

# Terminating BKZ

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**Abstract.** Strong lattice reduction is the key element for most attacks against lattice-based cryptosystems. Between the strongest but impractical HKZ reduction and the weak but fast LLL reduction, there have been several attempts to find efficient trade-offs. Among them, the BKZ algorithm introduced by Schnorr and Euchner [FCT’91] seems to achieve the best time/quality compromise in practice. However, no reasonable complexity upper bound is known for BKZ, and Gama and Nguyen [Eurocrypt’08] observed experimentally that its practical runtime seems to grow exponentially with the lattice dimension. In this work, we show that BKZ can be terminated long before its completion, while still providing bases of excellent quality. More precisely, we show that if given as inputs a basis  $(\mathbf{b}_i)_{i \leq n} \in \mathbb{Q}^{n \times n}$  of a lattice  $L$  and a block-size  $\beta$ , and if terminated after  $\Omega\left(\frac{n^3}{\beta^2}(\log n + \log \log \max_i \|\mathbf{b}_i\|)\right)$  calls to a  $\beta$ -dimensional HKZ-reduction (or SVP) subroutine, then BKZ returns a basis whose first vector has norm  $\leq 2\gamma_\beta^{\frac{n-1}{2(\beta-1)} + \frac{3}{2}} \cdot (\det L)^{\frac{1}{n}}$ , where  $\gamma_\beta \leq \beta$  is the maximum of Hermite’s constants in dimensions  $\leq \beta$ . To obtain this result, we develop a completely new elementary technique based on discrete-time affine dynamical systems, which could lead to the design of improved lattice reduction algorithms.

**Keywords.** Euclidean lattices, BKZ, lattice-based cryptanalysis.

## 1 Introduction

A (full-rank)  $n$ -dimensional lattice  $L \subseteq \mathbb{R}^n$  is the set of integer linear combinations  $\sum_{i=1}^n x_i \mathbf{b}_i$  of some linearly independent vectors  $(\mathbf{b}_i)_{i \leq n}$ . Such vectors are called a basis and we write  $L = L[(\mathbf{b}_i)_i]$ . Since  $L$  is discrete, it contains a shortest non-zero lattice vector, whose norm  $\lambda_1(L)$  is called the lattice minimum. Computing such a vector given a basis is referred to as the (computational) Shortest Vector Problem (SVP), and is NP-hard under randomized reductions [1,12]. The complexities of the best known SVP solvers are no less than exponential [22,23,2,15] (the record is held by the algorithm from [22], with complexity  $2^{2n+o(n)} \cdot \text{Poly}(\log \max_i \|\mathbf{b}_i\|)$ ). Finding a vector reaching  $\lambda_1(L)$  is polynomial-time equivalent to computing a basis of  $L$  that is reduced in the sense of Hermite-Korkine-Zolotarev (HKZ). The aforementioned SVP solvers can all be used to compute HKZ-reduced bases, in exponential time. On the other hand, bases reduced in the sense of Lenstra-Lenstra-Lovász (LLL) can be computed in polynomial time [16], but the first vector is only guaranteed to satisfy the weaker inequality  $\|\mathbf{b}_1\| \leq (4/3 + \varepsilon)^{\frac{n-1}{2}} \cdot \lambda_1(L)$  (for an arbitrary  $\varepsilon > 0$ ). In 1987, Schnorr introduced time/quality trade-offs between LLL and HKZ [33]. In the present work, we propose the first analysis of the BKZ algorithm [36,37], which is currently the most practical such trade-off [40,9].

Lattice reduction is a popular tool in cryptanalysis [27]. For many applications, such as Coppersmith’s method for computing the small roots of polynomials [5], LLL-reduction suffices. However, reductions of much higher quality seem required to break lattice-based cryptosystems. Lattice-based cryptography originated with Ajtai’s seminal hash function [1], and the GGH and NTRU encryption schemes [10,14]. Thanks to its excellent asymptotic performance, provable security guarantees, and flexibility, it is currently attracting wide interest and developing at a steady pace. We refer to [21,31] for recent surveys. A major obstacle to the real-life deployment of lattice-based cryptography is the lack of a precise understanding of the limits of the best practical attacks, whose main component is the computation of strongly reduced lattice bases. This prevents from having a precise correspondence between specific security levels and practical parameters. Our work is a step towards a clearer understanding of BKZ, and thus of the best known attacks.

Strong lattice reduction has been studied for about 25 years (see [33,37,34,7,32,9,8] among others). From a theoretical perspective, the best known time/quality trade-off is due to Gama and Nguyen [8]. By building upon the proof of Mordell’s inequality on Hermite’s constant, they devised the notion of *slide reduction*,

and proposed an algorithm computing slide-reduced bases: Given an arbitrary basis  $(\mathbf{b}_i)_{i \leq n}$  of a lattice  $L$ , the slide-reduction algorithm finds a basis  $(\mathbf{c}_i)_{i \leq n}$  of  $L$  such that

$$\|\mathbf{c}_1\| \leq ((1 + \varepsilon)\gamma_\beta)^{\frac{n-\beta}{\beta-1}} \cdot \lambda_1(L), \quad (1)$$

within  $\tau_{\text{slide}} := O\left(\frac{n^4}{\beta \cdot \varepsilon} \cdot \log \max_i \|\mathbf{b}_i\|\right)$  calls<sup>1</sup> to a  $\beta$ -dimensional HKZ-reduction algorithm and a  $\beta$ -dimensional (computational-)SVP solver, where  $\gamma_\beta \approx \beta$  is the  $\beta$ -dimensional Hermite constant. If  $L \subseteq \mathbb{Q}^n$ , the overall cost of the slide-reduction algorithm is  $\leq \text{Poly}(n, \text{size}(B)) \cdot \mathcal{C}_{\text{HKZ}}(\beta)$ , where  $\mathcal{C}_{\text{HKZ}}(\beta) = 2^{O(\beta)}$  is the cost of HKZ-reducing in dimension  $\beta$ . The higher  $\beta$ , the lower the achieved SVP approximation factor, but the higher the runtime. Slide reduction also provides a constructive variant of Minkowski's inequality, as (letting  $\det L$  denote  $\text{vol}(\mathbb{R}^n/L)$ ):

$$\|\mathbf{c}_1\| \leq ((1 + \varepsilon)\gamma_\beta)^{\frac{n-1}{2(\beta-1)}} \cdot (\det L)^{\frac{1}{n}}, \quad (2)$$

From a practical perspective, however, slide reduction seems to be (significantly) outperformed by the BKZ algorithm [9]. BKZ also relies on a  $\beta$ -dimensional HKZ-reduction algorithm (resp. SVP-solver). The worst-case quality of the bases it returns has been studied in [34] and is comparable to that of the slide reduction algorithm. The first vector of the output basis  $(\mathbf{c}_i)_{i \leq n}$  satisfies  $\|\mathbf{c}_1\| \leq ((1 + \varepsilon)\gamma_\beta)^{\frac{n-1}{\beta-1}} \cdot \lambda_1(L)$ . Note that this bound essentially coincides with (1), except for large values of  $\beta$ . A bound similar to that of (2) also holds.<sup>2</sup> In practice, the quality of the computed bases seems much higher with BKZ than with the slide-reduction algorithm [9]. With respect to run-time, no reasonable bound is known on the number of calls to the  $\beta$ -dimensional HKZ reduction algorithm it needs to make before termination.<sup>3</sup> In practice, this number of calls does not seem to be polynomially bounded [9] and actually becomes huge when  $\beta \geq 25$ . Because of its large (and somewhat unpredictable) runtime, it is folklore practice to terminate BKZ before the end of its execution, when the solution of the problem for which it is used for is already provided by the current basis [38,24].

**OUR RESULT.** We show that if terminated within polynomially many calls to HKZ/SVP, a slightly modified version of BKZ (see Section 3) returns bases whose first vectors satisfy a slightly weaker variant of (2).

**Theorem 1.** *There exists<sup>4</sup>  $C > 0$  such that the following holds for all  $n$  and  $\beta$ . Let  $B = (\mathbf{b}_i)_{i \leq n}$  be a basis of a lattice  $L$ , given as input to the modified BKZ algorithm of Section 3 with block-size  $\beta$ . If terminated after  $\tau_{\text{BKZ}} := C \frac{n^3}{\beta^2} \left( \log n + \log \log \max_i \frac{\|\mathbf{b}_i\|}{(\det L)^{1/n}} \right)$  calls to an HKZ-reduction (or SVP solver) in dimension  $\beta$ , the output  $(\mathbf{c}_i)_{i \leq n}$  is a basis of  $L$  that satisfies (with  $\gamma_\beta \leq \beta$  defined as the maximum of Hermite's constants in dimensions  $\leq \beta$ ):*

$$\|\mathbf{c}_1\| \leq 2(\gamma_\beta)^{\frac{n-1}{2(\beta-1)} + \frac{3}{2}} \cdot (\det L)^{\frac{1}{n}}.$$

If  $L$  is a rational lattice, then the overall cost is  $\leq \text{Poly}(n, \text{size}(B)) \cdot \mathcal{C}_{\text{HKZ}}(\beta)$ .

By using [18, p. 25], this provides an algorithm with runtime  $\leq \text{Poly}(n, \text{size}(B)) \cdot \mathcal{C}_{\text{HKZ}}(\beta)$  that returns a basis whose first vector satisfies  $\|\mathbf{c}_1\| \leq 4(\gamma_\beta)^{\frac{n-1}{\beta-1} + 3} \cdot \lambda_1(L)$ , which is only slightly worse than (1). These results indicate that BKZ can be used to achieve essentially the same quality guarantees as slide reduction, within a number of calls to HKZ in dimension  $\beta$  that is no larger than that of slide reduction. Actually, note that  $\tau_{\text{BKZ}}$  is significantly smaller than  $\tau_{\text{slide}}$ , in particular with a dependence with respect to  $\max_i \|\mathbf{b}_i\|$

<sup>1</sup> The component  $\frac{n^4}{\beta}$  of this upper bound is derived by adapting the results from [8] to our notations. A more thorough analysis leads to a smaller term.

<sup>2</sup> In [9], the bound  $\|\mathbf{c}_1\| \leq (\gamma_\beta)^{\frac{n-1}{2(\beta-1)} + \frac{1}{2}} \cdot (\det L)^{\frac{1}{n}}$  is claimed to hold, but without proof nor reference. We prove a (slightly) weaker bound, but we are able to improve on it if  $\gamma_n$  is replaced by any linear function. See appendix.

<sup>3</sup> A bound  $(n\beta)^n$  is mentioned in [9]. For completeness, we give a proof of a similar result in appendix.

<sup>4</sup> The constant  $C$  is used to absorb lower-order terms in  $n$ , and could be taken small.

that is *exponentially* smaller. It may be possible to obtain a similar bound for the slide-reduction algorithm by adapting our analysis.

To achieve our result, we use a completely new approach for analyzing lattice reduction algorithms. The classical approach to bound their runtimes was to introduce a quantity, sometimes called potential, involving the current Gram-Schmidt norms  $\|\mathbf{b}_i^*\|$ , which always strictly decreases every time some elementary step is performed. This technique was introduced by Lenstra, Lenstra and Lovász [16] for analyzing their LLL algorithm, and is still used in all complexity analyzes of (variants of) LLL we are aware of. It was later adapted to stronger lattice reduction algorithms [33,7,32,8]. We still measure progress with the  $\|\mathbf{b}_i^*\|$ 's, but instead of considering a single scalar combining them all, we look at the *full vector*  $(\|\mathbf{b}_i^*\|)_i$ . More specifically, we observe that each call to HKZ within BKZ has the effect of applying an affine transformation to the vector  $(\log \|\mathbf{b}_i^*\|)_i$ : instead of providing a lower bound to the progress made on a “potential”, we are then led to analyze a discrete-time dynamical affine system. Its fixed-points encode information on the output quality of BKZ, whereas its speed of convergence provides an upper bound on the number of times BKZ calls HKZ.

Intuitively, the effect of a call to HKZ on the vector  $(\log \|\mathbf{b}_i^*\|)_{i \leq n}$  is to essentially replace  $\beta$  consecutive coefficients by their average. We formalize this intuition by making a specific assumption (see Section 4). Under this assumption, the execution of BKZ exactly matches with a dynamical system that we explicit and fully analyze. However, we cannot prove that this intuition is always correct (counter-examples can actually be constructed). To circumvent this difficulty, we instead consider the vector  $\boldsymbol{\mu} = (\frac{1}{i} \sum_{j=1}^i \log \|\mathbf{b}_j^*\|)_{i \leq n}$ . This amortization (also used in [11] for analyzing HKZ-reduced bases) allows us to *rigorously* bound the evolution of  $\boldsymbol{\mu}$  by the orbit of a vector under another dynamical system. Since this new dynamical system happens to be a modification of the dynamical system used in the idealized model, the analysis performed for the idealized model can be adapted to the rigorous set-up.

This approach is likely to prove useful for analyzing other lattice reduction algorithms. As an illustration of its power, we provide two new results on LLL. First, we show that the SVP approximation factor  $\sqrt{4/3}^{n-1}$  can be reached in polynomial time using only Gauss reductions. This is closely related to the question whether the “optimal LLL” (i.e., using LLL parameter  $\delta = 1$ ) terminates in polynomial time [3,17]. Second, we give a LLL-reduction algorithm of bit-complexity  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(B))$ . Such a complexity bound was only very recently achieved, with a completely different approach [29]. Note that close-by results on LLL have been concurrently and independently obtained by Schnorr [35].

**PRACTICAL ASPECTS.** Our result is a (maybe pessimistic) worst-case quality bound on BKZ with early termination. In itself, this does not give a precise explanation of the practical behavior of BKZ. In particular, it does not explain why it outperforms slide reduction, but only why it does not behave significantly worse. However, this study illustrates the usefulness of early termination in BKZ: Much progress is done at the beginning of the execution, and quickly the basis quality becomes excellent; the rest of the execution takes much longer, for a significantly less dramatic quality improvement. This behavior is very clear in practice, as illustrated by Figure 1 of Section 2. Since most of the work performed by BKZ is completed within the first few calls to HKZ, it shows that the BKZ performance extrapolations used to estimate the hardness of cryptographic instances should focus only on the cost of a single call to HKZ and on the achieved basis quality after a few such calls. For instance, it indicates that the strategy (adopted, e.g., in [14,13]) consisting in measuring the full run-time of BKZ might be reconsidered.

Additionally, parts of the analysis might prove useful to better understand BKZ and devise reduction algorithms with improved practical time/quality trade-offs. In particular, the heuristic modelisation of BKZ as a discrete-time affine dynamical system suggests that the block of vectors on which HKZ-reduction is to be applied could be chosen adaptively, so that the system converges faster to its limit. It would not improve the output quality for BKZ, but it is likely to accelerate its convergence. Also, the second phase of BKZ, the one that takes longer but during which some little progress is still made, could be understood by introducing some randomness in the model: most of the time, the norm of the first vector found by the HKZ-reduction sub-routine is around its expected value (a constant factor smaller than its worst-case

bound), but it is significantly smaller every now and then. If such a model could predict the behavior of BKZ during its second phase, then maybe it would explain why it outperforms slide reduction. It might give indications on the optimal time for stopping BKZ with block-size  $\beta$  before switching to a larger block-size.

**Notations.** All vectors will be denoted in bold, and matrices in capital letters. If  $\mathbf{b} \in \mathbb{R}^n$ , the notation  $\|\mathbf{b}\|$  will refer to its Euclidean norm. If  $B \in \mathbb{R}^{n \times n}$ , we define  $\|B\|_2 = \max_{\|\mathbf{x}\|=1} \|B \cdot \mathbf{x}\|$  and we denote the spectral radius of  $B$  by  $\rho(B)$ . If  $B$  is a rational matrix, we define  $\text{size}(B)$  as the sum of the bit-sizes of the numerators and denominators of its entries. All complexity statements refer to elementary operations on bits. We will use the Landau notations  $o(\cdot)$ ,  $O(\cdot)$ ,  $\tilde{O}(\cdot)$  and  $\Omega(\cdot)$ . The notations  $\log(\cdot)$  and  $\ln(\cdot)$  respectively stand for the base 2 and natural logarithms.

## 2 Reminders

For an introduction to lattice reduction algorithms, we refer to [28].

**Successive Minima.** Let  $L$  be an  $n$ -dimensional lattice. Its  $i$ -th minimum  $\lambda_i(L)$  is defined as the minimal radius  $r$  such that  $\mathcal{B}(\mathbf{0}, r)$  contains  $\geq i$  linearly independent vectors of  $L$ .

**Hermite's constant.** The  $n$ -dimensional Hermite constant  $\gamma_n$  is defined as the maximum taken over all lattices  $L$  of dimension  $\leq n$  of the quantity  $\frac{\lambda_1(L)^2}{(\det L)^{2/\dim(L)}}$ . (The usual definition restricts  $L$  to dimension  $n$  instead of  $\leq n$ . We use this modified definition to ensure that  $\gamma_n$  increases with  $n$ .) Very few values of  $\gamma_n$  are known, but we have  $\gamma_n \leq 1 + \frac{n}{4}$  for all  $n$  (see [20, Re 2.7.5]).

**Gram-Schmidt orthogonalisation.** Let  $(\mathbf{b}_i)_{i \leq n}$  be a lattice basis. Its Gram-Schmidt orthogonalization  $(\mathbf{b}_i^*)_{i \leq n}$  is defined recursively by  $\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j < i} \mu_{i,j} \mathbf{b}_j^*$  with  $\mu_{i,j} = (\mathbf{b}_i^*, \mathbf{b}_j^*) / \|\mathbf{b}_j^*\|^2$  for  $i > j$ . The  $\mathbf{b}_i^*$ 's are mutually orthogonal. For  $i \leq j$ , we define  $\mathbf{b}_j^{(i)}$  as the projection of  $\mathbf{b}_j$  orthogonally to  $\text{Span}(\mathbf{b}_k)_{k < i}$ . Note that if  $L$  is an  $n$ -dimensional lattice, then  $\det L = \prod_{i=1}^n \|\mathbf{b}_i^*\|$ , for any basis  $(\mathbf{b}_i)_{i \leq n}$  of  $L$ .

**A few notions of reduction.** Given a basis  $(\mathbf{b}_i)_{i \leq n}$ , we say that it is *size-reduced* if the Gram-Schmidt coefficients  $\mu_{i,j}$  satisfy  $|\mu_{i,j}| \leq 1/2$  for all  $j < i \leq n$ . We say that  $(\mathbf{b}_i)_{i \leq n}$  is  $\delta$ -LLL-reduced for  $\delta \leq 1$  if it is size-reduced and the Lovász conditions  $\delta \|\mathbf{b}_i^*\|^2 \leq \|\mathbf{b}_{i+1}^*\|^2 + \mu_{i+1,i}^2 \|\mathbf{b}_i^*\|^2$  are satisfied for all  $i < n$ . For any  $\delta < 1$ , a  $\delta$ -LLL-reduced basis of a rational lattice  $L$  can be computed in polynomial time, given an arbitrary basis of  $L$  as input [16]. We say that  $(\mathbf{b}_i)_{i \leq n}$  is HKZ-reduced if it is size-reduced and for all  $i < n$ , we have  $\|\mathbf{b}_i^*\| = \lambda_1(L[(\mathbf{b}_j^{(i)})_{i \leq j \leq n}])$ . An HKZ-reduced basis of a lattice  $L \subseteq \mathbb{Q}^n$  can be computed in time  $2^{2n+o(n)} \cdot \text{Poly}(\text{size}(B))$ , given an arbitrary basis  $B$  of  $L$  as input [22]. The following is a direct consequence of the definitions of the HKZ-reduction and Hermite constant.

**Lemma 1.** For any HKZ-reduced basis  $(\mathbf{b}_i)_{i \leq n}$ , we have:  $\forall i < n, \|\mathbf{b}_i^*\| \leq \sqrt{\gamma_{n-i+1}} \cdot (\prod_{j=i}^n \|\mathbf{b}_j^*\|)^{\frac{1}{n-i+1}}$ .

**The BKZ algorithm.** We recall the original BKZ algorithm from [37] in Algorithm 1. BKZ was originally proposed as a mean of computing bases that are almost  $\beta$ -reduced.  $\beta$ -Reduction was proposed by Schnorr in [33], but without an algorithm for achieving it. The BKZ algorithm proceeds by iterating tours consisting of  $n - 1$  calls to a  $\beta$ -dimensional SVP solver called on the lattices  $L[(\mathbf{b}_i^{(k)})_{k \leq i \leq \min(k+\beta-1, n)}]$ . Its execution stops when no change occurs during a tour.

**Input** : A (LLL-reduced) basis  $(\mathbf{b}_i)_{i \leq n}$ , a blocksize  $\beta$  and a constant  $\delta < 1$ .  
**Output** : A basis of  $L[(\mathbf{b}_i)_{i \leq n}]$ .  
**repeat**  
    **for**  $k \leftarrow 1$  to  $n - 1$  **do**  
        Find  $\mathbf{b}$  such that  $\mathbf{b}^{(k)}$  is a shortest non-zero vector of  $L[(\mathbf{b}_i^{(k)})_{k \leq i \leq \min(k+\beta-1, n)}]$ ;  
        **if**  $\delta \cdot \|\mathbf{b}_k^*\| > \|\mathbf{b}\|$  **then**  
            LLL-reduce( $\mathbf{b}_1, \dots, \mathbf{b}_{k-1}, \mathbf{b}, \mathbf{b}_k, \dots, \mathbf{b}_{\min(k+\beta, n)}$ ).  
        **else**  
            LLL-reduce( $\mathbf{b}_1, \dots, \mathbf{b}_{\min(k+\beta, n)}$ ).  
**until** no change occurs.

**Algorithm 1:** The Schnorr and Euchner BKZ algorithm.

### 3 Terminating BKZ

In this article, we will not analyze the original BKZ algorithm, but we will focus on a slightly modified variant instead, which is given in Algorithm 2. It also performs BKZ tours, and during a tour it makes  $n - \beta + 1$  calls to a  $\beta$ -dimensional HKZ-reduction algorithm. It fits more closely to what would be the simplest BKZ-style algorithm, aiming at producing a basis  $(\mathbf{b}_i)_{i \leq n}$  such that the projected basis  $(\mathbf{b}_i^{(k)})_{k \leq i \leq k + \beta - 1}$  is HKZ-reduced for all  $k \leq n - \beta + 1$ .

**Differences between the two variants of BKZ.** The differences between the two algorithms are the following:

- In Algorithm 2, the execution can be terminated at the end of any BKZ tour.
- In the classical BKZ algorithm, the vector  $\mathbf{b}$  found by the SVP solver is kept only if  $\|\mathbf{b}^{(k)}\|$  is smaller than  $\delta \cdot \|\mathbf{b}_k^*\|$ . Such a factor  $\delta < 1$  does not appear in Algorithm 2. It is unnecessary for our analysis to hold, complicates the algorithm, and leads to output bases of lesser quality.
- For each  $k$  within a tour, Algorithm 1 only requires an SVP solver while Algorithm 2 calls an HKZ-reduction algorithm, which is more complex. We use HKZ-reductions for the ease of the analysis. Our analysis would still hold if the loop was done for  $k$  from 1 to  $n - 1$  and if the HKZ-reductions were replaced by calls to any algorithm that returns bases whose first vector reaches the minimum (which can be obtained by calling any SVP solver, putting the output vector in front of the input basis and calling LLL to remove the linear dependency).
- Finally, to insert  $\mathbf{b}$  in the current basis, Algorithm 1 performs an LLL-reduction. Applying LLL inside the projected block (i.e., to  $\mathbf{b}^{(k)}, \mathbf{b}_k^{(k)}, \dots, \mathbf{b}_{k+\beta-1}^{(k)}$ ) would be sufficient to remove the linear dependency while keeping  $\mathbf{b}^{(k)}$  in first position, but instead it runs LLL from the beginning of the basis until the end of the next block to be considered (i.e., up to index  $\min(k + \beta, n)$ ). This reduction is performed even if the block is already reduced and no vector is inserted. Experimentally, this seems to improve the speed of convergence of the algorithm by a small factor, but it does not seem easy to use our techniques to analyze this effect.

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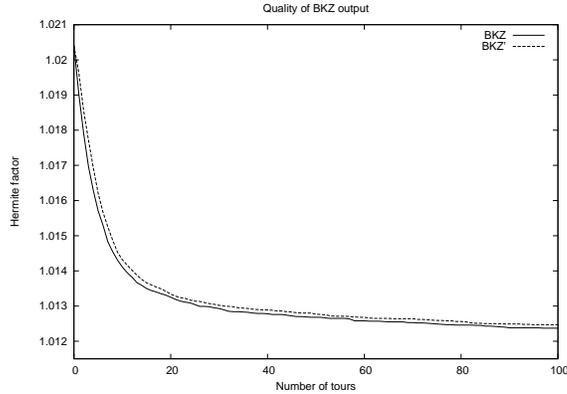
Input  : A basis  $(\mathbf{b}_i)_{i \leq n}$  and a blocksize  $\beta$ .
Output : A basis of  $L[(\mathbf{b}_i)_{i \leq n}]$ .
repeat
  for  $k \leftarrow 1$  to  $n - \beta + 1$  do
    Modify  $(\mathbf{b}_i)_{k \leq i \leq k + \beta - 1}$  so that  $(\mathbf{b}_i^{(k)})_{k \leq i \leq k + \beta - 1}$  is HKZ-reduced;
    Size-reduce( $\mathbf{b}_1, \dots, \mathbf{b}_n$ ).
until no change occurs or termination is requested.

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**Algorithm 2:** BKZ', the modified BKZ algorithm.

**On the practical behavior of BKZ.** In order to give an insight on the practical behavior of BKZ and BKZ', we give experimental results on the evolution of the quantity  $\frac{\|\mathbf{b}_1\|}{(\det L)^{1/n}}$  (the so-called Hermite factor) during their executions. The experiment corresponding to Figure 1 is as follows: We generated 64 knapsack-like lattice bases [25] of dimension  $n = 108$ , with non-trivial entries of bit-length  $100n$ ; Each was LLL-reduced using `fp111` [4] (with parameters  $\delta = 0.99$  and  $\eta = 0.51$ ); Then for each we ran NTL's BKZ [40] and an implementation of BKZ' in NTL, with blocksize 24. Figure 1 only shows the beginning of the executions. For both algorithms, the executions of about half the samples consisted in  $\simeq 600$  tours, whereas the longest execution stopped after  $\simeq 1200$  tours. The average value of  $\frac{\|\mathbf{b}_1\|}{(\det L)^{1/n}}$  at the end of the executions was  $\simeq 1.012$ .

**Cost of BKZ'.** In order to bound the bit-complexities of BKZ and BKZ', it is classical to consider several cost components separately. In this article, we will focus on the number of tours. The number of calls to an SVP solver (for BKZ) or an HKZ-reduction algorithm (in the case of BKZ') is  $\leq n$  times larger. A tour consists of efficient operations (LLL, size-reductions, etc) and of the more costly calls to SVP/BKZ. The cost of the SVP solver or the HKZ-reduction algorithm is often bounded in terms of the number of



**Fig. 1.** Evolution of the Hermite factor  $\frac{\|\mathbf{b}_1\|}{(\det L)^{1/n}}$  during the execution of BKZ and BKZ'.

arithmetic operations it performs: For all known algorithms, this quantity is (at least) exponential in the block-size  $\beta$ . Finally, one should also take into account the bit-costs of the arithmetic operations performed to prepare the calls to SVP/HKZ, during these calls, and after these calls (when applying the computed transforms to the basis, and calling LLL or a size-reduction). These arithmetic costs are classically bounded by considering the bit-sizes of the quantities involved. They can easily be shown to be polynomial in the input bit-size, by relying on rational arithmetic and using standard tools from the analyses of LLL and HKZ [16,15]. It is likely that these costs can be lowered further by relying on floating-point approximations to these rational numbers, using the techniques from [26,30]. To conclude, the overall cost is upper bounded by  $\text{Poly}(n, \log \|B\|) \cdot 2^{O(\beta)} \cdot \tau$ , where  $\tau$  is the number of tours.

## 4 Analysis of BKZ' in the Sandpile Model

In this section, we (rigorously) analyze a *heuristic model of BKZ'*. In the following section, we will show how this analysis can be adapted to allow for a (rigorous) study of the *genuine BKZ'* algorithm.

We first note that BKZ' can be studied by looking at the way the vector  $\mathbf{x} := (\log \|\mathbf{b}_i^*\|)_i$  changes during the execution, rather than considering the whole basis  $(\mathbf{b}_i)_i$ . This simplification is folklore in the analyses of lattice reduction algorithms, and allows for an interpretation in terms of sandpiles [19]. The study in the present section is heuristic in the sense that we assume the effect of a call to  $\text{HKZ}_\beta$  on  $\mathbf{x}$  is determined by  $\mathbf{x}$  only, in a deterministic fashion.

### 4.1 The sandpile model and its dynamical system interpretation

Before describing the model, let us consider the shape of a  $\beta$ -dimensional HKZ-reduced basis. Let  $(\mathbf{b}_i)_{i \leq \beta}$  be an HKZ-reduced basis, and  $x_i = \log \|\mathbf{b}_i^*\|$ . Then, by Lemma 1, we have:

$$\forall i \leq \beta, x_i \leq \frac{1}{2} \log \gamma_{\beta-i+1} + \frac{1}{\beta-i+1} \sum_{j=i}^{\beta} x_j. \quad (3)$$

Our heuristic assumption consists in replacing these inequalities by equalities.

**Heuristic Sandpile Model Assumption (SMA).** We assume for any HKZ-reduced basis  $(\mathbf{b}_i)_{i \leq \beta}$ , we have  $x_i = \frac{1}{2} \log \gamma_{\beta-i+1} + \frac{1}{\beta-i+1} \sum_{j=i}^{\beta} x_j$  for all  $i \leq \beta$ , with  $\mathbf{x} = (\log \|\mathbf{b}_i^*\|)_{i \leq \beta}$ .

Under SMA, once  $\sum_i x_i$  (i.e.,  $|\det(\mathbf{b}_i)_i|$ ) is fixed, an  $\mathbf{x}$  of an HKZ-reduced basis is uniquely determined.

**Lemma 2.** Let  $(\mathbf{b}_i)_{i \leq \beta}$  be HKZ-reduced,  $\mathbf{x} = (\log \|\mathbf{b}_i^*\|)_i$  and  $\mathbb{E}[\mathbf{x}] = \sum_{i \leq \beta} \frac{x_i}{\beta}$ . Then, under SMA,  $x_\beta = \mathbb{E}[\mathbf{x}] - \Gamma_\beta(\beta - 1)$  and:





**Lemma 4.** We have  $\bar{\mathbf{x}} = A \cdot \bar{\mathbf{x}} + \mathbf{g}$ .

*Proof.* Note first that for any  $\alpha$  and any  $\mathbf{x}$ , we have  $\sum_{i=1}^n (A^{(\alpha)} \cdot \mathbf{x})_i = \sum_{i=1}^n x_i$  and  $\sum_{i=1}^n g_i^{(\alpha)} = 0$ . This implies that:

$$\sum_{i=1}^n (A^{(\alpha)} \cdot \mathbf{x} + \mathbf{g}^{(\alpha)})_i = \sum_{i=1}^n x_i. \quad (4)$$

Let  $\bar{\mathbf{x}}^{(0)} = \bar{\mathbf{x}}$  and  $\bar{\mathbf{x}}^{(\alpha)} = A^{(\alpha)} \cdot \bar{\mathbf{x}}^{(\alpha-1)} + \mathbf{g}^{(\alpha)}$ , for  $\alpha \in [1, n - \beta + 1]$ . We prove by induction that:

$$\sum_{i=\alpha+1}^{\alpha+\beta-1} \bar{x}_i^{(\alpha)} = \sum_{i=\alpha+1}^{\alpha+\beta-1} \bar{x}_i \quad \text{and} \quad \bar{x}_i^{(\alpha)} = \bar{x}_i \quad \text{if} \quad i \notin [\alpha + 1, \alpha + \beta - 1]. \quad (*)$$

This holds for  $\alpha = 0$  since  $\bar{\mathbf{x}}^{(0)} = \bar{\mathbf{x}}$ . Let  $\alpha \geq 1$ . By the induction hypothesis and equality of the columns  $\alpha, \dots, \alpha + \beta - 1$  of  $A^{(\alpha)}$ , we have  $A^{(\alpha)} \cdot \bar{\mathbf{x}}^{(\alpha-1)} = A^{(\alpha)} \cdot \bar{\mathbf{x}}$  and hence  $\bar{\mathbf{x}}^{(\alpha)} = A^{(\alpha)} \cdot \bar{\mathbf{x}} + \mathbf{g}^{(\alpha)}$ . This directly implies that  $\bar{x}_i^{(\alpha)} = \bar{x}_i$  when  $i \notin [\alpha, \alpha + \beta - 1]$ . Combining this with (4) gives:

$$\sum_{i=\alpha}^{\alpha+\beta-1} \bar{x}_i^{(\alpha)} = \sum_{i=\alpha}^{\alpha+\beta-1} \bar{x}_i. \quad (5)$$

Since  $\bar{x}_\alpha^{(\alpha)} = \frac{1}{2} \log \gamma_\beta + \frac{1}{\beta} \sum_{j=\alpha}^{\alpha+\beta-1} \bar{x}_j^{(\alpha)}$ , we obtain (using (5) and the definition of  $\bar{\mathbf{x}}$ ):

$$\bar{x}_\alpha^{(\alpha)} = \frac{1}{2} \log \gamma_\beta + \frac{1}{\beta} \sum_{j=\alpha}^{\alpha+\beta-1} \bar{x}_j = \bar{x}_\alpha.$$

Combining this equality and (5) allows to complete the proof of (\*).

It remains to prove that  $\bar{x}_i^{(n-\beta+1)} = \bar{x}_i$  for  $i \geq n - \beta + 2$ . For  $i \geq n - \beta + 1$ , we have:

$$\bar{x}_i^{(n-\beta+1)} = \frac{1}{2} \log \gamma_{n-i+1} + \frac{1}{n-i+1} \sum_{j=i}^n \bar{x}_j^{(n-\beta+1)}.$$

By Lemma 2 and the definition of  $\mathbf{g}^{(n-\beta+1)}$ , this implies that  $\bar{x}_i^{(n-\beta+1)} = \frac{1}{\beta} \sum_{j=n-\beta+1}^n \bar{x}_j^{(n-\beta+1)} + g_i^{(n-\beta+1)}$ . As a consequence (using (5) and the definition of  $\bar{\mathbf{x}}$ ):

$$\bar{x}_i^{(n-\beta+1)} = \frac{1}{\beta} \sum_{j=n-\beta+1}^n \bar{x}_j + g_i^{(n-\beta+1)} = \frac{1}{\beta} \sum_{j=n-\beta+1}^n g_j^{(n-\beta+1)} + g_i^{(n-\beta+1)} = g_i^{(n-\beta+1)} = \bar{x}_i.$$

Overall, we have proven that  $A \cdot \bar{\mathbf{x}} + \mathbf{g} = \bar{\mathbf{x}}^{(n-\beta+1)} = \bar{\mathbf{x}}$ . □

**Fact.** Given  $M_k \in \mathbb{R}^{k \times k}$ ,  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$  and  $c \in \mathbb{R}$ , we define  $M_n \in \mathbb{R}^{n \times n}$  for  $n \geq k$ , as follows:

$$M_n = \left( \begin{array}{ccc|ccc} c & \cdots & c & \cdots & \mathbf{a}^T & \cdots \\ \vdots & \ddots & \vdots & & \vdots & \\ c & \cdots & c & \cdots & \mathbf{a}^T & \cdots \\ \hline \vdots & \vdots & \vdots & & & \\ \mathbf{b} & \cdots & \mathbf{b} & & & M_k \\ \vdots & \vdots & \vdots & & & \end{array} \right)$$

Then, for any  $n \geq k$ , we have  $\chi(M_n)(t) = (n - k)t^{n-k-1} \cdot \chi(M_{k+1}) - (n - k - 1)t^{n-k} \cdot \chi(M_k)$ .

**Proof of the fact.** We prove the result by induction. It clearly holds for  $n = k$  and  $n = k + 1$ . Assume now that  $n > k + 1$ . We have:

$$\begin{aligned} \chi(M_n)(t) &= \begin{vmatrix} (t-c) & -c & \cdots & -c & \cdots & -\mathbf{a}^T & \cdots \\ -c & (t-c) & \cdots & -c & \cdots & -\mathbf{a}^T & \cdots \\ \vdots & \vdots & & & & & \\ -c & -c & & & & & \\ \vdots & \vdots & & & & & \\ -\mathbf{b} & -\mathbf{b} & & & & & \\ \vdots & \vdots & & & & & \end{vmatrix} = \begin{vmatrix} 2t & -t & & & & & 0 \\ -t & (t-c) & \cdots & -c & \cdots & -\mathbf{a}^T & \cdots \\ \vdots & \vdots & & & & & \\ -c & -c & & & & & \\ 0 & \vdots & & & & & \\ -\mathbf{b} & -\mathbf{b} & & & & & \\ \vdots & \vdots & & & & & \end{vmatrix} \\ &= 2t \cdot \chi(M_{n-1}) - t^2 \cdot \chi(M_{n-2}). \end{aligned}$$

The result follows by elementary calculations.  $\square$

We now provide explicit lower and upper bounds for the coordinates of the solution  $\bar{\mathbf{x}}$ .

**Lemma 5.** For all  $i \leq n - \beta + 1$ , we have  $\left(\frac{n-i}{\beta-1} - \frac{3}{2}\right) \log \gamma_\beta \leq \bar{x}_i - \bar{x}_{n-\beta+1} \leq \frac{n-i}{\beta-1} \log \gamma_\beta$ .

*Proof.* We prove these bounds by induction on  $i$  for  $i = n - \beta, \dots, 1$ . Recall that

$$\forall i \leq n - \beta, \quad \bar{x}_i = \frac{\beta}{2(\beta-1)} \log \gamma_\beta + \frac{1}{\beta-1} \sum_{j=i+1}^{i+\beta-1} \bar{x}_j.$$

We first consider the upper bound on  $\bar{x}_i - \bar{x}_{n-\beta+1}$ . Since we defined Hermite's constant so that  $(\gamma_i)_i$  is increasing, we have  $\bar{x}_{n-\beta+1} \geq \cdots \geq \bar{x}_n$ . Therefore:

$$\forall i > n - \beta, \quad \bar{x}_i - \bar{x}_{n-\beta+1} \leq 0 \leq \frac{n-i}{\beta-1} \log \gamma_\beta.$$

Using the induction hypothesis, we obtain:

$$\bar{x}_i \leq \frac{\beta}{2(\beta-1)} \log \gamma_\beta + \frac{1}{\beta-1} \sum_{j=i+1}^{i+\beta-1} \left( \frac{n-j}{\beta-1} \log \gamma_\beta + \bar{x}_{n-\beta+1} \right) = \frac{n-i}{\beta-1} \log \gamma_\beta + \bar{x}_{n-\beta+1}.$$

We now consider the lower bound on  $\bar{x}_i - \bar{x}_{n-\beta+1}$ . It clearly holds for  $i = n - \beta + 1$ . We now prove it for  $i \in [n - 2(\beta - 1), n - \beta]$ . For that specific situation, we use the identity:

$$\forall i \in [n - 2(\beta - 1), n - \beta], \quad \bar{x}_i = \frac{\beta}{2(\beta-1)} \log \gamma_\beta + \frac{1}{\beta-1} \left( \sum_{j=i+1}^{n-\beta} \bar{x}_j + \sum_{j=n-\beta+1}^{i+\beta-1} \bar{x}_j \right). \quad (6)$$

As  $(\bar{x}_j)_j$  decreases, we have  $\frac{1}{i+2\beta-n-1} \sum_{j=n-\beta+1}^{i+\beta-1} \bar{x}_j \geq \frac{1}{\beta-1} \sum_{j=n-\beta+1}^n \bar{x}_j = \bar{x}_{n-\beta+1} - \frac{1}{2} \log \gamma_\beta$ . This implies:

$$\frac{1}{i+2\beta-n-1} \sum_{j=n-\beta+1}^{i+\beta-1} \bar{x}_j \geq \bar{x}_{n-\beta+1} + \frac{\log \gamma_\beta}{i+2\beta-n-1} \sum_{j=n-\beta+1}^{i+\beta-1} \left( \frac{n-j}{\beta-1} - \frac{3}{2} \right). \quad (7)$$

Using the induction hypothesis, we also have:

$$\frac{1}{n-\beta-i} \sum_{j=i+1}^{n-\beta} \bar{x}_j \geq \bar{x}_{n-\beta+1} + \frac{\log \gamma_\beta}{n-\beta-i} \sum_{j=i+1}^{n-\beta} \left( \frac{n-j}{\beta-1} - \frac{3}{2} \right). \quad (8)$$

Now, plugging (7) and (8) into (6) gives:

$$\bar{x}_i \geq \frac{\beta}{2(\beta-1)} \log \gamma_\beta + \bar{x}_{n-\beta+1} + \frac{\log \gamma_\beta}{\beta-1} \sum_{j=i+1}^{i+\beta-1} \left( \frac{n-j}{\beta-1} - \frac{3}{2} \right) = \left( \frac{n-i}{\beta-1} - \frac{3}{2} \right) \log \gamma_\beta + \bar{x}_{n-\beta+1}.$$

When  $i < n - 2(\beta - 1)$ , the proof for the lower bound is similar to that of the upper bound.  $\square$

As the set of solutions to  $\mathbf{x} = A \cdot \mathbf{x} + \mathbf{g}$  is  $\bar{\mathbf{x}} + \text{Span}(1, \dots, 1)^T$ , the value of  $\bar{\mathbf{x}}$  is only interesting up to a constant vector, which is why we bound  $\bar{x}_i - \bar{x}_{n-\beta+1}$  rather than  $\bar{x}_i$ . In other words, since  $\mathbf{x}^\infty$  of Theorem 1 is  $\bar{\mathbf{x}} - (\mathbb{E}[\bar{\mathbf{x}}])_i$ , the Lemma also applies to  $\mathbf{x}^\infty$ . It is also worth noting that the difference between the upper and lower bounds  $\frac{3}{2} \log \gamma_\beta$  is much smaller than the upper bound  $\frac{n-i}{\beta-1} \log \gamma_\beta$  (for most values of  $i$ ). If we replace  $\gamma_\beta$  by  $\beta$ , then, via a tedious function analysis, we can improve both bounds so that their difference is lowered to  $\frac{1}{2} \log \beta$ . In the special case  $\beta = 2$ , the expression of  $\bar{\mathbf{x}}$  is  $\bar{x}_i = \bar{x}_n + (n - i) \log \gamma_2$ .

### 4.3 Speed of convergence of the dynamical system

The classical approach to study the speed of convergence (with respect to  $k$ ) of a discrete-time dynamical system  $\mathbf{x}_{k+1} := A_n \cdot \mathbf{x}_k + \mathbf{g}_n$  (where  $A_n$  and  $\mathbf{g}_n$  are the  $n$ -dimensional values of  $A$  and  $\mathbf{g}$  respectively) consists in providing an upper bound to the largest eigenvalue of  $A_n^T A_n$ . It is relatively easy to prove that it is 1 (note that  $A_n$  is doubly stochastic). We are to show that the second largest singular value is  $< 1 - \frac{\beta^2}{2n^2}$ , and that this bound is sharp, up to changing the constant  $1/2$  and as long as  $n - \beta = \Omega(n)$ .

The asymptotic speed of convergence of the sequence  $(A_n^k \cdot \mathbf{x})_k$  is in fact determined by the eigenvalue(s) of  $A_n$  of largest module (this is the principle of the power iteration algorithm). However, this classical fact provides no indication on the dependency with respect to  $\mathbf{x}$ , which is crucial in the present situation. As we use the bound  $\|A_n^k \cdot \mathbf{x}\| \leq \|A_n\|_2^k \cdot \|\mathbf{x}\|$ , we are led to studying the largest singular values of  $A_n^T A_n$ .

We first explicit the characteristic polynomial  $\chi_n$  of  $A_n^T A_n$ . The following lemma shows that it satisfies a second order recurrence formula.

**Lemma 6.** *We have  $\chi_\beta(t) = t^{\beta-1}(t-1)$ ,  $\chi_{\beta+1}(t) = t^{\beta-1}(t-1)(t - \frac{1}{\beta^2})$  and, for any  $n \geq \beta$ :*

$$\chi_{n+2}(t) = \frac{(2\beta(\beta-1)+1)t-1}{\beta^2} \cdot \chi_{n+1}(t) - \left( \frac{\beta-1}{\beta} \right)^2 t^2 \cdot \chi_n(t).$$

*Proof.* We have  $A_\beta^T A_\beta = A_\beta$  and  $\dim \ker(A_\beta) = \beta - 1$ , thus  $t^{\beta-1} | \chi_\beta(t)$ . Since  $\text{Tr}(A_\beta) = 1$  we have  $\chi_\beta(t) = t^\beta(t-1)$ . The computation of  $A_{\beta+1}^T A_{\beta+1}$  gives:

$$A_{\beta+1}^T A_{\beta+1} = \left( \begin{array}{c|c} \frac{\beta+(\beta-1)^2}{\beta^3} & \begin{array}{c} \vdots \\ \frac{\beta-1}{\beta^2} \end{array} \\ \hline \dots \frac{\beta-1}{\beta^2} \dots & \frac{1}{\beta} \end{array} \right).$$

If  $y_1 + \dots + y_\beta = 0$  and  $y_{\beta+1} = 0$ , then  $A_{\beta+1}^T A_{\beta+1} \cdot \mathbf{y} = \mathbf{0}$ , hence  $\dim \ker(A_{\beta+1}^T A_{\beta+1}) \geq \beta - 1$  and  $t^{\beta-1} | \chi_{\beta+1}(t)$ . It can be checked that  $A_{\beta+1}^T A_{\beta+1} \cdot (1, \dots, 1)^T = (1, \dots, 1)^T$ . Finally, since  $\text{Tr}(A_{\beta+1}^T A_{\beta+1}) = 1 + \frac{1}{\beta^2}$  we have  $\chi_{\beta+1}(t) = t^{\beta-1}(t-1)(t - \frac{1}{\beta^2})$ .

For  $n \geq 1$ , let  $C_n$  be the  $n \times n$  bottom-right corner of  $A_{n+\beta-1}^T A_{n+\beta-1}$ . Note that for  $n, i, j > 1$ , we have  $c_{nij} = c_{n-1, i-1, j-1}$ , which means that we can write  $C_n$  as:

$$C_n = \left( \begin{array}{c|ccc} c_{n11} & c_{n12} & \dots & c_{n1n} \\ c_{n21} & & & \\ \vdots & & C_{n-1} & \\ c_{nn1} & & & \end{array} \right).$$

Moreover, we have  $c_{n11} = \left(\frac{\beta-1}{\beta}\right)^2 c_{n22} + \frac{1}{\beta^2}$ ,  $c_{ni1} = \frac{\beta-1}{\beta} c_{ni2}$  and  $c_{n1i} = \frac{\beta-1}{\beta} c_{n2i}$  for all  $i > 1$ . Subtracting  $\frac{\beta-1}{\beta}$  times the second column of  $tI_n - C_n$  from the first column and subtracting  $\frac{\beta-1}{\beta}$  times the second row from the first row gives:

$$\chi(C_n)(t) = \left| \begin{array}{c|ccc} \frac{2\beta^2-2\beta+1}{\beta^2}t - \frac{1}{\beta^2} & -\frac{\beta-1}{\beta}t & 0 & \dots & 0 \\ \hline -\frac{\beta-1}{\beta}t & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & tI_{n-1} - C_{n-1} \end{array} \right|.$$

By expansion on the first column and then on the first row we obtain:

$$\chi(C_n)(t) = \frac{(2\beta^2 - 2\beta + 1)t - 1}{\beta^2} \cdot \chi(C_{n-1})(t) - \left(\frac{\beta-1}{\beta}\right)^2 t^2 \cdot \chi(C_{n-2})(t).$$

Since the  $\beta$  first columns (resp. rows) of  $A_{n+\beta-1}^T A_{n+\beta-1}$  are identical, we obtain, by the previous Fact, that  $\chi_{n+\beta-1}(t) = \beta t^{\beta-1} \cdot \chi(C_n)(t) - (\beta-1)t^\beta \cdot \chi(C_{n-1})(t)$ . This implies that the  $\chi_n$ 's satisfy the same second order relation as the  $\chi(C_n)$ 's.  $\square$

We finally study the roots of  $\chi_n(t)$ . The proof of the following result relies on several changes of variables to link the polynomials  $\chi_n(t)$  to the Chebyshev polynomials of the second kind.

**Lemma 7.** *For any  $n \geq \beta \geq 2$ , the largest root of  $\frac{\chi_n(t)}{t-1}$  is in  $\left[1 - \frac{\pi^2 \beta^2}{(n-\beta)^2}, 1 - \frac{\beta^2}{2n^2}\right]$ .*

*Proof.* Let  $\bar{\chi}_n(t)$  be the polynomial  $t^n \chi_n(1/t)$ . Then, by Lemma 6, we have  $\bar{\chi}_\beta(t) = 1 - t$ ,  $\bar{\chi}_{\beta+1}(t) = (1-t)\left(1 - \frac{t}{\beta^2}\right)$ , and, for  $n \geq \beta$ :

$$\begin{aligned} \bar{\chi}_{n+2}(t) &= t^{n+2} \frac{(2\beta(\beta-1)+1)\frac{1}{t} - 1}{\beta^2} \cdot \chi_{n+1}\left(\frac{1}{t}\right) - t^{n+2} \left(\frac{\beta-1}{\beta}\right)^2 \frac{1}{t^2} \cdot \chi_n\left(\frac{1}{t}\right) \\ &= \frac{(2\beta(\beta-1)+1) - t}{\beta^2} \cdot \bar{\chi}_{n+1}(t) - \left(\frac{\beta-1}{\beta}\right)^2 \cdot \bar{\chi}_n(t). \end{aligned}$$

Let  $\tau(t') = 2\beta(\beta-1)(t'-1)$  and  $\psi_n(t') = \left(\frac{\beta}{\beta-1}\right)^{n-\beta} \cdot \frac{\bar{\chi}_n(1-\tau(t'))}{\tau(t')}$ . We have  $\psi_\beta(t') = 1$ ,  $\psi_{\beta+1}(t') = 2t' - \frac{\beta-1}{\beta}$  and, for  $n \geq \beta$ :

$$\begin{aligned} \psi_{n+2}(t') &= 2t' \left(\frac{\beta}{\beta-1}\right)^{n+1-\beta} \cdot \frac{\bar{\chi}_{n+1}(1-\tau(t'))}{\tau(t')} - \left(\frac{\beta}{\beta-1}\right)^{n-\beta} \cdot \frac{\bar{\chi}_n(1-\tau(t'))}{\tau(t')} \\ &= 2t' \cdot \psi_{n+1}(t') - \psi_n(t'). \end{aligned}$$

As a consequence, the  $\psi_n$ 's are polynomials (in  $t'$ ). Now, let  $(U_n)_{n \geq 0}$  be the sequence of Chebyshev polynomials of the second kind, i.e.,  $U_0 = 0$ ,  $U_1 = 1$  and  $U_{n+2}(t') = 2t' \cdot U_{n+1}(t') - U_n(t')$  for  $n \geq 0$ . These polynomials satisfy the following property:

$$\forall n \geq 0, \forall x \in \mathbb{R} \setminus \{2k\pi; k \in \mathbb{Z}\}, U_n(\cos x) = \frac{\sin(nx)}{\sin x}.$$

It can be proven by induction that  $\psi_n = U_{n-\beta+1} - \frac{\beta-1}{\beta} U_{n-\beta}$  for all  $n \geq \beta$ . By the Fact given below, this implies that there exists  $t'_0 \in \left[\cos \frac{\pi}{n-\beta}, \cos \frac{\pi}{2(n-\beta+1)}\right]$  such that  $\psi_n(t'_0) = 0$  and  $\psi_n(t') > 0$  for all  $t' \in (t'_0, 1)$ . We have  $\bar{\chi}_n(1 - \tau(t'_0)) = \left(\frac{\beta-1}{\beta}\right)^{n-\beta} \tau(t'_0) \psi_n(t'_0) = 0$ , hence  $t_0 = (1 - \tau(t'_0))^{-1}$  is a root of  $\chi_n(t)$ . Since

the image of  $(t'_0, 1)$  by  $t' \mapsto (1 - \tau(t'))^{-1}$  is  $(t_0, 1)$ , we obtain that  $t_0$  is the largest root of  $\chi_n(t)$  smaller than 1. We now compute bounds for  $t_0$ . We have  $2(n - \beta + 1) \leq 2n$  so  $\cos \frac{\pi}{n-\beta} \leq t'_0 \leq \cos \frac{\pi}{2n}$ . It can be checked that for  $u \leq \frac{\pi}{4}$ , we have  $\cos u \leq 1 - \frac{8}{17}u^2$ , so  $1 - \frac{\pi^2}{(n-\beta)^2} \leq t'_0 \leq 1 - \frac{2\pi^2}{17n^2}$ . This leads to  $1 + \frac{\pi^2\beta^2}{(n-\beta)^2} \geq 1 - \tau(t'_0) \geq 1 + 2\beta(\beta - 1)\frac{2\pi^2}{17n^2} \geq 1 + \frac{2\pi^2\beta^2}{17n^2}$ , and thus  $1 - \frac{\pi^2\beta^2}{(n-\beta)^2} \leq t_0 \leq 1 - \frac{1}{2}\frac{\beta^2}{n^2}$ .

To conclude, let  $\phi_n(t)$  be the polynomial  $\frac{\chi_n(t)}{t-1}$ . By using Lemma 6, it can be checked that  $\phi_n(1) = \left(\frac{\beta-1}{\beta}\right)^{n-\beta} \frac{n}{\beta}$ , which implies that  $\phi_n(1) \neq 0$ . This proves that 1 is never a multiple root of  $\chi_n$ , which completes the proof.  $\square$

**Fact.** Let  $n \geq 2$  and  $f(x) = \frac{\sin((n+1)x)}{\sin x} - \frac{\beta-1}{\beta} \cdot \frac{\sin(nx)}{\sin x}$ . The smallest positive root of  $f$  belongs to  $\left[\frac{\pi}{2(n+1)}, \frac{\pi}{n}\right]$ .

*Proof of the fact.* Since  $\sin$  is an increasing function on  $[0, \frac{\pi}{2}]$ , we have  $\sin(nx) < \sin((n+1)x)$  for all  $0 < x \leq \frac{\pi}{2(n+1)}$ . This implies that  $f(x) > 0$  on this interval. We also have  $f(\frac{\pi}{n}) = -1 < 0$ . The result follows from the intermediate value theorem.  $\square$

*Proof of Theorem 2.* The unicity and existence of  $\mathbf{x}^\infty$  come from Lemmata 3 and 4.

Let  $(\mathbf{b}_i^{(k)})_{i \leq n}$  be the basis after  $k$  tours of  $\text{BKZ}'_\beta$  and  $\mathbf{x}_i^{(k)} = \log \frac{\|\mathbf{b}_i^{(k)*}\|}{(\det L)^{1/n}}$ . The definition of  $\mathbf{x}^\infty$  and a simple induction imply that  $\mathbf{x}^{(k)} - \mathbf{x}^\infty = A^k(\mathbf{x}^{(0)} - \mathbf{x}^\infty)$ . Both  $\mathbf{x}^{(0)}$  and  $\mathbf{x}^\infty$  live in the subspace  $\mathcal{E} := \text{Span}(1, \dots, 1)^\perp$ , which is stabilized by  $A$ . Let us denote by  $A_\mathcal{E}$  the restriction of  $A$  to this subspace. Then the largest eigenvalue of  $A_\mathcal{E}^T A_\mathcal{E}$  is bounded in Lemma 7 by  $\left(1 - \frac{\beta^2}{2n^2}\right)$ . Taking the norm in the previous equation gives:

$$\|\mathbf{x}^{(k)} - \mathbf{x}^\infty\|_2 \leq \|A_\mathcal{E}\|_2^k \cdot \|\mathbf{x}^{(0)} - \mathbf{x}^\infty\|_2 = \rho(A_\mathcal{E}^T A_\mathcal{E})^{k/2} \cdot \|\mathbf{x}^{(0)} - \mathbf{x}^\infty\|_2 \leq \left(1 - \frac{\beta^2}{2n^2}\right)^{k/2} \|\mathbf{x}^{(0)} - \mathbf{x}^\infty\|_2.$$

The term  $\|\mathbf{x}^{(0)} - \mathbf{x}^\infty\|_2$  is bounded by  $\|\mathbf{x}^{(0)}\|_2 + \|\mathbf{x}^\infty\|_2 \leq \left(\log \max_i \frac{\|\mathbf{b}_i^*\|}{(\det L)^{1/n}}\right) n + n^{O(1)}$ . Taking the logarithm, we obtain  $\log \|\mathbf{x}^{(0)} - \mathbf{x}^\infty\|_2 = O(\log n + \log \log \max_i \frac{\|\mathbf{b}_i^*\|}{(\det L)^{1/n}})$ . This implies that there exists  $C$  such that  $\|\mathbf{x}^{(k)} - \mathbf{x}^\infty\|_2 \leq 1$  when  $k \geq C \frac{n^2}{\beta^2} (\log n + \log \log \max_i \frac{\|\mathbf{b}_i^*\|}{(\det L)^{1/n}})$ .

We now prove the last inequality of the theorem. By Lemma 5 and the fact that  $\sum_{i=n-\beta+1}^n x_i^\infty = \beta \left(x_{n-\beta+1}^\infty - \frac{1}{2} \log \gamma_\beta\right) \geq \beta x_{n-\beta+1}^\infty + \sum_{i=n-\beta+1}^n \left(\frac{\log \gamma_\beta}{\beta-1} (n-i) - \frac{3}{2} \log \gamma_\beta\right)$ , we have:

$$x_1^\infty = x_1^\infty - \sum_{j=1}^n \frac{x_j^\infty}{n} \leq (n-1) \frac{\log \gamma_\beta}{\beta-1} - \frac{1}{n} \sum_{i=1}^n \left(\frac{\log \gamma_\beta}{\beta-1} (n-i) - \frac{3}{2} \log \gamma_\beta\right) = \left(\frac{n-1}{2(\beta-1)} + \frac{3}{2}\right) \log \gamma_\beta.$$

Using the inequality  $x_1^{(k)} \leq x_1^\infty + 1$  and taking the exponential (in base 2) leads to the result.  $\square$

## 5 Analysis of BKZ'

We now show how the heuristic analysis of the previous section can be made rigorous. The main difficulty stems from the lack of control on the  $\|\mathbf{b}_i^*\|$ 's of an HKZ-reduced basis  $(\mathbf{b}_i)_{i \leq \beta}$ . More precisely, once the determinant and  $\|\mathbf{b}_\beta^*\|$  are fixed, the  $\|\mathbf{b}_i^*\|$ 's are all below a specific curve (explicitly given in Lemma 2). However, if only the determinant is fixed, the pattern of the  $\|\mathbf{b}_i^*\|$ 's can vary significantly: as an example, taking orthogonal vectors of increasing norms shows that  $\|\mathbf{b}_1^*\|$  (resp.  $\|\mathbf{b}_\beta^*\|$ ) can be arbitrarily small (resp. large). Unfortunately, when applying HKZ within BKZ', it seems we only control the determinant of the HKZ-reduced basis of the considered block, although we would prefer to have an upper bound for each Gram-Schmidt norm individually. We circumvent this difficulty by *amortizing* the analysis over the  $\|\mathbf{b}_i^*\|$ 's: as observed in [11], we have a sharp control *on each average* of the first  $\|\mathbf{b}_i^*\|$ 's. For an arbitrary basis  $B := (\mathbf{b}_i)_{i \leq n}$ , we define  $\mu_k^{(B)} = \frac{1}{k} \sum_{1 \leq i \leq k} \log \|\mathbf{b}_i^*\|$ , for  $k \leq n$ .

**Lemma 8 ([11, Le. 3]).** *If  $B = (\mathbf{b}_i)_{i \leq \beta}$  is HKZ-reduced, then  $\mu_k^{(B)} \leq \frac{\beta-k}{k} \log \Gamma_\beta(k) + \mu_\beta^{(B)}$  for all  $k \leq \beta$ .*

## 5.1 A dynamical system for (genuine) BKZ' tours

We now reformulate the results of the previous section with the  $\mu_i^{(B)}$ 's instead of the  $\log \|\mathbf{b}_i^*\|$ 's. This amounts to a base change in the discrete-time dynamical system of Subsection 4.1. We define:

$$P = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ \vdots & & & \ddots & \\ \frac{1}{n} & \dots & & & \frac{1}{n} \end{pmatrix}, \quad \tilde{A} = PAP^{-1} \quad \text{and} \quad \tilde{\mathbf{g}} = P \cdot \mathbf{g}.$$

Note that  $\boldsymbol{\mu}^{(B)} = P \cdot \mathbf{x}^{(B)}$ , where  $\mathbf{x}^{(B)} = (\log \|\mathbf{b}_i^*\|)_i$  and  $\boldsymbol{\mu}^{(B)} = (\mu_i^{(B)})_i$ .

**Lemma 9.** *Let  $B'$  be the basis obtained after a BKZ' tour given an  $n$ -dimensional basis  $B$  as input. Then  $\boldsymbol{\mu}^{(B')} \leq \tilde{A} \cdot \boldsymbol{\mu}^{(B)} + \tilde{\mathbf{g}}$ , where the inequality holds componentwise.*

*Proof.* Let  $\alpha \leq n - \beta + 1$ . We define  $\tilde{A}^{(\alpha)} = PA^{(\alpha)}P^{-1}$  and  $\tilde{\mathbf{g}}^{(\alpha)} = P \cdot \mathbf{g}^{(\alpha)}$ . Let  $B^{(\alpha)}$  be the basis after the first  $\alpha$  calls to  $\beta$ -HKZ (starting with indices  $1, \dots, \alpha$ ). We first prove that we have:

$$\boldsymbol{\mu}^{(B^{(\alpha)})} \leq \tilde{A}^{(\alpha)} \cdot \boldsymbol{\mu}^{(B^{(\alpha-1)})} + \tilde{\mathbf{g}}^{(\alpha)}. \quad (9)$$

This vectorial inequality can be checked by making  $\tilde{A}^{(\alpha)}$  and  $\tilde{\mathbf{g}}^{(\alpha)}$  explicit:

$$\tilde{A}_{ij}^{(\alpha)} = \begin{cases} 1 & \text{if } i = j \text{ with } i < \alpha \text{ or } i \geq \alpha + \beta - 1 \\ \frac{\alpha-1}{i} \left(1 - \frac{i-\alpha+1}{\beta}\right) & \text{if } i \in [\alpha, \alpha + \beta - 2] \text{ and } j = \alpha - 1 \\ \frac{(\alpha+\beta-1)(i-\alpha+1)}{\beta i} & \text{if } i \in [\alpha, \alpha + \beta - 2] \text{ and } j = \alpha + \beta - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{g}_i^{(\alpha)} = \begin{cases} \frac{\beta-i+\alpha-1}{i} \log \Gamma_\beta(i - \alpha + 1) & \text{if } i \in [\alpha, \alpha + \beta - 2] \\ 0 & \text{otherwise.} \end{cases}$$

We provide more details on the proof of (9) in appendix.

Now, let  $\boldsymbol{\nu}^{(0)} = \boldsymbol{\mu}^{(B^{(0)})} = \boldsymbol{\mu}$  and  $\boldsymbol{\nu}^{(\alpha)} = \tilde{A}^{(\alpha)} \cdot \boldsymbol{\nu}^{(\alpha-1)} + \tilde{\mathbf{g}}^{(\alpha)}$ . We prove by induction that  $\boldsymbol{\mu}^{(B^{(\alpha)})} \leq \boldsymbol{\nu}^{(\alpha)}$ . For  $\alpha \geq 1$ , we have (successively using (9), the induction hypothesis and the fact that  $\tilde{A}^{(\alpha)} \geq 0$ ):

$$\boldsymbol{\mu}^{(B^{(\alpha)})} \leq \tilde{A}^{(\alpha)} \cdot \boldsymbol{\mu}^{(B^{(\alpha-1)})} + \tilde{\mathbf{g}}^{(\alpha)} \leq \tilde{A}^{(\alpha)} \cdot \boldsymbol{\nu}^{(\alpha-1)} + \tilde{\mathbf{g}}^{(\alpha)} = \boldsymbol{\nu}^{(\alpha)}.$$

The result follows, by taking  $\alpha = d - \beta + 1$ . □

## 5.2 Analysis of the updated dynamical system

Similarly to the analysis of the previous section, it may be possible to obtain information on the speed of convergence of BKZ' by estimating the eigenvalues of  $\tilde{A}^T \cdot \tilde{A}$ . However, the latter eigenvalues seem significantly less amenable to study than those of  $A^T A$ . The following lemma shows how to short-circuit the study of the modified dynamical system. For a basis  $B \in \mathbb{R}^{n \times n}$  given as input to  $\text{BKZ}'_\beta$ , we define  $B^{[0]} = B$  and  $B^{[i]}$  as the current basis after the  $i$ -th BKZ' tour. We also define  $\boldsymbol{\mu}^\infty = P \cdot \mathbf{x}^\infty$ .

**Lemma 10.** *Let  $B \in \mathbb{R}^{n \times n}$  a basis given as input to  $\text{BKZ}'_\beta$ . Wlog we assume that  $\mu_n^{(B)} = \mu_n^\infty$  (since  $\mu_n^{(B)} = \frac{1}{n} \log |\det B|$ , this can be achieved by multiplying  $B$  by a scalar). We have:*

$$\forall k \geq 0, \forall i \leq n, \quad \mu_i^{(B^{[k]})} \leq \mu_i^\infty + (1 + \log n)^{1/2} \cdot \left(1 - \frac{\beta^2}{2n^2}\right)^{k/2} \|\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty\|_2.$$

*Proof.* First, by using Lemma 9 and noting that  $\tilde{A} \cdot \boldsymbol{\mu}^\infty = \boldsymbol{\mu}^\infty + \tilde{\mathbf{g}}$ , it can be shown by induction that

$$\boldsymbol{\mu}^{(B^{[k]})} - \boldsymbol{\mu}^\infty \leq \tilde{A}^k \cdot (\boldsymbol{\mu}^{(B^{[0]})} - \boldsymbol{\mu}^\infty). \quad (10)$$

Now, we have  $\|\tilde{A}^k \cdot (\boldsymbol{\mu}^{(B^{[0]})} - \boldsymbol{\mu}^\infty)\|_2 = \|PA^kP^{-1} \cdot (\boldsymbol{\mu}^{(B^{[0]})} - \boldsymbol{\mu}^\infty)\|_2 \leq \|P\|_2 \cdot \|A^k \cdot (\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty)\|_2$ . Thanks to the assumption on  $\mu_n^{(B)}$ , we know that  $\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty \in \text{Span}(1, \dots, 1)^\perp$ , which is stable under  $A$ . As in theorem 2, we introduce the restriction  $A_\mathcal{E}$  of  $A$  to this subspace. By the results of Subsection 4.3, we know that the largest eigenvalue of  $A_\mathcal{E}^T \cdot A_\mathcal{E}$  is  $\leq (1 - \frac{\beta^2}{2n^2})$ . Therefore:

$$\begin{aligned} \|\tilde{A}^k \cdot (\boldsymbol{\mu}^{(B^{[0]})} - \boldsymbol{\mu}^\infty)\|_2 &\leq \|P\|_2 \cdot \|A_\mathcal{E}^k \cdot (\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty)\|_2 \leq \|P\|_2 \cdot \|A_\mathcal{E}\|_2^k \cdot \|\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty\|_2 \\ &\leq \rho(P^T P)^{1/2} \cdot \left(1 - \frac{\beta^2}{2n^2}\right)^{k/2} \cdot \|\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty\|_2, \end{aligned}$$

where  $\rho$  denotes the spectral radius. Now, the sum of the coordinates of any row of  $P^T P$  is  $\leq \sum_{i=1}^n \frac{1}{i} \leq 1 + \ln n \leq 1 + \log n$ . This gives  $\rho(P^T P) \leq 1 + \log n$ . The result follows.  $\square$

**Lemma 11.** *There exists  $C > 0$  such that the following holds for all integers  $n \geq \beta$ , and  $\varepsilon \in (0, 1]$ . Let  $(\mathbf{b}_i)_{i \leq n}$  be a basis of a lattice  $L$ , given as input to the modified BKZ' algorithm of Section 2 with block-size  $\beta$ . If terminated after  $C \frac{n^3}{\beta^2} (\log \frac{n}{\varepsilon} + \log \log \max_i \frac{\|\mathbf{b}_i^*\|}{(\det L)^{1/n}})$  calls to an HKZ-reduction (resp. SVP solver) in dimension  $\beta$ , the output  $(\mathbf{c}_i)_{i \leq n}$  is a basis of  $L$  that satisfies:*

$$\|\mathbf{c}_1\| \leq (1 + \varepsilon) \gamma_\beta^{\frac{n-1}{2(\beta-1)} + \frac{3}{2}} \cdot (\det L)^{\frac{1}{n}}.$$

*Proof.* Wlog we assume that  $\mu_n(B^{[0]}) = \mu_n^\infty$ . The proof is similar to that of theorem 2. We know that:

$$\mu_1^\infty - \mu_n^\infty = x_1^\infty - \frac{1}{n}(x_1^\infty + \dots + x_n^\infty) \leq \left(\frac{n-1}{2(\beta-1)} + \frac{3}{2}\right) \log \gamma_\beta \quad (11)$$

We have  $\log\left(\frac{(1+\log n)^{\frac{1}{2}} \|\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty\|_2}{\log(1+\varepsilon)}\right) = O(\log \frac{n}{\varepsilon} + \log \log \max_i \|\mathbf{b}_i\|)$  so there exists  $C \geq 0$  (independent of  $\beta$ ) such that for any  $k \geq C \frac{n^2}{\beta^2} (\log \frac{n}{\varepsilon} + \log \log \max \|\mathbf{b}_i\|)$ , we have:

$$(1 + \log n)^{\frac{1}{2}} \left(1 - \frac{\beta^2}{2n^2}\right)^{\frac{k}{2}} \|\mathbf{x}^{(B^{[0]})} - \mathbf{x}^\infty\|_2 \leq \log(1 + \varepsilon).$$

This gives  $\mu_1(B^{[k]}) \leq \mu_1^\infty + \log(1 + \varepsilon) \leq \left(\mu_n(B^{[0]}) + \frac{n-1}{2(\beta-1)} + \frac{3}{2}\right) \log \gamma_\beta + 1$ . Taking the exponential (in base 2) leads to the result.  $\square$

Theorem 1 corresponds to taking  $\varepsilon = 1$  in Lemma 11. Also, when  $\beta = 2$ , plugging the explicit expression of  $\mathbf{x}^\infty$  into (11) leads to the improved bound  $\|\mathbf{c}_1\| \leq (1 + \varepsilon) \cdot (\gamma_2)^{\frac{n-1}{2}} \cdot (\det L)^{\frac{1}{n}}$ .

## 6 Applications to LLL-Reduction

In this section, we investigate the relationship between BKZ'<sub>2</sub> reduction and the notion of LLL-reduction [16]. Note that analogues of some of the results of this section have been concurrently and independently obtained by Schnorr [35].

**Reminders on the LLL algorithm.** The LLL algorithm with parameter  $\delta$  proceeds by successive loop iterations. Each iteration has a corresponding index  $k$ , defined as the smallest such that  $(\mathbf{b}_i)_{i \leq k}$  is not  $\delta$ -LLL-reduced. The iteration consists in size-reducing  $(\mathbf{b}_i)_{i \leq k}$  and then checking Lovász's condition  $\delta \|\mathbf{b}_{k-1}^*\|^2 \leq \|\mathbf{b}_k^*\|^2 + \mu_{k,k-1}^2 \|\mathbf{b}_{k-1}^*\|^2$ . If it is satisfied, then we proceed to the next loop iteration, and otherwise, we swap the vectors  $\mathbf{b}_k$  and  $\mathbf{b}_{k-1}$ . Any such swap decreases the quantity  $\Pi((\mathbf{b}_i)_i) = \prod_{i=1}^n \|\mathbf{b}_i^*\|^{2(n-i+1)}$  by

a factor  $\geq 1/\delta$  whereas it remains unchanged during size-reductions. Since  $\Pi((\mathbf{b}_i)_i) \leq 2^{O(n^2 \text{size}(B))}$  and since for any integer basis  $\Pi((\mathbf{b}_i)_i)$  is an integer, this allows to prove termination within  $O(n^2 \text{size}(B))$  loop iterations when  $\delta < 1$ . When  $\delta = 1$ , we obtain the so-called *optimal LLL algorithm*. Termination can still be proven by using different arguments, but with a much larger bound  $2^{\text{Poly}(n)} \cdot \text{Poly}(\text{size}(B))$  (see [3,17]).

**An iterated version of BKZ<sub>2</sub>.** We consider the algorithm Iterated-BKZ<sub>2</sub> (described in Algorithm 3) which given as input a basis  $(\mathbf{b}_i)_{i \leq n}$  successively applies BKZ<sub>2</sub> to the bases  $(\mathbf{b}_i)_{i \leq n}$ ,  $(\mathbf{b}_i^{(2)})_{2 \leq i \leq n}$ ,  $\dots$ ,  $(\mathbf{b}_i^{(n-1)})_{n-1 \leq i \leq n}$ . By using a quasi-linear time Gauss reduction algorithm (see [39,42]) as the HKZ<sub>2</sub> algorithm within BKZ<sub>2</sub>, Algorithm Iterated-BKZ<sub>2</sub> can be shown to run in quasi-linear time.

**Input** : A basis  $(\mathbf{b}_i)_{i \leq n}$  of a lattice  $L$ .  
**Output** : A basis of  $L$ .  
**for**  $k := 1$  to  $n - 1$  **do**  
    Apply BKZ<sub>2</sub> to the basis  $(\mathbf{b}_i^{(k)})_{k \leq i \leq n}$ ;  
    Let  $T$  be the corresponding transformation matrix;  
    Update  $(\mathbf{b}_i)_{i \leq n}$  by applying  $T$  to  $(\mathbf{b}_i)_{k \leq i \leq n}$ .  
**Return**  $(\mathbf{b}_i)_{i \leq n}$ .

**Algorithm 3:** Iterated-BKZ<sub>2</sub> Algorithm

**Lemma 12.** *Let  $B$  be a basis of an  $n$ -dimensional lattice, and  $\varepsilon > 0$  be arbitrary. Then, using Algorithm Iterated-BKZ<sub>2</sub>, one can compute, in time  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(B))$ , a basis  $(\mathbf{b}'_i)_{i \leq n}$  such that*

$$\forall i \leq n, \|\mathbf{b}'_i\| \leq (1 + \varepsilon) \left(\frac{4}{3}\right)^{\frac{n-i}{2}} \cdot \left(\prod_{j=i}^n \|\mathbf{b}'_j\|\right)^{\frac{1}{n-i+1}}. \quad (12)$$

*Proof.* We first prove that (12) holds for the output of Iterated-BKZ<sub>2</sub>. The remark at the end of Section 5 shows that (12) holds for  $i = 1$  after the first step of the algorithm. The following steps do not modify the first vector of the basis, nor do they modify the right hand side of (12), hence the inequality holds. Now, Iterated-BKZ<sub>2</sub> starting from Step 2 is equivalent to applying Iterated-BKZ<sub>2</sub> to the basis  $(\mathbf{b}_i^{(2)})_{2 \leq i \leq n}$ . It follows from the case  $i = 1$  and a direct induction that (12) holds for all  $i$ .

We turn to analyzing the complexity. First, note that HKZ in dimension 2, i.e., Gauss' reduction, can be performed in time  $\tilde{O}(\text{size}(C))$  given basis  $C \in \mathbb{Q}^{2 \times 2}$  as input (see [39,42]). Standard techniques allow one to bound the bit-sizes of all the vectors occurring during an execution of BKZ<sub>2</sub> (and hence Iterated-BKZ<sub>2</sub>), by a linear function of the bit-size of the input. This completes the proof.  $\square$

**A close analogue of the optimal LLL.** Let  $B = (\mathbf{b}_i)_{i \leq n}$  an integral basis output by Iterated-BKZ<sub>2</sub>. For  $i \leq n$ , we let  $p_i, q_i$  be coprime rational integers such that  $\frac{p_i}{q_i} = \left(\frac{3}{4}\right)^{(n-i+1)(n-i)} \cdot \frac{\|\mathbf{b}'_i\|^{2(n-i+1)}}{\prod_{j=i}^n \|\mathbf{b}'_j\|^2}$ . By (12), we know that  $p_i/q_i \leq (1 + \varepsilon)^{n-i+1}$ . Note that  $p_i/q_i$  is a rational number with denominator  $\leq 2^{O(n^2 + \text{size}(B))}$ . We can thus find a constant  $c$  such that, for all  $i$ , the quantity  $|p_i/q_i - 1|$  is either 0 or  $\geq 2^{-c(n^2 + \text{size}(B))}$ . Hence, if we choose  $\varepsilon < \frac{1}{2n} \cdot 2^{-c(n^2 + \text{size}(B))}$ , all the inequalities from (12) must hold with  $\varepsilon = 0$ . Overall, we obtain, in polynomial time and using only swaps and size-reductions, a basis for which (12) holds with  $\varepsilon = 0$ .

**A quasi-linear time LLL-reduction algorithm.** BKZ<sub>2</sub> can be used to obtain a variant of LLL which given as input an integer basis  $(\mathbf{b}_i)_{i \leq n}$  and  $\delta < 1$  returns a  $\delta$ -LLL-reduced basis of  $L[(\mathbf{b}_i)_{i \leq n}]$  in time  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(B))$ . First, we apply the modification from [18, p. 25] to a terminated BKZ<sub>2</sub> so that the modified algorithm, when given as input an integer basis  $(\mathbf{b}_i)_{i \leq n}$  and  $\varepsilon > 0$ , returns in time  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(B))$  a basis  $(\mathbf{b}'_i)_{i \leq n}$  of  $L[(\mathbf{b}_i)_{i \leq n}]$  such that  $\|\mathbf{b}'_1\| \leq (1 + \varepsilon)^2 (4/3)^{n-1} \lambda_1(L)$ . The complexity bound holds because the transformation from [18, p. 25] applies BKZ<sub>2</sub>  $n$  times on bases whose bit-sizes are  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(B))$ .

We then iterate this algorithm  $n$  times on the projected lattices  $(\mathbf{b}_i^{(k)})_{k \leq i \leq n}$  so that the output basis  $(\mathbf{c}_i)_{i \leq n}$  of  $L[(\mathbf{b}_i)_{i \leq n}]$  satisfies:

$$\forall i \leq n, \|\mathbf{c}_i\| \leq (1 + \varepsilon)^2 (4/3)^{n-i} \lambda_1(L[(\mathbf{b}_j^{(i)})_{i \leq j \leq n}]). \quad (13)$$

It follows from inequalities and the size-reducedness of  $(\mathbf{c}_i)_{1 \leq i \leq n}$  that  $\text{size}(C) = \text{Poly}(n) \cdot \text{size}(B)$ .

We call  $\delta$ -LLL' the successive application of the above algorithm based on BKZ'2 and LLL with parameter  $\delta$ . We are to prove that the number of loop iterations performed by  $\delta$ -LLL is  $\text{Poly}(n)$ .

**Theorem 3.** *Given as inputs a basis  $B \in \mathbb{Z}^{n \times n}$  of a lattice  $L$  and  $\delta < 1$ , algorithm  $\delta$ -LLL' algorithm outputs a  $\delta$ -LLL-reduced basis of  $L$  within  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(B))$  bit operations.*

*Proof.* With the same notations as above, it suffices to prove that given  $(\mathbf{c}_i)_{i \leq n}$  as input, algorithm  $\delta$ -LLL terminates within  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(C))$  bit operations. Let  $(\mathbf{c}'_i)_{i \leq n}$  be the output basis. As size-reductions can be performed in time  $\text{Poly}(n) \cdot \tilde{O}(\text{size}(C))$ , it suffices to show that the number of loop iterations of  $\delta$ -LLL given  $(\mathbf{c}_i)_{i \leq n}$  as input is  $\text{Poly}(n)$ . To do this, it suffices to bound  $\frac{\Pi((\mathbf{c}_i)_{i \leq n})}{\Pi((\mathbf{c}'_i)_{i \leq n})}$  by  $2^{\text{Poly}(n)}$ .

First of all, we have  $\lambda_1(L[(\mathbf{c}_j^{(i)})_{i \leq j \leq n}]) \leq \lambda_i(L)$ , for all  $i \leq n$ . Indeed, let  $\mathbf{v}_1, \dots, \mathbf{v}_i \in L$  be linearly independent such that  $\max_{j \leq i} \|\mathbf{v}_j\| \leq \lambda_i(L)$ ; at least one of them, say  $\mathbf{v}_1$ , remains non-zero when projected orthogonally to  $\text{Span}(\mathbf{c}_j)_{j < i}$ . We thus have  $\lambda_1(L[(\mathbf{c}_j^{(i)})_{i \leq j \leq n}]) \leq \|\mathbf{v}_1\| \leq \lambda_i(L)$ . Now, using (13), we obtain:

$$\Pi((\mathbf{c}_i)_{i \leq n}) = \prod_{i=1}^n \|\mathbf{c}_i^*\|^{2(n-i+1)} \leq 2^{O(n^3)} \prod_{i=1}^n \lambda_i(L)^{2(n-i+1)}.$$

On the other hand, we have (see [16, (1.7)])  $\lambda_i(L) \leq \max_{j \leq i} \|\mathbf{c}'_j\| \leq (\frac{1}{\sqrt{\delta-1/4}})^{i-1} \|\mathbf{c}'_i^*\|$ , for all  $i \leq n$ . As a consequence, we have  $\Pi((\mathbf{c}'_i)_{i \leq n}) \geq 2^{-O(n^3)} \cdot \prod_{i=1}^n \lambda_i(L)^{2(n-i+1)}$ . This completes the proof.  $\square$

**Acknowledgments** We thank N. Gama and P. Nguyen for explaining to us their bound on the number of tours of the original BKZ algorithm. We also thank C.-P. Schnorr for helpful discussions. The authors were partly supported by the LaRedA ANR grant and an ARC Discovery Grant DP110100628.

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## A Bounding the number of tours in the original BKZ algorithm

A bound  $(n\beta)^n$  is claimed in [9]. The authors kindly explained to us how to prove a similar upper bound. We give the proof, for the sake of completeness.

First, note that during the execution of BKZ (Algorithm 1), the basis  $(\mathbf{b}_i^{(k)})_{k \leq i \leq \min(k+\beta-1, n)}$  given as input to the SVP solver is always LLL-reduced. Now, we *modify* the call to LLL following the call to the

SVP, as follows. If the SVP solver did not find a sufficiently short vector (i.e.,  $\delta \cdot \|\mathbf{b}_k^*\| \leq \|\mathbf{b}\|$  in Algorithm 1), then we proceed as in Algorithm 1. Otherwise, we first call LLL on  $\mathbf{b}, \mathbf{b}^{(k)}, \mathbf{b}_k^{(k)}, \dots, \mathbf{b}_{\min(k+\beta-1, n)}^{(k)}$  to remove the linear dependency, we apply the appropriate transformation matrix to  $\mathbf{b}_1, \dots, \mathbf{b}_n$ , and then we call LLL again on the vectors  $\mathbf{b}_1, \dots, \mathbf{b}_{\min(k+\beta, n)}$ .

Suppose the call to the SVP solver is successful. The modification above ensures that the projected basis  $\mathbf{b}_k^{(k)}, \dots, \mathbf{b}_{\min(k+\beta-1, n)}^{(k)}$  is reduced both before the call to the SVP solver and before the second call to LLL. Furthermore, by a standard property of LLL, the vector found by the SVP solver is the first vector of the basis before the second call to LLL. Overall, the effect on the  $\|\mathbf{b}_i^*\|$ 's of a call to the SVP solver and the first call to LLL is as follows:

- $\|\mathbf{b}_k^*\|$  decreases by a factor  $\leq \delta$ ,
- $\|\mathbf{b}_j^*\|$  remains constant if  $j \notin [k, \min(k+\beta-1, n)]$ ,
- $\|\mathbf{b}_j^*\|$  does not increase by a factor  $\geq 2^\beta$  if  $j \in [k+1, \min(k+\beta-1, n)]$  (because the former and new  $\|\mathbf{b}_j^*\|$ 's approximate the successive minima of  $L[(\mathbf{b}_i^{(k)})_{k \leq i \leq \min(k+\beta-1, n)}]$  (see, e.g., [6, Th. 18.12.1]).

To conclude, consider the quantity  $\prod_{i \leq n} \|\mathbf{b}_i^*\|^{[\frac{3\beta}{\log(1/\delta)}]^{n-i+1}}$ . From the above, it always decreases by a factor  $\leq \frac{1}{2}$  during a successful call to the SVP solver followed by the first call to LLL. It also always decreases during a LLL swap (see [16]). Finally, it never increases during the execution of BKZ. As the input and output bases of BKZ are LLL-reduced, it always belongs to the interval

$$\left[ \prod_{i \leq n} (\lambda_i 2^{-n})^{[\frac{3\beta}{\log(1/\delta)}]^{n-i+1}}, \prod_{i \leq n} (\lambda_i 2^n)^{[\frac{3\beta}{\log(1/\delta)}]^{n-i+1}} \right],$$

where the  $\lambda_i$ 's are the successive minima of the lattice under scope. This implies that the number of calls to the SVP oracle is  $O(\beta)^n$ .  $\square$

## B Improving the constant $\frac{3}{2}$ in Theorems 1 and 2

Theorem 1 asserts the following bound on the output of the modified BKZ algorithm:

$$\|\mathbf{c}_1\| \leq 2(\gamma_\beta)^{\frac{n-1}{2(\beta-1)} + \frac{3}{2}} \cdot (\det L)^{\frac{1}{n}}.$$

We show that there exists a universal (and efficiently computable) constant  $K$  such that for sufficiently large  $\beta$  and  $n \geq \beta$ , we have:

$$\|\mathbf{c}_1\| \leq K \cdot \beta^{\frac{n-1}{2(\beta-1)} + \frac{1-\ln 2}{2}} \cdot (\det L)^{\frac{1}{n}}.$$

The base  $\beta$  of the power could be replaced by  $\alpha\beta$  ( $\alpha < 1$ ) provided that  $\gamma_\beta < \alpha\beta$  holds for sufficiently large  $\beta$ .

*Proof.* In the present work, we only used the facts that  $\gamma_n$  is an upper bound on the Hermite constant and that  $\gamma_n \leq \gamma_{n+1}$ . Since  $\gamma_n \leq n$ , the proofs also hold with  $\gamma_n$  replaced by  $n$ .

Let  $y_1 = 0$  and  $y_{i+1} = \frac{1}{i} \sum_{j=1}^i y_j + \frac{i+1}{2i} \log(i+1)$  for  $i \geq 2$ . We have:

$$\begin{aligned} y_{i+1} - y_i &= \frac{1}{i} \sum_{j=1}^{i-1} y_j + \frac{i+1}{2i} \log(i+1) - \frac{i-1}{i} y_i \\ &= \frac{1}{i} \sum_{j=1}^{i-1} y_j + \frac{i+1}{2i} \log(i+1) - \frac{i-1}{i} \left( \frac{1}{i-1} \sum_{j=1}^{i-1} y_j + \frac{i}{2(i-1)} \log i \right) \\ &= \frac{i+1}{2i} \log(i+1) - \frac{1}{2} \log i \\ &= \frac{1}{2} (\log(i+1) - \log i) + \frac{1}{2i} \log(i+1). \end{aligned}$$

Let  $(\mathbf{b}_1, \dots, \mathbf{b}_n)$  be an BKZ-reduced basis and  $\bar{x}_i = \log \|\mathbf{b}_i^*\|$ . Wlog, we may assume that  $\|\mathbf{b}_n^*\| = 1$ . Under the SMA, we have  $\bar{x}_i = y_{n-i+1}$  for all  $i \in [n - \beta + 1, n]$ . We proceed as for Lemma 5: we compute upper and lower bounds for any fixed point  $(\bar{x}_i)_i$  of the dynamical system  $\mathbf{x} \leftarrow A \cdot \mathbf{x} + \mathbf{g}$ . It then suffices to combine them, as in the proof of Theorem 2.

**A lower bound on  $\bar{x}_i$ .** We prove by induction on  $i = n - \beta + 1, \dots, 1$  that we have:

$$\forall i \in [1, n - \beta + 1], \quad \bar{x}_i - \bar{x}_{n-\beta+1} \geq \left( \frac{n-i}{\beta-1} - 1 \right) \log \beta.$$

This trivially holds for  $i = n - \beta + 1$ . As in the proof of Lemma 5, we now consider  $i \in [n - 2(\beta - 1), n - \beta]$ , for which we have (Eq. (6)):

$$\bar{x}_i = \frac{\beta}{2(\beta-1)} \log \beta + \frac{1}{\beta-1} \left( \sum_{j=i+1}^{n-\beta} \bar{x}_j + \sum_{j=n-\beta+1}^{i+\beta-1} \bar{x}_j \right). \quad (14)$$

The following sequences are concave:

- $(y_k)_{1 \leq k \leq \beta}$ : It suffices to show that  $y_{k+1} - y_k = \frac{1}{2}(\log(k+1) - \log k) + \frac{1}{2k} \log(k+1)$  is non-increasing; For  $k \geq 3$ , both  $(\log(k+1) - \log k)$  and  $\frac{1}{2k} \log(k+1)$  are non-increasing; It can be checked by hand that  $y_4 - y_3 \leq y_3 - y_2 \leq y_2 - y_1$ .
- $(y_{\beta-k+1})_{1 \leq k \leq \beta}$ : By symmetry.
- $\left( \frac{1}{k} \sum_{j=1}^k y_{\beta-j+1} \right)_{1 \leq k \leq \beta}$ : See [41, Le. 5] for example.
- $(z_k)_{k \in [1, \beta]}$  defined by  $z_k = \frac{1}{k} \sum_{j=n-\beta+1}^{n-\beta+k} \bar{x}_j$ : This is a simple translation of indices.

Since  $(z_k)_k$  is concave, we obtain

$$z_k \geq z_1 + (z_\beta - z_1) \frac{k-1}{\beta-1} = \bar{x}_{n-\beta+1} - \frac{\log \beta}{2} \cdot \frac{k-1}{\beta-1} = \bar{x}_{n-\beta+1} - \frac{\log \beta}{k} \cdot \sum_{j=1}^k \frac{j-1}{\beta-1}. \quad (15)$$

Using the previous equation with  $k = i + 2\beta - n - 1$  gives:

$$\frac{1}{i + 2\beta - n - 1} \sum_{j=n-\beta+1}^{i+\beta-1} \bar{x}_j \geq \bar{x}_{n-\beta+1} + \frac{\log \beta}{i + 2\beta - n - 1} \sum_{j=n-\beta+1}^{i+\beta-1} \left( \frac{n-j}{\beta-1} - 1 \right). \quad (16)$$

Using the induction hypothesis (on each  $\bar{x}_j$  for  $j \in [n - \beta + 1, i + \beta - 1]$ ), we also have:

$$\frac{1}{n - \beta - i} \sum_{j=i+1}^{n-\beta} \bar{x}_j \geq \bar{x}_{n-\beta+1} + \frac{\log \beta}{n - \beta - i} \sum_{j=i+1}^{n-\beta} \left( \frac{n-j}{\beta-1} - 1 \right). \quad (17)$$

Then, we plug (16) and (17) into (14). The end of the proof is similar to that of Lemma 5 (where the constant  $\frac{3}{2}$  is replaced by 1).

**An Upper bound on  $\bar{x}_i$ .** Starting from the equation  $y_{i+1} - y_i = \frac{1}{2}(\log(i+1) - \log i) + \frac{1}{2i} \log(i+1)$ , we obtain:

$$\begin{aligned} y_i &= \frac{1}{2} \log i + \sum_{j=2}^i \frac{\log j}{2(j-1)} \\ &= \frac{1}{2} \log i + \sum_{j=2}^i \left( \frac{\log j}{2j} + \frac{\log j}{2j(j-1)} \right) \\ &\leq \frac{1}{2} \log i + \int_{x=1}^i \frac{\log x}{2x} dx + C_1 \\ &\leq \frac{\ln 2}{4} \log^2 i + \frac{1}{2} \log i + C_2, \end{aligned}$$

