer.

We have $d^\circ(x^{2^i+1}) = 2$ and it is proven in [28] that $d^\circ(x^{\frac{1}{2^i+1}}) = \frac{m+1}{2}$. We show below that $d^\circ(F') = 4$ for $m \geq 9$. Since the function F' has algebraic degree different from 2 and $\frac{m+1}{2}$ then it is EA-inequivalent to the Gold functions and to their inverses.

Since $F'(x) = F_1^{-1}(x) = [F(x^{\frac{1}{2^i+1}})]^{-1} = [F^{-1}]^{2^i+1}$ then to get the representation of the function F' we need the representation of the function F^{-1}. The following computations are helpful to show that $F^{-1} = F \circ F$.

$$\begin{aligned}\mathrm{tr}_{m/3}[(x + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^i+1}] &= \mathrm{tr}_{m/3}(x^{2^i+1}) + \mathrm{tr}_{m/3}(x^{2^s})\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) \\ &+ \mathrm{tr}_{m/3}(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^s(2^i+1)}) + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)})\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^s(2^i+1)}),\end{aligned}$$

since

$$\begin{aligned}&\mathrm{tr}_{m/3}((x^{2^i+1} + x^{2^{2s}(2^i+1)})^{2^i}) = \mathrm{tr}_{m/3}((x^{2^i+1} + x^{2^{2s}(2^i+1)})^{2^s}) \\ &= \mathrm{tr}_{m/3}(x^{2^s(2^i+1)} + x^{2^{3s}(2^i+1)}) = \mathrm{tr}_{m/3}(x^{2^s(2^i+1)} + x^{2^i+1}).\end{aligned}$$

Then

$$\begin{aligned}&\mathrm{tr}_{m/3}[(x + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^i+1} + (x + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^{2s}(2^i+1)}] \\ &= \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + \mathrm{tr}_{m/3}(x^{2^s})\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) \\ &+ \mathrm{tr}_{m/3}(x)\,\mathrm{tr}_{m/3}(x^{2^{2s}(2^i+1)} + x^{2^s(2^i+1)}) + \mathrm{tr}_{m/3}(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^s(2^i+1)}) \\ &+ \mathrm{tr}_{m/3}(x^{2^{2s}})\,\mathrm{tr}_{m/3}(x^{2^{2s}(2^i+1)} + x^{(2^i+1)}) + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)})\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^s(2^i+1)}) \\ &+ \mathrm{tr}_{m/3}(x^{2^{2s}(2^i+1)} + x^{2^s(2^i+1)})\,\mathrm{tr}_{m/3}(x^{2^{2s}(2^i+1)} + x^{(2^i+1)}) = \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) \\ &+ \mathrm{tr}_{m/3}(x + x^{2^s} + x^{2^{2s}})\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2 \\ &= \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2\end{aligned}$$

and

$$F \circ F(x) = x + \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2$$

and, since $\mathrm{tr}_m(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)})) = 0$,

$$\begin{aligned}&(F \circ F) \circ F(x) = x + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + \mathrm{tr}_m(x)[\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) \\ &+ \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2] + [\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) \\ &+ \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2]^2 \\ &= x + \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2 + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^4 \\ &= x + \mathrm{tr}_3(tr_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)})) = x + \mathrm{tr}_m(x^{2^i+1} + x^{2^{2s}(2^i+1)})) = x.\end{aligned}$$

Therefore,

$$F^{-1}(x) = F \circ F(x) = x + \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^2.$$

Thus, we have

$$\begin{aligned}
&F'(x) = [F^{-1}(x)]^{2^i+1} = [x + \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + (\mathrm{tr}_{m/3}(x^{2^i+1} \\
&+x^{2^{2s}(2^i+1)}))^2]^{2^i+1} = x^{2^i+1} + \mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^s+1} \\
&+(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2(2^s+1)} + x^{2^i}\,\mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) \\
&+x\ \ \mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^s} + x^{2^i}\,\mathrm{tr}_{m/3}(x^{2(2^i+1)} + x^{2^{2s+1}(2^i+1)}) \\
&+x\ (\mathrm{tr}_{m/3}(x^{2(2^i+1)} + x^{2^{2s+1}(2^i+1)}))^{2^s} + \mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^s+2} \\
&+\mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^{s+1}+1} = x^{2^i+1} + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2(2^s+1)} \\
&+x^{2^i}\,\mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}) + x\ \ \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^s(2^i+1)}) \\
&+x^{2^i}\,\mathrm{tr}_{m/3}(x^{2(2^i+1)} + x^{2^{2s+1}(2^i+1)}) + x\ \ \mathrm{tr}_{m/3}(x^{2(2^i+1)} + x^{2^{s+1}(2^i+1)}) + \mathrm{tr}_m(x)[(\mathrm{tr}_{m/3}(x^{2^i+1} \\
&+x^{2^{2s}(2^i+1)}))^{2^s+1} + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^s+2} + (\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^{s+1}+1}].
\end{aligned}$$

The only item in this sum which can give algebraic degree greater than 4 is the last item. We have

$$(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^s+1}+(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^s+2}+(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^{s+1}+1}$$

$$= (\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^s+1}+(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{4(2^s+1)}+(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^{2s}},$$

since

$$2^s+2 = \begin{cases} 4 & \text{if } s=1 \\ 6 & \text{if } s=2 \end{cases}, \qquad 4(2^s+1) = \begin{cases} 12 = 5 \pmod{2^3-1} & \text{if } s=1 \\ 20 = 6 \pmod{2^3-1} & \text{if } s=2 \end{cases},$$

$$2^{s+1}+1 = \begin{cases} 5 & \text{if } s=1 \\ 9 = 2 \pmod{2^3-1} & \text{if } s=2 \end{cases}, \quad 2^{2s} = \begin{cases} 4 & \text{if } s=1 \\ 16 = 2 \pmod{2^3-1} & \text{if } s=2 \end{cases}.$$

On the other hand,

$$\begin{aligned}
&(\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)}))^{2^s+1} = \mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^{2s}(2^i+1)})\,\mathrm{tr}_{m/3}(x^{2^i+1} + x^{2^s(2^i+1)}) \\
&= \mathrm{tr}_{m/3}(x^{2^i+1})^2 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^{2^{2s}+1} + (\mathrm{tr}_{m/3}(x^{2^i+1}))^{2^s+1} + (\mathrm{tr}_{m/3}(x^{2^i+1}))^{2^{2s}+2^s} \\
&= (\mathrm{tr}_{m/3}(x^{2^i+1}))^6 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^5 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^3 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^2 \qquad (1)
\end{aligned}$$

Using (1) we get

$$(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^s+1}+(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{4(2^s+1)}+(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)}))^{2^{2s}}$$

$$= (\mathrm{tr}_{m/3}(x^{2^i+1}))^6 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^5 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^3 + (\mathrm{tr}_{m/3}(x^{2^i+1}))^2 + [(\mathrm{tr}_{m/3}(x^{2^i+1}))^3$$

$$+(\mathrm{tr}_{m/3}(x^{2^i+1}))^6+(\mathrm{tr}_{m/3}(x^{2^i+1}))^5+\mathrm{tr}_{m/3}(x^{2^i+1})]+(\mathrm{tr}_{m/3}(x^{2^i+1}))^2+(\mathrm{tr}_{m/3}(x^{2^i+1}))^4$$
$$=\mathrm{tr}_{m/3}(x^{2^i+1})+(\mathrm{tr}_{m/3}(x^{2^i+1}))^4. \tag{2}$$

Hence, applying (1) and (2) we get

$$\begin{aligned}F'(x) &= x^{2^i+1}+[(\mathrm{tr}_{m/3}(x^{2^i+1}))^6+(\mathrm{tr}_{m/3}(x^{2^i+1}))^5+(\mathrm{tr}_{m/3}(x^{2^i+1}))^3+(\mathrm{tr}_{m/3}(x^{2^i+1}))^2]^2\\
&+x^{2^i}\,\mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)})+x\ \ \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^s(2^i+1)})\\
&+x^{2^i}\,\mathrm{tr}_{m/3}(x^{2(2^i+1)}+x^{2^{2s+1}(2^i+1)})+x\ \ \mathrm{tr}_{m/3}(x^{2(2^i+1)}+x^{2^{s+1}(2^i+1)})\\
&+\mathrm{tr}_m(x)[\mathrm{tr}_{m/3}(x^{2^i+1})+(\mathrm{tr}_{m/3}(x^{2^i+1}))^4]=x^{2^i+1}+(\mathrm{tr}_{m/3}(x^{2^i+1}))^6+(\mathrm{tr}_{m/3}(x^{2^i+1}))^5\\
&+(\mathrm{tr}_{m/3}(x^{2^i+1}))^3+(\mathrm{tr}_{m/3}(x^{2^i+1}))^4+x^{2^i}\,\mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)})\\
&+x\ \ \mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^s(2^i+1)})+x^{2^i}\,\mathrm{tr}_{m/3}(x^{2(2^i+1)}+x^{2^{2s+1}(2^i+1)})\\
&+x\ \ \mathrm{tr}_{m/3}(x^{2(2^i+1)}+x^{2^{s+1}(2^i+1)})+\mathrm{tr}_m(x)\,\mathrm{tr}_{m/3}(x^{2^i+1}+x^{4(2^i+1)}).\end{aligned}$$

Below we consider all items in the sum presenting the function F' which may give the algebraic degree 4:

$$[(\mathrm{tr}_{m/3}(x^{2^i+1}))^6+(\mathrm{tr}_{m/3}(x^{2^i+1}))^5+(\mathrm{tr}_{m/3}(x^{2^i+1}))^3]$$
$$+[x^{2^i}\,\mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^{2s}(2^i+1)})+x\ \ \mathrm{tr}_m(x)(\mathrm{tr}_{m/3}(x^{2^i+1}+x^{2^s(2^i+1)})].$$

For simplicity we take $i=1$. Obviously, all the items in the second bracket of the algebraic degree 4 have the form $x^{2^j+2^k+2^l+2^r}$, where $r<l<k<j\le m-1$, $r\le 1$. Therefore, if we find an item of algebraic degree 4 in the first bracket of the form $x^{2^j+2^k+2^l+2^r}$, where $2\le r<l<k<j\le m-1$, which does not cancel, then this item does not vanish in the whole sum.

We have

$$\begin{aligned}\mathrm{tr}_{m/3}(x^3) &= x^{2+1}+x^{2^4+2^3}+\ldots+x^{2^{m-5}+2^{m-6}}+x^{2^{m-2}+2^{m-3}}=\sum_{k=0}^{\frac{m}{3}-1}x^{2^{3k+1}+2^{3k}},\\
(\mathrm{tr}_{m/3}(x^3))^2 &= x^{2^2+2}+x^{2^5+2^4}+\ldots+x^{2^{m-4}+2^{m-5}}+x^{2^{m-1}+2^{m-2}}=\sum_{k=0}^{\frac{m}{3}-1}x^{2^{3k+2}+2^{3k+1}},\\
(\mathrm{tr}_{m/3}(x^3))^4 &= x^{2^3+2^2}+x^{2^6+2^5}+\ldots+x^{2^{m-3}+2^{m-4}}+x^{2^m+2^{m-1}}=\sum_{k=0}^{\frac{m}{3}-2}x^{2^{3k+3}+2^{3k+2}}+x^{2^{m-1}+1},\end{aligned}$$

$$(\mathrm{tr}_{m/3}(x^3))^3 = (\mathrm{tr}_{m/3}(x^3))^2\,\mathrm{tr}_{m/3}(x^3)=\sum_{i,k=0}^{\frac{m}{3}-1}x^{2^{3k+1}+2^{3k}+2^{3i+2}+2^{3i+1}}, \tag{3}$$

$$(\mathrm{tr}_{m/3}(x^3))^5 = \sum_{j=0}^{\frac{m}{3}-2}\sum_{k=0}^{\frac{m}{3}-1}x^{2^{3j+3}+2^{3j+2}+2^{3k+1}+2^{3k}}+\sum_{k=0}^{\frac{m}{3}-1}x^{2^{m-1}+1+2^{3k+1}+2^{3k}}, \tag{4}$$

$$(\mathrm{tr}_{m/3}(x^3))^6 = \sum_{j=0}^{\frac{m}{3}-2}\sum_{k=0}^{\frac{m}{3}-1} x^{2^{3j+3}+2^{3j+2}+2^{3k+2}+2^{3k+1}} + \sum_{k=0}^{\frac{m}{3}-1} x^{2^{m-1}+1+2^{3k+2}+2^{3k+1}}. \qquad (5)$$

Note that all exponents of weight 4 in (3)-(5) are smaller than 2^m. If $m \geq 9$ then it is obvious that the item $x^{2^6+2^5+2^4+2^3}$ does not vanish in (4) and it definitely differs from all items in (3) and (5).

Hence, the function F' has the algebraic degree 4 when $m \geq 9$ and that completes the proof of the theorem. $\square$

It is proven in [5] that the Gold functions are CCZ-inequivalent to the Welch function for all $m \geq 9$. Therefore, the function F' of Theorem 1 is CCZ-inequivalent to the Welch function. Further, the inverse and the Dobbertin APN functions are not AB (see [13, 11]) and, therefore, the AB function F' is CCZ-inequivalent to them. The algebraic degree of the Kasami function $x^{4^i-2^i+1}$, $2 \leq i \leq \frac{n-1}{2}, \gcd(i,m) = 1$, is equal to $i+1$. Thus, its algebraic degree equals 4 if and only if $i = 3$. Since the function F' is defined only for m divisible by 3 then for $i = 3$ we would have $\gcd(i,m) \neq 1$. On the other hand, if Gold and Kasami functions are CCZ-equivalent then it follows from the proof of Theorem 5 of [5] that the Gold function is EA-equivalent to the inverse of the Kasami function which must be quadratic in this case. Thus, if F' was EA-equivalent to the inverse of a Kasami function then F' would be quadratic. Thus, F' cannot be EA-equivalent to the Kasami functions or to their inverses.

Proposition 2 *The function of Theorem* 1 *is EA-inequivalent to the Welch, Kasami, inverse, Dobbertin functions and to their inverses.*

For $m = 2t+1$ the Niho function has the algebraic degree $t+1$ if t is odd and the algebraic degree $(t+2)/2$ if t is even. Therefore, its algebraic degree equals 4 if and only if $m = 7, 13$.

Proposition 3 *The function of Theorem* 1 *is EA-inequivalent to the Niho function.*

We do not have a general proof of EA-inequivalence of F' and the inverse of the Niho function but for $m = 9$ the Niho function coincides with the Welch functions and therefore its inverse cannot be EA-equivalent to the function F'. Thus, for $m = 9$ the function F' is EA-inequivalent to any power function.

Corollary 1 *For $m = 9$ the function of Theorem* 1 *is EA-inequivalent to any power function.*

When m is odd and divisible by 3 the APN functions from Table 2 have the algebraic degree different from 4. Thus we get the following proposition.

Proposition 4 *The function of Theorem* 1 *is EA-inequivalent to any APN function from Table* 2.

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