

Cryptanalysis of a Hash Function Proposed at ICISC 2006

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Abstract. A simple method for constructing collisions for Shpilrain’s polynomial-based hash function from ICISC 2006 is presented. The attack relies on elementary linear algebra and can be considered as practical: For the parameters suggested, we give a specific collision, computed by means of a computer algebra system.

Keywords: cryptanalysis, hash function

1 Introduction

In [Shp06] Shpilrain proposes a hash function H which builds on the Merkle-Damgård construction [Dam90, Mer90] and relies on computations in the quotient of a polynomial ring. In [Cha06] Chang reports that the underlying compression function is easy to invert and that a meet-in-the-middle attack enables a preimage attack on H . According to Chang’s complexity estimate, for the specific parameters proposed in [Shp06] the computational effort for mounting such a preimage attack appears to be in the magnitude of 2^{80} operations.

The collision attack we describe below can be considered as practical—for the specific parameters proposed in [Shp06] we give a collision of two equal length bitstrings with about 10.2 KByte each. Shpilrain’s proposed hash function H does not involve padding, but the collision given below remains valid if the usual Merkle-Damgård strengthening is applied to H .

2 The proposal from ICISC 2006

Let $p(x) \in \mathbb{F}_2[x]$ be a univariate polynomial of degree n over the finite field with two elements. Moreover, let α be the residue class of x in the quotient $R := \mathbb{F}_2[x]/(p(x))$, thus $p(\alpha) = 0$. We remark that [Shp06] writes “ $R = \mathbb{F}_{2^n} = \mathbb{F}_2[x]/(p(x))$ ” which suggests $p(x)$ to be irreducible, but the specific polynomial $p(x)$ proposed is reducible.

2.1 General construction

To define the hash function H , two elements $h_0, h_1 \in R$ are fixed, and the hash value of an individual bit is defined as

$$\begin{aligned} H(0) &:= h_0 \\ H(1) &:= h_1 \end{aligned} \tag{1}$$

Next, a triple $(u_0, u_1, u_2) \in R^3$ is used to fix a binary operation $\circ$ on R :

$$\begin{aligned} \circ : R^2 &\longrightarrow R \\ (r_1, r_2) &\longmapsto r_1 \circ r_2 := u_0 + r_1 \cdot r_2 + r_1^2 \cdot u_1 + r_2^2 \cdot u_2 \end{aligned} \tag{2}$$

To hash a bitstring M , the following procedure is used:

1. Going from left to right, the bitstring M is split into 32-bit blocks $M = B_1 \parallel B_2 \cdots \parallel B_\ell$, where the last block B_ℓ has less than 32 bit, if the length of M is not a multiple of 32. There is no padding.
2. The hash value of each single 32-bit block $B_i = B_{i,0} \parallel \cdots \parallel B_{i,31}$ is computed by applying the above operation $\circ$ one bit at a time, going from left to right:

$$H(B_i) := (\dots ((H(B_{i,0}) \circ H(B_{i,1})) \circ H(B_{i,2})) \dots) \circ H(B_{i,31})$$

(where the hash value $H(B_{i,j})$ of a single bit $B_{i,j}$ is given by (1)).

3. The hash value $H(M)$ of M is computed by applying the operation $\circ$ one block at a time, going from left to right:

$$H(M) := (\dots ((H(B_0) \circ H(B_1)) \circ H(B_2)) \dots) \circ H(B_\ell)$$

The value $H(M)$ is the output of the hash function for input M .

2.2 Suggested parameters

As specific parameter choice, [Shp06] suggests the following:

$$\begin{aligned} p(x) &:= x^{163} + x^7 + x^6 + x^5 + x^4 + x + 1 \\ h_0 &:= \alpha^7 + 1 \\ h_1 &:= \alpha^8 + 1 \\ (u_0, u_1, u_2) &:= (1, \alpha^2, \alpha) \end{aligned}$$

To demonstrate the practicality of the attack proposed below, in Section 3.3 we construct a specific collision for this parameter choice.

3 Finding collisions

As already indicated above, the notation “ $R = \mathbb{F}_{2^n} = \mathbb{F}_2[x]/(p(x))$ ” in [Shp06] suggests the considered polynomial $p(x)$ to be irreducible. However, with a computer algebra system like Magma [BCP97] one easily checks that the proposed polynomial splits into four irreducible factors from $\mathbb{F}_2[x]$. Namely, for $p(x) = x^{163} + x^7 + x^6 + x^5 + x^4 + x + 1$ we have $p(x) = q_1(x) \cdot q_2(x) \cdot q_3(x) \cdot q_4(x)$, where

$$\begin{aligned} q_1(x) &:= x^9 + x^7 + x^5 + x + 1 \\ q_2(x) &:= x^{18} + x^{14} + x^{12} + x^{11} + x^6 + x^4 + 1 \\ q_3(x) &:= x^{38} + x^{36} + x^{33} + x^{31} + x^{30} + x^{28} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} \\ &\quad + x^{17} + x^{16} + x^{12} + x^{10} + x^8 + x^7 + x^4 + x^3 + x^2 + 1 \\ q_4(x) &:= x^{98} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} + x^{84} + x^{82} + x^{73} + x^{69} \\ &\quad + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{58} + x^{55} + x^{54} + x^{53} + x^{46} \\ &\quad + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{31} + x^{29} + x^{28} \\ &\quad + x^{26} + x^{25} + x^{24} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^9 + x^8 + x^7 \\ &\quad + x^6 + x^5 + x^3 + x^2 + 1 \end{aligned}$$

Thus, before discussing the core part of our attack, it is worth discussing briefly how to exploit such a factorization for a collision search.

3.1 Using the Chinese Remainder Theorem

According to the Chinese Remainder Theorem, any factorization of the polynomial $p(x)$ into coprime factors $q_1(x) \dots, q_s(x)$ yields a decomposition of the ring $R = \mathbb{F}_2[x]/(p(x))$ into a direct product of rings $R_i := \mathbb{F}_2[x]/(q_i(x))$:

$$R \simeq R_1 \times \dots \times R_s$$

As the hash function H composes the hash values of the individual 32-bit blocks with simple ring operations, it looks tempting to exploit this isomorphism of rings to perform the collision search “one R_i at a time”. Suppose we have found two bitstrings M_1, M_2 whose lengths are multiples of 32 and which satisfy

$$H(M_1) \equiv H(M_2) \pmod{q_s(x)},$$

i.e., we have a collision in the R_s -component. Owing to the Merkle-Damgård structure of H , we then have

$$H(M_1 \parallel T) \equiv H(M_2 \parallel T) \pmod{q_s(x)}$$

for arbitrary bitstrings T appended to M_1 and M_2 . Thus, if we heuristically (though actually incorrectly) take the values $H(M_1 \parallel T)$ and $H(M_2 \parallel T)$ as being uniformly and independently distributed modulo $q_{s-1}(x)$, we would expect that within $O(2^{\deg(q_{s-1}(x))})$ random attempts for T , we encounter a pair of messages $M_1 \parallel T_{s-1}$, $M_2 \parallel T_{s-1}$ whose hash values coincide in the $R_{s-1} \times R_s$ -component of R . If the degree of q_{s-1} is small, this approach can be efficient enough. In our experiments we used the linear algebra technique described in the next section to reduce the computational effort for finding a matching T_{s-1} .

Now assume we have found a matching “tail” T_{s-1} and that the length of T_{s-1} is a multiple of 32. Then we can apply the same reasoning as before to extend the collision

$$H(M_1 \parallel T_{s-1}) \equiv H(M_2 \parallel T_{s-1}) \pmod{q_{s-1}(x) \cdot q_s(x)}$$

from $R_{s-1} \times R_s$ to $R_{s-2} \times R_{s-1} \times R_s$: Analogously as before, now we test bitstrings T_{s-2} until

$$H(M_1 \parallel T_{s-1} \parallel T_{s-2}) \equiv H(M_2 \parallel T_{s-1} \parallel T_{s-2}) \pmod{q_{s-2} \cdot q_{s-1} \cdot q_s}$$

holds. In this way, we can process the components $R_s, R_{s-1}, \dots, R_1$ one by one, starting from a collision in a single component.

Example 1. For the specific parameters from Section 2.2 we have $s = 4$, and the degrees of $q_1(x)$, $q_2(x)$ and $q_3(x)$ are rather small—namely 9, 18 and 38. Thus, once we know a pair of messages colliding in the larger R_4 -component (of size 2^{98}), deriving a full collision that is valid in R should be straightforward. Indeed, in our actual computations this worked as expected.

3.2 Using linear algebra

In view of the above discussion, the parameter choice in [Shp06] does not seem to offer an adequate security level, and constructing a collision in the component R_4 (of size 2^{98}) seems to be the most time-consuming task for mounting such an attack. In this section we show that such a collision can be found easily, without implementing a full birthday attack in R_4 .

Remark 1. We describe the attack for an irreducible polynomial $p(x)$ of degree n , i. e., for $R \simeq \mathbb{F}_{2^n}$. For the specific parameter set from Section 2.2, this linear algebra based part is exploited for R_4 and R_3 only, but the attack technique as such does not rely on the described shortcut via the Chinese Remainder Theorem. In particular, simply imposing

$p(x)$ to be irreducible of degree 163 does not appear to be an adequate countermeasure to rule out the attack.

Let $R' \subseteq R$ be the image of H when being restricted to messages whose length is a multiple of 32 (i.e., we have no incomplete last blocks). To each 32-bit block B , we can assign the following map ϕ_B , which captures the update of H 's internal state when appending B to a message whose length is a multiple of 32.

$$\begin{aligned}\phi_B : R' &\longrightarrow R' \\ h &\longmapsto h \circ H(B)\end{aligned}$$

The map ϕ_B , is affine in the sense that it splits into the sum of the $\mathbb{F}_2$ -linear map $h \mapsto h \cdot H(B) + h^2 \cdot u_1$ and the constant shift $H(B)^2 \cdot u_2 + u_0$. If we consider a sequence of blocks $B_1, \dots, B_t$, then the composition

$$\phi_{B_1 \| B_2 \| \dots \| B_t} := \phi_{B_t}(\phi_{B_{t-1}}(\dots \phi_{B_1}(h)) \dots)$$

computes the hash value obtained by appending $B_1 \| B_2 \| \dots \| B_t$ to a preimage of $h \in R'$. As each of the ϕ_{B_i} is affine in the sense just described, the same holds for $\phi_{B_1 \| B_2 \| \dots \| B_t}$ —with the constant shift depending on $B_1, \dots, B_t$. The linear part of $\phi_{B_1 \| B_2 \| \dots \| B_t}$ is just the functional composition of the linear parts of the ϕ_{B_i} s.

Once we know a sequence of 32-bit blocks $B_1, \dots, B_t$ and two different values $h_1, h_2 \in R'$ with

$$\phi_{B_1 \| B_2 \| \dots \| B_t}(h_1) = \phi_{B_1 \| B_2 \| \dots \| B_t}(h_2) \quad ,$$

or equivalently

$$\phi_{B_1 \| B_2 \| \dots \| B_t}(h_1) + \phi_{B_1 \| B_2 \| \dots \| B_t}(h_2) = 0 \quad , \quad (3)$$

we have a collision for H —provided we know preimages of h_1 and h_2 . As the left-hand side of Equation (3) is $\mathbb{F}_2$ -linear in $h_1 + h_2$ —the constant shifts cancel out in the summation—we can rewrite (3) in the form

$$\overline{(h_1 + h_2)} \cdot \mathcal{M}_{B_1 \| B_2 \| \dots \| B_t} = 0 \quad .$$

Here $\mathcal{M}_{B_1 \| B_2 \| \dots \| B_t}$ is an $n \times n$ matrix over $\mathbb{F}_2$, and $\overline{(h_1 + h_2)} \in \mathbb{F}_2^n$ is comprised of the coefficients of $h_1 + h_2$ when being expressed in the appropriate $\mathbb{F}_2$ -vector space basis. Now, if we can find $B_1, \dots, B_t$ such that $\mathcal{M}_{B_1 \| B_2 \| \dots \| B_t}$ is of low rank (i.e., has a large kernel) we can simply try to choose messages $M_1 \neq M_2$ at random until the sum of their hash values $(H(M_1) + H(M_2))$ yields a vector $\overline{(H(M_1) + H(M_2))}$ in the kernel of $\mathcal{M}_{B_1 \| B_2 \| \dots \| B_t}$.

Remark 2. It is worth noting that there is no particular requirement on the messages M_1, M_2 . This seems a useful feature when aiming at meaningful collisions: Suppose we have a message/file format of interest, where it is possible to append “garbage” at the end of a valid message (up to some fixed end-of-message delimiter).

Then we could fix two meaningful messages M'_1, M'_2 which we want to collide and choose our candidates as $M_1 := M'_1 \parallel N_1, M_2 := M'_2 \parallel N_2$ with random bitstrings N_1, N_2 . The final colliding messages then had the form

$$\begin{aligned} M_1 &= M'_1 \parallel N_1 \parallel B_1 \parallel \cdots \parallel B_t \parallel E \\ M_2 &= M'_2 \parallel N_2 \parallel B_1 \parallel \cdots \parallel B_t \parallel E \end{aligned}$$

where E can be a message-independent (possibly empty) end-of-message delimiter.

Expediting the computation of a kernel element In our experiments with the parameters from Section 2.2, finding a small, say ≈ 16 , number t of blocks $B_1, \dots, B_t$ such that $\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t}$ has a rank defect of $\approx t$ required no particular effort. Already a trivial enumeration of some 32-bit blocks B_1 quickly yields a candidate where choosing all t blocks equal to B_1 results in a matrix $\mathcal{M}_{B_1 \parallel B_1 \parallel \dots \parallel B_1}$ with rank defect t . For larger rank defects, however, the heuristics we used required a significantly larger number of blocks (see below). Aiming at collisions of moderate length, it seems worthwhile to improve the simple guessing strategy for finding kernel elements:

Suppose our $n \times n$ matrix $\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t}$ over $\mathbb{F}_2$ has rank defect d . Taking the candidate vectors $\overline{H(M_1) + H(M_2)}$ for independently and uniformly at random chosen elements from $\mathbb{F}_2^n$, we could expect that after $O(2^{n-d})$ attempts a kernel vector is found. If we do not mandate M_1 and M_2 to have a particular form, we can easily improve on this as follows:

1. Using a computer algebra system, we can easily find a vector space basis of the (d -dimensional) kernel $\ker(\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t})$ of $\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t}$.
2. Using a birthday attack we search for messages M_1, M_2 such that the projections of $\overline{H(M_1)}, \overline{H(M_2)}$ on $\ker(\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t})$ coincide. In other words we want $\overline{H(M_1)}$ and $\overline{H(M_2)}$ to be in the same residue class of $\mathbb{F}_2^n / \ker(\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t})$. Then

$$\overline{H(M_1)} + \overline{H(M_2)} = \overline{H(M_1) + H(M_2)} \in \ker(\mathcal{M}_{B_1 \parallel B_2 \parallel \dots \parallel B_t})$$

as desired.

Taking the $\overline{(H(M_i))}$ for independently and uniformly at random chosen elements from $\mathbb{F}_2^n$, we expect to find the desired messages M_1 and M_2 after $O(2^{(n-d)/2})$ attempts.

Example 2. For Shpilrain’s specific parameter proposal (see Section 2.2), in the largest component obtained from the Chinese Remainder Theorem, we have $n = 98$. Here we used a matrix with a rank defect of $d = 42$, constructed from $t = 2882$ blocks B_i .

Finding a low rank matrix By construction, we have

$$\mathcal{M}_{B_1 \| B_2 \| \dots \| B_t} = \mathcal{M}_{B_1} \cdot \mathcal{M}_{B_2} \cdot \dots \cdot \mathcal{M}_{B_{t-1}} \cdot \mathcal{M}_{B_t},$$

with $\mathcal{M}_{B_i}$ being the $n \times n$ matrix over $\mathbb{F}_2$ representing the linear part of ϕ_{B_i} . Thus, the task of finding a matrix $\mathcal{M}_{B_1 \| B_2 \| \dots \| B_t}$ of low rank reduces to finding 32-bit blocks B_i such that we can form products of the respective matrices $\mathcal{M}_{B_i}$ with the product having low rank. Also, from a practical perspective it seems desirable that the number t of blocks is not too large, so that the resulting collision fits into, say, a few KByte.

In our experiments with the parameter set from Section 2.2, simple heuristics turned out to yield adequate blocks $B_1, \dots, B_t$, and we did not attempt a thorough theoretical analysis or optimization of the task:

- For small values of t , say $t \approx 16$, already by just enumerating some 32-bit blocks B_i we quickly obtain candidates such that t identical blocks B_i yield a matrix $\mathcal{M}_{B_i \| B_i \| \dots \| B_i}$ with rank defect t .
- Knowing a product $\mathcal{M}_{B_1} \cdot \dots \cdot \mathcal{M}_{B_{t'}}$ of low rank, one can try to exhaust 32-bit blocks $B_{t'+1}$ until multiplying $\mathcal{M}_{B_1} \cdot \dots \cdot \mathcal{M}_{B_{t'}}$ with $\mathcal{M}_{B_{t'+1}}$ reduces the rank further. Experimentally, this worked nicely for up to around $t' \approx 20$ blocks.
- If we have found a small number of matrix products $\mathcal{P}_1, \dots, \mathcal{P}_v$ with a certain rank defect, we can try to form short products of these $\mathcal{P}_i$ s and hope that the multiplication reduces the rank.

This procedure can be applied repeatedly and in our experiments worked quite nicely. The main drawback is that each $\mathcal{P}_i$ can already be derived from a number of 32-bit blocks B_j : if we form a product

$$\mathcal{P}' := \mathcal{P}_1 \cdot \dots \cdot \mathcal{P}_{n_1}$$

of n_1 matrices $\mathcal{P}_i$ where each $\mathcal{P}_i$ is a product of n_2 matrices $\mathcal{M}_{B_j}$, then $\mathcal{P}'$ corresponds to $n_1 \cdot n_2$ 32-bit blocks B_j .

The next section shows that the above attack can be considered as practical: We use it to derive a collision for the parameter choice proposed in [Shp06] (see Section 2.2).

3.3 A collision for the proposed parameters

As already mentioned, the specific polynomial $p(x)$ suggested by Shpilrain in [Shp06] splits into a product $p(x) = q_1(x) \cdot q_2(x) \cdot q_3(x) \cdot q_4(x)$ as specified in Section 3. Therefore we made use of the Chinese Remainder Theorem as discussed in Section 3.1.

A collision in $\mathbb{F}_{2^{98}}$ To construct a collision in $\mathbb{F}_2[x]/(q_4(x)) \simeq \mathbb{F}_{2^{98}}$ we applied the techniques from the previous section: Using a sequence

$$T'_4 := B_1 \parallel \cdots \parallel B_{2882}$$

of 2882 suitably chosen 32-bit blocks, we derived a matrix $\mathcal{M}_{B_1 \parallel \cdots \parallel B_{2882}}$ of rank 56, i.e., with rank defect $d = 98 - 56 = 42$. To specify T'_4 , we define the following bitstrings (to be read from left to right, line by line):

$A_1 :=$ ‘003FF003 06B80000 06B20000 06B20000 06BA0000 06B0C000
06BA6800 06B4F400 06B6F400 06B52A00 06BB9600 06B9DC80
06BD1180 06B6AB20 06BEF3B0 06B2B470 06BDCAF0 06B11ACC
06B90F3C 06B3B432 06B49CCA 06BB6E03’ (22 · 32 bit)

$A_2 :=$ ‘003FF003 06A80000 06AA0000 06A50000 06A84000 06A24000
06A22000 06A16800 06AE8400 06AE1C00 06ADAE00 06A9D500
06A3B780 06AC29C0 06AD93C0 06A7E260 06A874C2 06A85DCA
06A7A3B9 06ABAF95 06A84DFD’ (21 · 32 bit)

$A_3 :=$ ‘003FF003 06CC0000 06C20000 06CA8000 06CF8000 06C84000
06C64000 06C17000 06CF4800 06C98400 06CB2900 06CE8080
06C79080 06C95080 06C2A948 06CBCE28 06C00214 06CC572C
06C70021’ (19 · 32 bit)

$A_4 :=$ ‘003FF003 06F80000 06F80000 06F80000 06F88000 06F98000
06FBA000 06F13000 06F04800 06FE9C00 06F32E00 06FEEE00
06FA9180 06F4CDC0 06F88EB0 06F0BEF0 06FE26A8 06FB3B78’
(18 · 32 bit)

$A_5 :=$ ‘003FF003 00000000 00010000 00038000 0009C000 000CE000
00065000 0007D000 00033000 000D1400 00033C00 000D0900
00008080 000CD020 000A9FA0 0009EEF0 000BDE0C 000A944C
00031A4A 0007A5FE 001F97E7 004081C9 006AC9DC 008039BD
01C1E775 031A68F0 0E217B84’ (27 · 32 bit)

At this each hexadecimal digit represents a sequence of 4 bits (‘0’ – ‘0000’, ‘1’ – ‘0001’, ..., ‘E’ – ‘1110’, ‘F’ – ‘1111’). Using $A_1, \dots, A_5$ as building blocks, we define eight more bitstrings:

$$A_6 := A_5 \parallel A_5 \parallel A_4 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_5 \quad (226 \text{ blocks})$$

$$A_7 := A_5 \parallel A_5 \parallel A_4 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_5 \quad (199 \text{ blocks})$$

$$A_8 := A_5 \parallel A_5 \parallel A_4 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_5 \parallel A_5 \quad (226 \text{ blocks})$$

$$A_9 := A_5 \parallel A_5 \parallel A_4 \parallel A_5 \parallel A_2 \parallel A_1 \parallel A_5 \parallel A_5 \quad (196 \text{ blocks})$$

$$A_{10} := A_5 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_5 \parallel A_3 \parallel A_3 \parallel A_5 \parallel A_5 \quad (246 \text{ blocks})$$

$$A_{11} := A_5 \parallel A_3 \parallel A_3 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_3 \parallel A_5 \parallel A_5 \quad (265 \text{ blocks})$$

$$A_{12} := A_5 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_3 \parallel A_2 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_5 \quad (275 \text{ blocks})$$

$$A_{13} := A_5 \parallel A_4 \parallel A_5 \parallel A_5 \parallel A_5 \parallel A_3 \parallel A_3 \parallel A_5 \parallel A_5 \quad (218 \text{ blocks})$$

In terms of $A_6, \dots, A_{13}$, the bitstring T'_4 can be described as follows:

$$T'_4 := A_{11} \parallel A_{12} \parallel A_7 \parallel A_{11} \parallel A_{13} \parallel A_{10} \parallel A_9 \parallel A_6 \parallel A_{11} \parallel A_{12} \parallel A_8 \parallel A_6$$

Next, with a birthday attack as described we found two 32-bit blocks

$$M_1 := \text{‘2B99EF46’} \text{ and } M_2 := \text{‘02B6CF84’}$$

with $\overline{H(M_1) + H(M_2)}$ being in the kernel of $\mathcal{M}_{B_1 \parallel \dots \parallel B_{2882}}$. Consequently we obtain

$$H(M_1 \parallel T'_4) \equiv H(M_2 \parallel T'_4) \pmod{q_4(x)} \quad . \quad (4)$$

Pruning T'_4 Inspecting $M_1 \parallel T'_4$ and $M_2 \parallel T'_4$ more closely, it turns out that (4) remains valid, if we remove the last 300 blocks from T'_4 . We write T_4 for the bitstring of length $2582 \cdot 32 = 2882 \cdot 32 - 300 \cdot 32$ resulting from pruning T'_4 accordingly. In particular, we have

$$H(M_1 \parallel T_4) \equiv H(M_2 \parallel T_4) \pmod{q_4(x)} \quad . \quad (5)$$

Applying the Chinese Remainder Theorem Next, we want to identify bitstrings T_3, T_2, T_1 such that

$$H(M_1 \parallel T_4 \parallel \dots \parallel T_i) \equiv H(M_2 \parallel T_4 \parallel \dots \parallel T_i) \pmod{q_1(x) \cdot \dots \cdot q_i(x)}$$

holds for $1 \leq i \leq 4$.

The polynomial $q_3(x)$ is of degree 38. To extend the “ $\mathbb{F}_{2^{98}}$ -collision” in (5) accordingly, the linear algebra approach from before can be reused: First, we identify a short bitstring

$T'_3 :=$ ‘003FF003 06300000 06320000 063E8000 06394000 0638C000
0639A000 063C6000 0633D000 063A3400 063DBA00 0633BC80
06395B80 0637AC40 0635AF10 0636CB38 063CF824 063EEE8C’
(18 blocks)

which, when “hashing modulo $q_3(x)$ ”, corresponds to a matrix $\mathcal{M}_{T'_3}$ of low rank. Then we enumerate short bitstrings, until a candidate

$$T''_3 := \text{‘00171999’}$$

is found such that $H(M_1 \parallel T_4 \parallel T''_3) + H(M_2 \parallel T_4 \parallel T''_3) \pmod{q_3(x)}$ yields a vector in the kernel of $\mathcal{M}_{T'_3}$. Defining T_3 as $T_3 := T''_3 \parallel T'_3$, we have

$$H(M_1 \parallel T_4 \parallel T_3) \equiv H(M_2 \parallel T_4 \parallel T_3) \pmod{q_4(x) \cdot q_3(x)} \quad (6)$$

as desired. Extending the collision in (6) to the complete quotient ring $\mathbb{F}_2[x]/(q_1(x) \cdot q_2(x) \cdot q_3(x) \cdot q_4(x))$ turns out to be straightforward: Appending one more 32-bit block

$$T_2 := \text{‘0008D718’}$$

already yields the desired collision

$$H(M_1 \parallel T_4 \parallel T_3 \parallel T_2) = H(M_2 \parallel T_4 \parallel T_3 \parallel T_2) \quad .$$

Thus, we have found two different bitstrings of size $2603 \cdot 32$ bit (i. e., ≈ 10.2 KByte), both of which hash to the same value.

For computing this collision we used the computer algebra system Magma [BCP97] on a number of different hardware platforms. We estimate our computational effort to be in the magnitude of one CPU day on a standard PC with about 8 GByte RAM.

4 Conclusion

As explained in the above discussion and demonstrated through a specific collision, the hash function proposed in [Shp06] does not offer strong collision resistance. Consequently, for applications that rely on collision resistance, the use of this hash function does not seem to be advisable.

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