ERGODIC THEORY OVER $\mathbb{F}_2[[T]]$

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Abstract. In cryptography and coding theory, it is important to study the pseudo-random sequences and the ergodic transformations. We already have the 1-Lipshitz ergodic theory over $\mathbb{Z}_2$ established by V. Anashin and others. In this paper we present an ergodic theory over $\mathbb{F}_2[[T]]$ and some ideas which might be very useful in applications.

Keywords: Ergodic; Function Fields.

1. introduction

A dynamical system on a measurable space $\mathbb{S}$ is understood as a triple $(\mathbb{S}; \mu; f)$, where $\mathbb{S}$ is a set endowed with a measure μ, and

$$f : \mathbb{S} \to \mathbb{S}$$

is a measurable function, that is, an f-preimage of any measurable subset is a measurable subset.

A trajectory of the dynamical system is a sequence

$$x_0, f(x_0), f^{(2)}(x_0), f^{(3)}(x_0), \cdots$$

of points of the space $\mathbb{S}$, x_0 is called an initial point of the trajectory.

Definition 1. A mapping $F : \mathbb{S} \to \mathbb{Y}$ of a measurable space $\mathbb{S}$ into a measurable space $\mathbb{Y}$ endowed with probabilistic measure μ and ν, respectively, is said to be measure-preserving whenever $\mu(F^{-1}(S)) = \nu(S)$ for each measurable subset $S \subset \mathbb{Y}$. In case $S = \mathbb{Y}$ and $\mu = \nu$, a measure preserving mapping F is said to be ergodic if $F^{-1}(S) = S$ for a measurable set S implies either $\mu(S) = 1$ or $\mu(S) = 0$.

In the case $\mathbb{S} = \mathbb{Z}_p$, a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ has its Mahler expansion:

$$f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i}, \ a_i \in \mathbb{Z}_p, \qquad a_i \to 0 \quad \text{as } i \to \infty.$$

We say that such a function f satisfies the 1-Lipschitz condition if

$$|f(x+y) - f(x)| \leq |y|, \text{ for any } x, y \in \mathbb{Z}_p.$$

1-Lipshitz condition is also called "compatible" condition. V. Anashin gives some sufficient and necessary conditions on the Mahler coefficients for f to be 1-Lipschitz and measure-preserving. When p is odd ($p = 2$ respectively), he also gives the sufficient (sufficient and necessary, respectively) conditions on the Mahler coefficients and the Van der Put coefficients for f to be ergodic, i.e.

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Proposition 1 ([An1]). *A 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is measure-preserving (ergodic) if and only if it is bijective (transitive respectively) modulo p^k modulo p^k for all integers $k \geq 0$.*

Theorem 1 ([An2]). *(measure preserving property) function $f : \mathbb{Z}_p \to \mathbb{Z}_p$,*

$$f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i}$$

defines a 1-Lipschitz measure-preserving transformation on $\mathbb{Z}_p$ whenever the following conditions hold simultaneously:

$$\begin{array}{ll} a_1 \not\equiv 0 & (\operatorname{mod} p), \\ a_i \equiv 0 & (\operatorname{mod} p^{[\log_p(i)]+1}), \quad i = 2, 3, \cdots . \end{array}$$

The function f defines a 1-Lipschitz ergodic transformation on $\mathbb{Z}_p$ whenever the following conditions hold simultaneously:

$$\begin{array}{ll} a_0 \not\equiv 0 & (\operatorname{mod} p), \\ a_1 \equiv 1 & (\operatorname{mod} p), \quad for\ p\ odd, \\ a_1 \equiv 1 & (\operatorname{mod} 4), \quad for\ p = 2, \\ a_i \equiv 0 & (\operatorname{mod} p^{[\log_p(i+1)]+1}), \quad i = 2, 3, \cdots . \end{array}$$

Moreover, in the case $p = 2$ these conditions are necessary.

For any non-negative integer m, the Van der Put function $\chi(m, x)$ on $\mathbb{Z}_p$ is the characteristic function

$$\chi(m, x) = \begin{cases} 1, & \text{if } |x - m|_p \leq p^{-[\log_p m]-1}, \\ 0, & \text{otherwise,} \end{cases}$$

for $x \neq 0$, and

$$\chi(0, x) = \begin{cases} 1, & \text{if } |x|_p \leq 1/p, \\ 0, & \text{otherwise .} \end{cases}$$

The Van der Put functions $\chi(m, x)$ consist of an orthonormal basis of the space $C(\mathbb{Z}_p, \mathbb{Q}_p)$ of the continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ (see [Ma]). In terms of the Van der Put basis $\{\chi(m, x)\}_{m \geq 0}$, we have

Theorem 2 ([An4]). *A function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and preserves the Haar measure if and only if it can be represented as*

$$f(x) = b_0\chi(0, x) + b_1\chi(1, x) + \sum_{m=2}^{\infty} 2^{[\log_2 m]} b_m \chi(m, x),$$

where $b_m \in \mathbb{Z}_2$ for $m = 0, 1, 2, \cdots$, and

- $b_0 + b_1 \equiv 1 (\operatorname{mod} 2)$
- $|b_m|_2 = 1$, *for $m \geq 2$.*

Theorem 3 ([An4]). *A function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and ergodic if and only if it can be represented as*

$$f(x) = b_0\chi(0, x) + b_1\chi(1, x) + \sum_{m=2}^{\infty} 2^{[\log_2 m]} b_m \chi(m, x),$$

where $b_m \in \mathbb{Z}_2$ for $m = 0, 1, 2, \cdots$, and

- $b_0 \equiv 1 \pmod 2$, $b_0 + b_1 \equiv 3 \pmod 4$, $b_2 + b_3 \equiv 2 \pmod 4$,
- $|b_m|_2 = 1$, *for* $m \geq 2$,
- $\sum_{m=2^{n-1}}^{2^n-1} b_m \equiv 0 (\mathrm{mod}\, 4)$, *for* $n \geq 3$.

1-Lipschitz functions and ergodic functions over $\mathbb{Z}_p$ enjoy extensive applications in coding and cryptography theory.

The two discrete valuation rings $\mathbb{F}_p[[T]]$ and $\mathbb{Z}_p$ are homeomorphic. Therefore in the same way we define the Van der Put basis $\{\chi(\alpha, x)\}_{\alpha \in \mathbb{F}_p[T]}$ on the space of functions over $\mathbb{F}_p[[T]]$:

$$\chi(\alpha, x) = \begin{cases} 1, & \text{if } x \in B_{p^{-\deg(\alpha)-1}}(\alpha), \\ 0, & \text{otherwise,} \end{cases}$$

is the characteristic function of the ball $B_{p^{-\deg(\alpha)-1}}(\alpha) = \{x \in \mathbb{F}_p[[T]] : |x - \alpha|_T \leq p^{-\deg(\alpha)-1}\}$ of the center α and the radius $|T|^{\deg(\alpha)+1}$ if $\alpha \neq 0$, and

$$\chi(0, x) = \begin{cases} 1, & \text{if } x \in B_{p^{-1}}(0), \\ 0, & \text{otherwise,} \end{cases}$$

is the characteristic function of the ball $B_{p^{-1}}(0)$ of the center 0 and the radius p^{-1}. Then it is easy to see that every T-adic continuous function $f : \mathbb{F}_p[[T]] \to \mathbb{F}_p[[T]]$ can be expressed as

$$f(x) = \sum_{\alpha \in \mathbb{F}_p[T]} B_\alpha \chi(\alpha, x), \; B_\alpha \in \mathbb{F}_p[[T]], \quad \text{with } B_\alpha \to 0 \text{ as } \deg \alpha \to \infty.$$

The elementary theory of p-adic analysis occurred back in 1958 (see [Ma], Mahler). It is relatively recently that the theory is applied to the theory of cryptography and coding. In 2002, A. Klimov and A. Shamir used 'T-function' to produce long period pseudo-random sequences. The object 'T-function' is the 'Compatible mapping' in algebra, determined function in automata theory, triangle boolean mapping in Boolean function theory, 1-Lipschitz function in p-adic analysis. Since basic instructions of a processor, with the exception of rotations and shifts towards the low order bits, are all 'T-functions', it is important to study 1-Lipschitz ergodic functions. In this paper, we will extend the ergodic function theory over $\mathbb{Z}_p$ to the power series ring $\mathbb{F}_p[[T]]$. We are especially interested in the case $p = 2$ because of the potential application to the coding theory. We will first establish a useful lemma in determining the ergodic functions, then use the Van der Put basis over $\mathbb{F}_2[[T]]$ to describe the ergodic functions $f : \mathbb{F}_2[[T]] \to \mathbb{F}_2[[T]]$, and finally, we translate the results to the expansion coefficients of Carlitz basis. Our motivation comes from the same topology structure between $\mathbb{Z}_p$ and $\mathbb{F}_p[[T]]$, and the fact that the addition on $\mathbb{F}_r[[T]]$ is easier than that on $\mathbb{Z}_p$, which could make computations more effective in applications. The first and the second authors want to thank Professor V. Anashin for many inspirational talks about the topics of p-adic dynamical systems and related problems.

2. Ergodic Functions over $\mathbb{F}_2[[T]]$ and Van der Put Expansions

We will discuss the ergodic functions over $\mathbb{F}_2[[T]]$ and what the conditions of the expansion coefficients under the Van der Put basis should be satisfied.

The absolute value on the discrete valuation ring $\mathbb{F}_2[[T]]$ is normalized so that $|T| = \frac{1}{2}$.

Suppose $f : \mathbb{F}_2[T] \to \mathbb{F}_2[T]$ is a map. Then f can be expressed in terms of Van der Put basis:

$$f(x) = \sum_{\alpha \in \mathbb{F}_2[T]} B_\alpha \chi(\alpha, x). \tag{1}$$

If the function f is continuous under the T-adic topology, then $B_\alpha \to 0$ as $\deg(\alpha) \to \infty$, and f can be extended to a continuous function from $\mathbb{F}_2[[T]]$ to itself, which is still denoted by f. The coefficients B_α of the expansion (1) can be calculated as follows (see Mahler's book [Ma] for the detail):

- $B_0 = f(0), B_1 = f(1)$;
- $B_\alpha = f(\alpha) - f(\alpha - \alpha_n T^n)$, if $\alpha = \alpha_0 + \alpha_1 T + \cdots + \alpha_n T^n \in \mathbb{F}_2[T]$ with $\alpha_n \neq 0$ (that is, $\alpha_n = 1$) is of degree greater than or equal to 1.

Also for $\alpha = \alpha_0 + \alpha_1 T + \cdots + \alpha_n T^n \in \mathbb{F}_2[T]$, we denote by

$$\alpha_{[k]} = \alpha_0 + \alpha_1 T + \cdots + \alpha_k T^k$$

for any k between 0 and n.

Theorem 4. *A continuous function $f : \mathbb{F}_2[[T]] \to \mathbb{F}_2[[T]]$ is 1-Lipschitz (1-Lip) if and only if it can be expressed as*

$$f(x) = b_0 \chi(0, x) + \sum_{\alpha \in \mathbb{F}_2[T] \backslash \{0\}} T^{\deg_T \alpha} b_\alpha \chi(\alpha, x), \qquad \textit{with } b_\alpha \in \mathbb{F}_2[[T]],$$

where the sum over $\mathbb{F}_2[T]$ is added according to the order $(0, 1, T, 1+T, T^2, 1+T^2, T+T^2, 1+T+T^2, \cdots)$ of ascending degrees of $\alpha \in \mathbb{F}_2[T]$.

Proof. Suppose f is 1-Lip. Then it is clear that $b_0 = B_0 = f(0) \in \mathbb{F}_2[[T]]$ and $b_1 = B_1 = f(1) \in \mathbb{F}_2[[T]]$. For $\alpha = \alpha_0 + \alpha_1 T + \cdots + \alpha_n T^n \in \mathbb{F}_2[T]$ of degree $n \geq 1$, we have

$$|B_\alpha|_T = |f(\alpha) - f(\alpha - \alpha_n T^n)|_T \leq |\alpha_n T^n|_T = 2^{-n}.$$

Therefore B_α can be written as $B_\alpha = T^{\deg_T \alpha} b_\alpha$ with $b_\alpha \in \mathbb{F}_2[[T]]$.

Conversely, suppose $f(x) = b_0 \chi(0, x) + \sum_{\alpha \in \mathbb{F}_2[T] \backslash \{0\}} T^{\deg_T \alpha} b_\alpha \chi(\alpha, x)$ with $b_\alpha \in \mathbb{F}_2[[T]]$. If X, Y both belong to $\mathbb{F}_2[[T]]$ and $X \equiv Y (\operatorname{mod} T^n)$, then

$$\chi(\alpha, X) = \chi(\alpha, Y) \text{ for any } \alpha \in \mathbb{F}_2[[T]] \text{ with } \alpha = 0 \text{ or } \deg(\alpha) < n.$$

Therefore we have $f(X) \equiv f(Y) (\operatorname{mod} T^n)$, hence the function f is 1-Lip. $\square$

Proposition 2. *A 1-Lipschitz function $f : \mathbb{F}_p[[T]] \to \mathbb{F}_p[[T]]$ is measure-preserving (respectively, ergodic) if and only if f is bijective modulo T^k(respectively, transitive modulo T^k) for all integers $k \geq 0$.*

Proof. As measure-preserving(respectively,ergodic) and bijective modulo T^k(respectively, transitive modulo T^k) are all topological properties, their relationships can be completely described from topological structures. It is well known that $\mathbb{F}_p[[T]]$ and $\mathbb{Z}_p$ have the same non-archimedean topology(T-adic and p-adic topology). Hence the proof follows [An2], sections 4.4.1 to 4.4.3 of chapter 4. $\square$

Theorem 5. *(measure preserving property) A 1-Lip function $f : \mathbb{F}_2[[T]] \to \mathbb{F}_2[[T]]$,*

$$f(x) = b_0 \chi(0, x) + \sum_{\alpha \in \mathbb{F}_2[T] \backslash \{0\}} T^{\deg_T \alpha} b_\alpha \chi(\alpha, x), \qquad \textit{with } b_\alpha \in \mathbb{F}_2[[T]]$$

is measure preserving if and only if the following conditions hold simultaneously:

(1) $b_0 + b_1 \equiv 1(\bmod T)$;

(2) $|b_\alpha| = 1$, *for* $\deg(\alpha) \geq 1$.

Proof. Suppose f is bijective $\bmod T^n$ for all $n \in \mathbb{N}$, we need to show that the two conditions for the Van der Put coefficients are satisfied. At first, from the bijectivity of $\bmod T$, we get

$f(0) = b_0 \equiv 1 \,(\bmod T)$; $f(1) = b_1 \equiv 0 \,(\bmod T)$; or
$f(0) = b_0 \equiv 0 \,(\bmod T)$; $f(1) = b_1 \equiv 1 \,(\bmod T)$.

Thus we get $b_0 + b_1 \equiv 1 \,(\bmod T)$. Secondly, we consider the bijectivity of the function f mod T^2. As $f(T) - f(0) \not\equiv 0(\bmod T^2)$, we get $b_0\chi(0,T) + Tb_T\chi(T,T) - b_0\chi(0,0) = Tb_T \not\equiv 0(\bmod T^2)$, therefore $|b_T| = 1$; Also $f(1+T) - f(1) \not\equiv 0(\bmod T^2)$ implies $b_1\chi(1, 1+T) + Tb_{1+T}\chi(1+T, 1+T) - b_1\chi(1,1) = Tb_{1+T} \not\equiv 0(\bmod T^2)$, therefore $|b_{1+T}| = 1$. In the same way for the general case when $\deg(\alpha) = n$, we use the bijectivity of the function $\bmod T^{n+1}$, thus $f(\alpha) - f(\alpha - \alpha_n T^n) = T^{\deg_T \alpha} b_\alpha = T^n b_\alpha \not\equiv 0(\bmod T^{n+1})$, therefore $|b_\alpha| = 1$.

Conversely, suppose the two conditions for Van der Put coefficients are satisfied. As $f(0) = b_0, f(1) = b_1$, we see that the first condition implies the bijectivity $\bmod T$. To derive the bijectivity $\bmod T^n$, we choose

$$X = X_0 + X_1T + \cdots + X_{n-1}T^{n-1}, \quad Y = Y_0 + Y_1T + \cdots + Y_{n-1}T^{n-1}$$

such that $f(X) - f(Y) \equiv 0(\bmod T^n)$. If $X \not\equiv Y(\bmod T^n)$, then we denote the first integer m between 0 and n such that $X_m \neq Y_m$ and consider the equation $f(X) - f(Y) \equiv 0(\bmod T^{m+1})$. As for $i < m$, $X_i = Y_i$, thus $\chi(\alpha, X_{[m-1]}) = \chi(\alpha, Y_{[m-1]})$ for all $\alpha, \deg_T(\alpha) < m$. But $\chi(\alpha, X_{[m]}) \neq \chi(\alpha, Y_{[m]})$ for $\deg_T(\alpha) = m$. Denote by $\gamma = X_{[m-1]} = Y_{[m-1]}$, then $f(X) - f(Y) = T^m b_{\gamma+T^m} + (\text{ higher } T\text{-power terms}) \not\equiv 0(\bmod T^{m+1})$, as $|b_{\gamma+T^m}| = 1$. This contradicts to $f(X) - f(Y) \equiv 0(\bmod T^{m+1})$. Therefore $X \equiv Y(\bmod T^n)$, and so f is injective. Thus f is bijective $\bmod T^n$, as $\mathbb{F}_2[[T]]/T^n$ is a finite set. □

Lemma 1. *Suppose a 1-Lip measure-preserving function* $f : \mathbb{F}_2[[T]] \to \mathbb{F}_2[[T]]$ *is transitive(single orbit) over* $\mathbb{F}_2[[T]]/T^n$, $n > 2$. *Then* f *is transitive over* $\mathbb{F}_2[[T]]/T^{n+1}$ *if and only if* $\#\{x \in \mathbb{F}_2[T] : \deg_T(x) < n \text{ and } \deg_T(f(x)) = n\}$ *is an odd integer.*

Remark 1. We are going to give descriptions of ergodic functions on $\mathbb{F}_2[[T]]$ in terms of Van der Put basis (Theorem 6) and in terms of Carlitz basis (Theorem 9). But in applications to computer programming of cryptography, this lemma should provide a much more efficient method in creating psudo-random sequences.

Proof. Let $A = \mathbb{F}_2[T]$ be the polynomial ring, $A_n = \{x \in A : \deg(x) < n\}$ for any non-negative integer n.

"Necessity". As f is 1-Lip, when we consider the trajectory of f modulo T^k, we need only consider $\{x_0 \bmod T^k, f(x_0 \bmod T^k), \cdots, f^{(i)}(x_0 \bmod T^k), \cdots\}$ with representatives of image elements chosen in A_k. If f is transitive over $\mathbb{F}_2[[T]]/T^{n+1}$, then there exist $x_0, x_1 \in A_n$ such that $f(x_0) = x_1 + T^n$. We consider the trajectory of f modulo T^{n+1} starting with x_0:

$$\begin{array}{ccccccccc} & x_0 & \to & f(x_0) & \to & \cdots & \to & f^{(2^n-1)}(x_0) & \to \\ \to & f^{(2^n)}(x_0) & \to & f^{(2^n+1)}(x_0) & \to & \cdots & \to & f^{(2^{n+1}-1)}(x_0) & \to \\ \to & x_0 + T^{n+1}(1+*) & & & & (\bmod T^{n+1}) & & & \end{array} \tag{2}$$

where "$*$" $\in T\mathbb{F}_2[[T]]$, and an element of the second row is equal to the corresponding element of the first row in the the column plus a T^n, since the map f is 1-Lip and measure-preserving: $f^{(2^n+i)}(x_0) \equiv f^{(i)}(x_0) + T^n \mod T^{n+1}$ for $0 \leq i \leq 2^n - 1$. We look at the elements from the left to the right in the first row, if there is an element in A_n other than x_0 mapped by f to an element in the set $A_n + T^n$, then there would be some other element in the set $A_n + T^n$ mapped to A_n, and hence in the second row there would be an element in A_n mapped by f to an element in $A_n + T^n$. This implies that the total number of elements in A_n in the trajectory mapped by f to $A_n + T^n$ is an odd integer, that is, $\#\{x \in \mathbb{F}_2[T] : \deg_T(x) < n$ and $\deg_T(f(x)) = n\}$ is an odd integer.

"Sufficiency". By the condition, there must exist $x_0, x_1 \in A_n$ such that $f(x_0) = x_1 + T^n$. We consider diagram (2) again. Since f is transitive modulo T^n, the elements of the first row are distinct and so are the elements of the second row. It also implies that $f^{(2^n)}(x_0)$ is either equal to x_0 or $x_0 + T^n$. But if $f^{(2^n)}(x_0) = x_0$, then $\#\{x \in \mathbb{F}_2[T] : \deg_T(x) < n$ and $\deg_T(f(x)) = n\}$ would be an even integer. Therefore we must have $f^{(2^n)}(x_0) = x_0 + T^n$. And we get $f^{(2^n+i)}(x_0) \equiv f^{(i)}(x_0) + T^n \mod T^{n+1}$ for $0 \le i \le 2^n - 1$ by the 1-Lip measure-preserving assumption on f. Hence all the elements in the first row and the second of diagram (2) are distinct, that is, f is transitive modulo T^{n+1}. □

Theorem 6. *(ergodic property) A 1-Lip function $f : \mathbb{F}_2[[T]] \to \mathbb{F}_2[[T]]$*

$$f(x) = b_0\chi(0,x) + \sum_{\alpha \in \mathbb{F}_2[T]\setminus\{0\}} T^{\deg_T \alpha} b_\alpha \chi(\alpha, x), \qquad \textit{with } b_\alpha \in \mathbb{F}_2[[T]] \tag{3}$$

is ergodic if and only if the following conditions hold simultaneously:

(1) $b_0 \equiv 1 \pmod T$, $b_0 + b_1 \equiv 1 + T \pmod{T^2}$, $b_T + b_{1+T} \equiv T \pmod{T^2}$;
(2) $|b_\alpha| = 1$, *for* $\deg(\alpha) \ge 1$;
(3) $\sum_{\alpha = T^{n-1}}^{1+T+\cdots+T^{n-1}} b_\alpha \equiv T (\operatorname{mod} T^2)$.

Proof. Since f is a 1-Lip function, we have

$$f(x) = B_0\chi(0,x) + \sum_{\alpha \in \mathbb{F}_2\setminus\{0\}} B_\alpha \chi(\alpha, x) = b_0\chi(0,x) + \sum_{\alpha \in \mathbb{F}_2[T]\setminus\{0\}} T^{\deg_T \alpha} b_\alpha \chi(\alpha, x)$$

with $b_\alpha \in \mathbb{F}_2[[T]]$.

"Necessity". Suppose f is ergodic. By transitivity modulo T, we get

$$f(0) \equiv 1 \mod T, \quad f(1) \equiv 0 \mod T.$$

Therefore $b_0 = B_0 \equiv 1 \;(\bmod T)$, and also $f(0) + f(1) \equiv 1 \;(\bmod T)$. But $f(0) + f(1) \ne 1 (\operatorname{mod} T^2)$, otherwise we have $f(0) \equiv 1 (\operatorname{mod} T^2)$, $f(1) \equiv 0 (\operatorname{mod} T^2)$; or $f(0) \equiv 1 + T (\operatorname{mod} T^2)$, $f(1) \equiv T (\operatorname{mod} T^2)$, but by the transitivity $\operatorname{mod} T^2$, these two cases can not appear. So we get

$$f(0) + f(1) \equiv 1 + T (\operatorname{mod} T^2), \qquad \text{that is,} \quad b_0 + b_1 \equiv 1 + T (\operatorname{mod} T^2). \tag{4}$$

By the transitivity $\operatorname{mod} T^2$ and Lemma 1, we know that

$$f(0) + f(1) + f(T) + f(1+T) \equiv T^2 (\operatorname{mod} T^3),$$

which gives us

$$B_T + B_{1+T} \equiv T^2 (\operatorname{mod} T^3).$$

As $B_T = Tb_T$, $B_{1+T} = Tb_{1+T}$, we get

$$b_T + b_{1+T} \equiv T (\operatorname{mod} T^2).$$

Now consider

$$\sum_{x\in A_n} f(x) = \sum_{\beta\in A_{n-1}} f(\beta+T^{n-1}) + \sum_{\beta\in A_{n-1}} f(\beta) = \sum_{\alpha=T^{n-1}}^{1+T+\cdots+T^{n-1}} B_\alpha.$$

Lemma 1 gives us

$$\sum_{x\in A_n} f(x) \equiv T^n \bmod T^{n+1},$$

so we put these equations together to get

$$\sum_{\alpha=T^{n-1}}^{1+T+\cdots+T^{n-1}} b_\alpha \equiv T \bmod T^2.$$

"Sufficiency". Suppose the three conditions are satisfied, we want to prove f is transitive on every $\mathbb{F}_2[[T]]/T^n$ for all $n\in\mathbb{N}$. But this is just to apply Lemma 1 on the induction process for n, the first condition gives the first step of the induction. □

3. 1-Lip Functions over $\mathbb{F}_r[[T]]$ and Carlitz Expansions

We first recall some useful formulas in function field arithmetic, with all the details and expositions in [Go]. Let $A=\mathbb{F}_r[T]$ ($r=p^m$) with the normalized absolute value $|\cdot|_T$ such that $|T|=|T|_T=1/r$. The completion of A with respect to this absolute value is $\hat{A}=\mathbb{F}_r[[T]]$.

Definition 2. We set the following notations:

- $[i]=T^{r^i}-T$, where i is a positive integer;
- $L_i=1$ if $i=0$; and $L_i=[i]\cdot[i-1]\cdots[1]$ if i is a positive integer;
- $D_i=1$ if $i=0$; and $D_i=[i]\cdot[i-1]^r\cdots[1]^{r^{i-1}}$ if i is a positive integer;
- for any non-negative integer $n=n_0+n_1\,r+\cdots+n_s\,r^s$, the n-th Carlitz factorial $\Pi(n)$ is defined by
$$\Pi(n)=\prod_{j=0}^{s} D_j^{n_j};$$
- $e_d(x)=\begin{cases} x, & \text{if } d=0,\\ \displaystyle\prod_{\alpha\in A,\,\deg_T(\alpha)<d}(x-\alpha), & \text{if } d \text{ is a positive integer;}\end{cases}$
- $E_i(x)=e_i(x)/D_i$, for any non-negative integer i;
- $G_n(x)=\prod_{i=0}^{s}(E_i(x))^{n_i}$, $n=n_0+n_1r+\cdots+n_sr^s$ non-negative integers, $0\le n_i<r$;
- $G'_n(x)=\prod_{i=0}^{s}G'_{n_ir^i}(x)$, where $G_{n_ir^i}=\begin{cases}(E_i(x))^{n_i}, & \text{if } 0\le n_i<r-1,\\ (E_i(x))^{n_i}-1, & \text{if } n_i=r-1.\end{cases}$

The polynomials $G_n(x)$ and $G'_n(x)$ are called Carlitz polynomials.

Proposition 3 ([Ca]). *The following formulas hold for Caritz polynomials*

- $G_m(t+x)=\sum_{\substack{k+l=m\\k,l\ge0}}\binom{m}{k}G_k(t)G_l(x)$, $t,x\in\mathbb{F}_2[[T]]$.

- $G'_m(t+x) = \sum\limits_{\substack{k+l=m\\k,l\geq 0}} \binom{m}{k} G_k(t)G'_l(x)$.

Orthogonality property of $\{G_n(x)\}_{n\geq 0}$ *and* $\{G'_n(x)\}_{n\geq 0}$*:*

- *For any* $s < r^m$, l *an arbitrary non-negative integer,*

$$\sum_{\deg(\alpha)<m} G_l(\alpha)G'_s(\alpha) = \begin{cases} 0, & \text{if } l+s \neq r^m - 1; \\ (-1)^m, & \text{if } l+s = r^m - 1. \end{cases} \tag{5}$$

The polynomials $G_n(x)$ and $G'_n(x)$ map A to A. And it is well known that $\{G_n(x)\}_{n\geq 0}$ is an orthonormal basis of the space $C(\mathbb{F}_r[[T]], \mathbb{F}_r((T)))$ of continuous functions, that is, every T-adic continuous function can be written as:

$$f(x) = \sum_{n=0}^{\infty} a_n G_n(x), \quad \text{ where } |a_n| \to 0, \text{ as } n \to \infty,$$

with the sup-norm $||f|| = \max\limits_n\{|a_n|\}$. Moreover, the expansion coefficient a_n can be calculated as:

$$a_n = (-1)^m \sum_{\deg(\alpha)<m} G'_{r^m-1-n}(\alpha)f(\alpha), \text{ for any integer such that } r^m > n. \tag{6}$$

Following Wagner [Wa], we define a new sequence of polynomials $\{H_n(x)\}_{n\geq 0}$ by

$$H_0(x) = 1, \qquad \text{and}$$
$$H_n(x) = \frac{\Pi(n+1)G_{n+1}(x)}{\Pi(n)\, x} \qquad \text{for } n \geq 1.$$

Then we get

Lemma 2 (Wagner [Wa]). $\{H_n(x)\}_{n\geq 0}$ *is an orthonormal basis of* $C(\mathbb{F}_r[[T]], \mathbb{F}_r((T)))$.

To study the 1-Lip functions over $\hat{A} = \mathbb{F}_r[[T]]$, we recall the interpolation polynomials introduced by Amice [Am]. These polynomials $\{Q_n(x)\}_{n\geq 0}$ are constructed from which is called by Amice the "very well distributed sequence" $\{u_n\}_{n\geq 0}$ with $u_n \in A$:

$$Q_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{(x-u_0)(x-u_1)\cdots(x-u_{n-1})}{(u_n-u_0)(u_n-u_1)\cdots(u_n-u_{n-1})}, & \text{if } n \geq 1. \end{cases} \tag{7}$$

We choose the sequence $\{u_n\}_{n\geq 0}$ in the following way such that $u_n \neq 0$ for any $n \geq 0$. Let $S = \{\alpha_0, \alpha_1, \cdots, \alpha_{r-1}\}$ be a system of representatives of $A/(T\cdot A)$, and assume that $\alpha_0 = T$ (thus $0 \notin S$). Then any element $x \in \mathbb{F}_r((T))$ can be uniquely written as

$$x = \sum_{k \gg -\infty}^{\infty} \beta_k \, \pi^k$$

where $\beta_k \in S$, with x in $\hat{A}$ if and only if $\beta_k = 0$ for all $k < 0$. To each non-negative integer $n = n_0 + n_1 q + \cdots + n_s r^s$ in r-digit expansion, we assign the element

$$u_n = \alpha_{n_0} + \alpha_{n_1}T + \cdots + \alpha_{n_s}T^s.$$

We have

Theorem 7 ([Am]). *The interpolation polynomials $\{Q_n(x)\}_{n\geq 0}$ defined above is an orthonormal basis of $C(\mathbb{F}_r[[T]],\mathbb{F}_r((T)))$. That is, any continuous function $f(x)$ from $\mathbb{F}_r[[T]]$ to $\mathbb{F}_r((T))$ can be written as*

$$f(x)=\sum_{n=0}^{\infty} a_n Q_n(x), \tag{8}$$

where $a_n \to 0$ as $n\to\infty$, and the sup-norm of f is given by $||f|| = \max_n\{|a_n|\}$.

Since $Q_n(u_n)=1$ for all n, and $Q_m(u_n)=0$ for $m>n$ by the equation (7), we see that the expansion coefficients a_n can be deduced by the following induction formula

$$\begin{aligned} a_0 &= f(0)\\ a_n &= f(u_n)-\sum_{j=0}^{n-1} a_j Q_j(u_n). \end{aligned} \tag{9}$$

Equation (8) is valid not only for continuous functions on $\hat{A}=\mathbb{F}_r[[T]]$, but also for any function f from $A\backslash\{0\}$ to $\mathbb{F}_r((T))$, since the "very well distributed" sequence $\{u_n\}$ we choose is just $A\backslash\{0\}$ as a set, thus the summation of equation (8) is a finite sum when an element of $A\backslash\{0\}$ is plugged in. The element 0 is excluded because of the way the sequence $\{u_n\}$ is chosen. This idea comes from Mahler [Ma], and is important to deal with the 1-Lip functions on $\hat{A}$. If the function f is continuous on $\hat{A}\backslash\{0\}$, then f is certainly determined by the values of f at the points of $A\backslash\{0\}$.

Suppose $\{R_n(x)\}_{n\geq 0}$ is any orthonormal basis of $C(\hat{A},\mathbb{F}_r((T)))$ consisting of polynomials in the variable x with $\deg(R_n)=n$. Then we have

$$Q_n(x)=\sum_{j=0}^{n}\gamma_{n,j}R_j(x), \tag{10}$$

where $\gamma_{n,j}\in\hat{A}$ for all n,j, and $\gamma_{n,n}\in\hat{A}^\times$.

Lemma 3. *Let n be a positive integer, then*

$$\frac{\Pi(n-1)}{\Pi(n)}=\frac{1}{L_{\nu(n)}}$$

Proof. Straightforward computation. □

Lemma 4 ([Ya]). *If a function $f:\mathbb{F}_r[[T]]\backslash\{0\}\to\mathbb{F}_r((T))$ can be expressed as $f(x)=\sum_{n=0}^{\infty}a_nH_n(x)$ for all $x\in\mathbb{F}_r[[T]], x\neq 0$, that is, the summation converges to $f(x)$ for $x\neq 0$, then the sequence $\{a_n\}_{n\geq 0}$ is determined by the values $f(x)$ for all $x\in\mathbb{F}_r[T]\backslash\{0\}$. More precisely, for any non-negative integer n, we choose an integer w such that $n<r^w-1$ and set $S=\{\alpha\in\mathbb{F}_r[T]:\ \deg(\alpha)<w,\alpha\neq 0\}$, then a_n is determined by the values of f at the points of S:*

$$a_n=\frac{(-1)^w}{L_{\nu(n+1)}}\sum_{\alpha\in S}\alpha f(\alpha)G'_{r^w-2-n}(\alpha). \tag{11}$$

Proof. This is a refined statement of Lemma 5.5 of [Ya]. By definition and Lemma 3,

$$H_n(x)=\frac{\Pi(n+1)G_{n+1}(x)}{\Pi(n)x}=L_{\nu(n+1)}\frac{G_{n+1}(x)}{x}$$

for $n \geq 0$, thus

$$xf(x) = \sum_{n=0}^{\infty} a_n L_{\nu(n+1)} G_{n+1}(x)$$

for any $x \neq 0$ in $\hat{A}$. Since $G_{n+1}(0) = 0$, we sum up all elements $\alpha \in S$ in the above equation, and apply the equation (5) of orthogonality property to get

$$\begin{aligned}
& \sum_{\alpha \in S} \alpha f(\alpha) G'_m(\alpha) = \sum_{\alpha \in S} \sum_{n \geq 0} a_n L_{\nu(n+1)} G_{n+1}(\alpha) G'_m(\alpha) \\
= & \sum_{\deg(\alpha) < w} \sum_{n \geq 0} a_n L_{\nu(n+1)} G_{n+1}(\alpha) G'_m(\alpha) \\
= & \sum_{n \geq 0} a_n L_{\nu(n+1)} \sum_{\deg(\alpha) < w} G_{n+1}(\alpha) G'_m(\alpha) \\
= & (-1)^w a_{r^w - 2 - m} L_{\nu(r^w - 1 - m)}.
\end{aligned}$$

Notice that the summations on n are actually finite sums, thus we can change the order of summations on α and on n. Therefore we get the formula (11), and the conclusion. □

For any positive integer n, write $n = n_0 + n_1 r + \cdots + n_w r^w$ in r-digit expansion, with $n_w \neq 0$,

- denote $\nu(n)$ the largest integer such that $r^{\nu(n)} | n$;
- $l(n) = l_r(n) = n_w r^w$.

Lemma 5. *We have*

(1) $|L_n| = r^{-n} = |T|^n$ *for any non-negative integer* n*;*
(2) *For any non-negative integer* n*,* $\nu(n) \leq [\log_r n]$.

Proof. Immediate from definition. □

Denote

$$((i_1, i_2, \cdots, i_s)) = \frac{(i_1 + i_2 + \cdots + i_s)!}{i_1!\, i_2!\, \cdots\, i_s!}$$

for any integers $i_1, i_2, \cdots, i_s \geq 0$. We have the following assertion about the multinomial numbers by Lucas [Lu]:

Lemma 6 (Lucas)**.** *For non-negative integers* $n_0, n_1, \cdots, n_s$*,*

$$((n_0, n_1, \cdots, n_s)) \equiv \prod_{j \geq 0} ((n_{0,j}, n_{1,j}, \cdots, n_{s,j})) \mod p \tag{12}$$

where $n_i = \sum_{j \geq 0} n_{i,j}\, r^j$ *is the* r*-digit expansion for* $i = 0, 1, \cdots, s$.

Remark 2. Lemma 6 is useful when $s = 1$. In this case formula (12) is expressed in the form: let $n = \sum_j n_j\, r^j$ and $k = \sum_j k_j\, r^j$ be r-digit expansion for non-negative integers n and k, then

$$\binom{n}{k} \equiv \prod_{j \geq 0} \binom{n_j}{k_j} \mod p.$$

Lemma 7. *Let* $f(x) = \sum_{n=0}^{\infty} a_n H_n(x)$ *be a continuous function from* $\hat{A} \backslash \{0\}$ *to* $\mathbb{F}_r((T))$ *(this implies that the series converges for any* $x \in \hat{A} \backslash \{0\}$ *). Suppose that* $|f(x)| \leq 1$ *for any* $x \in \hat{A} \backslash \{0\}$*. Then* $|a_n| \leq 1$ *for* $n \geq 0$.

Proof. Since f is continuous, it is determined by the values of f at the points in $A\backslash\{0\}$. From the explanation in the paragraph after Theorem 7, we see that f can be written as

$$f(x)=\sum_{n=0}^{\infty} b_n Q_n(x). \tag{13}$$

Induction formula (9) and the condition that $|f(x)|\le 1$ for any $x\in \hat{A}\backslash\{0\}$ imply $|b_n|\le 1$ for all n.

Now we fix a non-negative integer w and let $N\ge r^w-1$ be an integer. Then for any $\alpha\neq 0$ with $\deg(\alpha)<w$, we can write the right hand of equation (13) as a finite sum:

$$f(\alpha)=\sum_{n=0}^{N} b_n Q_n(\alpha).$$

In the above equation, substitute $Q_n(\alpha)$ by the equation (10) with $R_n(x)=H_n(x)$ for any non-negative integer n, we get

$$f(\alpha)=\sum_{j=0}^{N}\left(\sum_{n=j}^{N} b_n\gamma_{n,j}\right)H_j(\alpha)=\sum_{j=0}^{N} a_j H_j(\alpha),$$

for any $\alpha\neq 0$ with $\deg(\alpha)<w$. Therefore for any non-negative integer $j<r^w-1$, Lemma 4 implies that

$$a_j=\sum_{n=j}^{N} b_n\gamma_{n,j}.$$

Hence we have $|a_j|\le 1$ for $j<r^w-1$. Since w is arbitrary, we get the conclusion. □

Theorem 8. *A continuous function $f(x)=\sum_{n=0}^{\infty} a_n G_n(x)$ from $\mathbb{F}_r[[T]]$ to $\mathbb{F}_r((T))$ is 1-Lip if and only if $|a_n|\le |T|^{[\log_r n]}$ for $n\ge 1$ and $|a_0|\le 1$.*

Proof. The proof is very similar to that on the C^n functions over positive characteristic local rings [Ya]. We can calculate for $y_1\neq 0$ by using the equation of Proposition 3,

$$\begin{aligned}\frac{1}{y_1}(f(y_1+x)-f(x)) &= \sum_{n_0=0}^{\infty} a_{n_0}\frac{1}{y_1}(G_{n_0}(y_1+x)-G_{n_0}(x))\\ &= \sum_{n_0=0}^{\infty}\sum_{j_1=0}^{\infty}\binom{n_0+j_1+1}{j_1+1}\frac{a_{n_0+j_1+1}}{L_{\nu(j_1+1)}}H_{j_1}(y_1)G_{n_0}(x),\end{aligned} \tag{14}$$

The order of summations can be exchanged since the sequence of the terms in the summation tends to 0 as $j_1+n_0\to\infty$ for any $y_1\neq 0$, and any x in $\mathbb{F}_r[[T]]$.

"Sufficiency". The absolute values of $G_{n_0}(x)$, $H_{j_1}(y_1)$, and the binomial numbers of equation (14) are all less than or equal to 1. By Lemma 5 and the condition on a_n, we can estimate that $|a_{n_0+j_1+1}/L_{\nu(j_1+1)}|\le 1$. Therefore the function f is 1-Lip.

"Necessity". Suppose f is 1-Lip. The function $\Psi_1 f(x,y_1)=\frac{1}{y_1}(f(y_1+x)-f(x))$ is continuous on $\mathbb{F}_r[[T]]\times(\mathbb{F}_r[[T]]\backslash\{0\})$. Since $\Psi_1 f(x,y_1)$ is continuous with respect to $x\in$

$\mathbb{F}_r[[T]]$, we get a function

$$F_{n_0}(y_1) = \sum_{j_1=0}^{\infty} \binom{n_0+j_1+1}{j_1+1} \frac{a_{n_0+j_1+1}}{L_{\nu(j_1+1)}} H_{j_1}(y_1)$$

for every $n_0 \geq 0$. We have $|F_{n_0}(y_1)| \leq 1$ for any $y_1 \in \mathbb{F}_r[[T]]\backslash\{0\}$, since f is 1-Lip. And $F_{n_0}(y_1)$ is continuous on $\hat{A}\backslash\{0\}$. Then Lemma 7 implies that

$$\left| \binom{n_0+j_1+1}{j_1+1} \frac{a_{n_0+j_1+1}}{L_{\nu(j_1+1)}} \right| \leq 1, \tag{15}$$

for any $n_0 \geq 0, j_1 \geq 0$.

It is clear that $|a_0| \leq 1$. For any integer $n \geq 1$, we write $n = m_0 + m_1 r + \cdots + m_w r^w$ in r-digit expansion, where $m_w \neq 0$. We choose non-negative integers n_0, j_1 by

$$j_1 + 1 = l(n) = m_w r^w, \quad \text{and} \quad n_0 = n - j_1 - 1.$$

Then from Lucas formula (12) and Lemma 5, we get

$$\binom{n_0+j_1+1}{j_1+1} = 1, \quad \text{and} \quad |L_{\nu(j_1+1)}| = r^{-w} = |T|^{[\log_r n]}.$$

And from equation (15), we see that

$$|a_n| \leq |T|^{[\log_r n]} \quad \text{for } n \geq 1.$$

□

4. Ergodic Functions over $\mathbb{F}_2[[T]]$ and Carlitz Expansions

In this section, we take $r = 2$. And the Carlitz polynomials $G_n(x)$ and $G'_n(x)$ are defined for $x \in \mathbb{F}_2[[T]]$ with coefficients in $\mathbb{F}_2((T))$. We will prove the ergodic property of functions over $\mathbb{F}_2[[T]]$ by translating the conditions of ergodicity under Van der Put basis to Carlitz basis. At first we notice that the polynomials $G_n(x)$ and $G'_n(x)$ have the following special values:

- $G_0(x) = 1$ for any x, $G_n(0) = 0$, if $n \geq 1$;
- $G_1(x) = x$, $G_n(1) = 0$, if $n \geq 2$;
- $G_2(T) = G_2(1+T) = 1$, $G_3(T) = T$, $G_3(1+T) = 1+T$, and $G_n(T) = G_n(1+T) = 0$ if $n \geq 4$;
- $G'_0(\alpha) = 1$, for any $\alpha \in \mathbb{F}_r[[T]]$.

We also recall that $A = \mathbb{F}_2[T]$, $\hat{A} = \mathbb{F}_2[[T]]$, $A_n = \{\alpha \in A : \deg_T(\alpha) = n\}$, and $A_{\leq n} = \{\alpha \in A : \deg_T(\alpha) \leq n\}$ for any non-negative integer n. Moreover, we notice that a function $f \in C(\hat{A}, \mathbb{F}_2((T)))$ is measure-preserving if and only if

$$\left| \frac{1}{y} \left(f(x+y) - f(x) \right) \right| = 1 \text{ for any } y \in \hat{A}\backslash\{0\} \text{ and any } x \in \hat{A}. \tag{16}$$

Theorem 9. *(ergodic property) A 1-Lip function $f : \mathbb{F}_2[[T]] \to \mathbb{F}_2[[T]]$*

$$f(x) = \sum_{n=0}^{\infty} a_n G_n(x)$$

is ergodic if and only if the following conditions are satisfied

(1) $a_0 \equiv 1(\bmod T)$, $a_1 \equiv 1+T(\bmod T^2)$, $a_3 \equiv T^2(\bmod T^3)$;
(2) $|a_n| < |T|^{[\log_2 n]} = 2^{-[\log_2 n]}$, *for* $n \geq 2$;
(3) $a_{2^n-1} \equiv T^n(\bmod T^{n+1})$ *for* $n > 2$.

Proof. We have $f(x) = \sum_{n=0}^{\infty} a_n G_n(x) = \sum_{\alpha\in\mathbb{F}_2[T]} B_\alpha \chi(\alpha, x)$. At first we translate the conditions (1) and (3) of Theorem 6 to those on the coefficients of the Carlitz basis. In the expansion (3) of Theorem 6, we also use the notation $B_\alpha = T^{\deg(\alpha)} b_\alpha$ for $\alpha \in \mathbb{F}_2[T]$.

(1) $B_0 = b_0 \equiv 1(\bmod T)$:
as $B_0 = f(0) = \sum_{n=0}^{\infty} a_n G_n(0)$, this condition is equivalent to

$$a_0 = \sum_{n=0}^{\infty} a_n G_n(0) = f(0) = B_0 \equiv 1(\bmod T).$$

$B_0 + B_1 = b_0 + b_1 \equiv 1 + T(\bmod T^2)$:
from $B_0 + B_1 = f(0) + f(1) = \sum_{n=0}^{\infty} a_n G_n(0) + \sum_{n=0}^{\infty} a_n G_n(1)$, this condition is equivalent to

$$\begin{aligned} a_1 &\equiv a_0 + a_0 + a_1 \equiv f(0) + f(1) \\ &\equiv B_0 + B_1 \equiv 1 + T(\bmod T^2). \end{aligned}$$

$b_T + b_{1+T} \equiv T(\bmod T^2)$:
this is the same as $B_T + B_{1+T} \equiv T^2(\bmod T^3)$. From the explicit calculation

$$\begin{aligned} B_T + B_{1+T} &= (f(T) - f(0)) + (f(1+T) - f(1)) \\ &= \sum_{n=0}^{\infty} a_n(G_n(T) - G_n(0)) + \sum_{n=0}^{\infty} a_n(G_n(1+T) - G_n(1)) \\ &= a_3, \end{aligned}$$

we see that this condition is equivalent to $a_3 \equiv T^2(\bmod T^3)$.

(3) The third condition of Theorem 6 (ergodic property under Van der Put basis) is $\sum_{\alpha\in A_{n-1}} b_\alpha \equiv T(\bmod T^2)$, which is equivalent to $\sum_{\alpha\in A_{n-1}} B_\alpha \equiv T^n(\bmod T^{n+1})$. We can calculate

$$\begin{aligned} \sum_{\alpha\in A_{n-1}} B_\alpha &= \sum_{\beta\in A_{\leq n-2}} (f(\beta + T^{n-1}) - f(\beta)) \\ &= \sum_{\beta\in A_{\leq n-2}} \sum_{m=0}^{\infty} (a_m G_m(\beta + T^{n-1}) - a_m G_m(\beta)) \\ &= \sum_{\beta\in A_{\leq n-2}} \sum_{m=1}^{\infty} a_m \sum_{j=0}^{m-1} \binom{m}{j} G_j(\beta) G_{m-j}(T^{n-1}) \\ &= \sum_{m=1}^{\infty} a_m \sum_{j=0}^{m-1} \binom{m}{j} G_{m-j}(T^{n-1}) \sum_{\beta\in A_{\leq n-2}} G_j(\beta) G_0'(\beta) \\ &= \sum_{m=1}^{\infty} a_m \sum_{j=0}^{m-1} \binom{m}{j} G_{m-j}(T^{n-1})(-1)^{n-1}\delta_{j,\,2^{n-1}-1} \\ &= a_{2^n-1}. \end{aligned}$$

The last equality of the above equations holds because $\binom{m}{2^{n-1}-1} \neq 0$ only when $m = (2^{n-1}-1) + l \cdot 2^{n-1}$ for some integer $l \geq 0$ (due to the Lucas formula (12)) and

$m-1 \geq 2^{n-1}-1$, and we have the special values of Carlitz polynomials:

$$G_{l \cdot 2^{n-1}}(T^{n-1}) = \begin{cases} 1, & \text{if } l = 1, \\ 0, & \text{if } l > 1. \end{cases}$$

The order of summation can be exchanged because the function f is assumed to be 1-Lip, thus $a_n \to 0$ as $n \to \infty$. Therefore the third condition of Theorem 6 is equivalent to $a_{2^n-1} \equiv T^n \pmod{T^{n+1}}$ for $n > 2$.

Now we scrutinize equation (14):

$$\begin{aligned} &\frac{1}{y_1}(f(y_1+x)-f(x)) \\ &= \sum_{n_0=0}^{\infty}\sum_{j_1=0}^{\infty}\binom{n_0+j_1+1}{j_1+1}\frac{a_{n_0+j_1+1}}{L_{\nu(j_1+1)}}H_{j_1}(y_1)G_{n_0}(x) \\ &= a_1 + \sum_{j_1=1}^{\infty}\frac{a_{j_1+1}}{L_{\nu(j_1+1)}}H_{j_1}(y_1)G_0(x) \\ &\quad + \sum_{n_0=1}^{\infty}\sum_{j_1=0}^{\infty}\binom{n_0+j_1+1}{j_1+1}\frac{a_{n_0+j_1+1}}{L_{\nu(j_1+1)}}H_{j_1}(y_1)G_{n_0}(x) \end{aligned} \tag{17}$$

for $x \in \hat{A}$ and $y_1 \in \hat{A}\backslash\{0\}$.

"Sufficiency". Assume the three conditions of the theorem are satisfied. Then we know that $a_1 \equiv 1 (\operatorname{mod} T)$ and we can deduce from equation (17) that

$$\frac{1}{y_1}\left(f(y_1+x)-f(x)\right) = 1 + Th(y_1,x), \tag{18}$$

where $h(y_1,x)$ is a continuous function from $\left(\hat{A}\backslash\{0\}\right) \times \hat{A}$ to $\hat{A}$. Therefore $|\frac{1}{y_1}\left(f(y_1+x)-f(x)\right)| = 1$ for any $y_1 \in \hat{A}/\{0\}$, and any $x \in \hat{A}$. Hence the function f is measure preserving and Theorem 5 implies that the condition (2) of Theorem 6, as well as conditions (1) and (3), is satisfied. Therefore the function f is ergodic.

"Necessity". Assume that the 1-Lip function f is ergodic. Then by Theorem 6 and the discussion at the beginning of this proof, we see that conditions (1) and (3) are satisfied. We do the same calculation to get equation (18), where

$$\begin{aligned} h(y_1,x) = \frac{1}{T}\Big(&a_1 - 1 + \sum_{j_1=1}^{\infty}\frac{a_{j_1+1}}{L_{\nu(j_1+1)}}H_{j_1}(y_1)G_0(x) \\ &+ \sum_{n_0=1}^{\infty}\sum_{j_1=0}^{\infty}\binom{n_0+j_1+1}{j_1+1}\frac{a_{n_0+j_1+1}}{L_{\nu(j_1+1)}}H_{j_1}(y_1)G_{n_0}(x)\Big). \end{aligned}$$

As the function f is assumed to be ergodic, $h(y_1,x)$ is a continuous function from $\left(\hat{A}\backslash\{0\}\right) \times \hat{A}$ to $\hat{A}$. Therefore we can apply the same proof as Theorem 8 to get the condition (2): $|a_n| < |T|^{[\log_2 n]} = 2^{-[\log_2 n]}$, for $n \geq 2$. $\square$

Example 1. In terms of Carlitz basis, the simplest ergodic function on $\mathbb{F}_2[[T]]$ would be

$$f(x) = 1 + (1+T)x + \sum_{n=2}^{\infty} T^n G_{2^n-1}(x).$$

By Theorem 9, we can easily write down as many ergodic functions as we want in terms of Carlitz polynomials. How are they related to the expressions as functions over $\mathbb{Z}_2$ as in [An1] [An2] [An3] [An4] is an interesting question. But it is more interesting to study how the idea of Lemma 1 can be implemented in applications.

Since cryptographic codes or keys are sequences of digits 0 and 1, we interpret them as elements of $\mathbb{Z}_2$ of $\mathbb{F}_2[[T]]$. So an ergodic transformation can be viewed either as a function on Z_2 or a function on $\mathbb{F}_2[[T]]$. In this paper, we get the sufficient and necessary conditions about ergodic functions over $\mathbb{F}_2[[T]]$. As there are no carry overs in the additions on $\mathbb{F}_2[[T]]$, the computation is much faster than the corresponding operations on $\mathbb{Z}_2$. In fact, the addition of two elements in $\mathbb{F}_2[[T]]$ can be seen as bitwise XOR over $\mathbb{Z}_2$. The multiplication is also slightly different on $\mathbb{F}_2[[T]]$ to that on $\mathbb{Z}_2$. The idea of this paper may provide a new way to design practical cryptography component after some good related analysis on the functions over $\mathbb{F}_2[[T]]$, which we hope will do in the near future.

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