Revisiting the BGE Attack on a White-Box AES Implementation

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Abstract. White-box cryptography aims to protect the secret key of a cipher in an environment in which an adversary has full access to the implementation of the cipher and its execution environment. In 2002, Chow, Eisen, Johnson and van Oorschot proposed a white-box implementation of AES. In 2004, Billet, Gilbert and Ech-Chatbi presented an efficient attack (referred to as the BGE attack) on this implementation, extracting its embedded AES key with a work factor of 2^{30}. In 2012, Tolhuizen presented an improvement of the most time-consuming phase of the BGE attack. This paper presents several improvements to the other phases of the BGE attack. The paper shows that the overall work factor of the BGE attack is reduced to 2^{22} when all improvements are implemented. In 2010, Karroumi presented a white-box AES implementation that is designed to withstand the BGE attack. This paper shows that the implementations of Karroumi and Chow *et al.* are the same. As a result, Karroumi's white-box AES implementation is vulnerable to the attack it was designed to resist.

Keywords. White-box cryptography, dual cipher, AES, cryptanalysis.

1 Introduction

In 2002, Chow *et al.* introduced the concept of white-box cryptography by presenting a white-box implementation of AES [5]. White-box cryptography aims to protect the confidentiality of the secret key of a cipher in a white-box environment. In a white-box environment, an adversary is assumed to have full access to the implementation of the cipher and its execution environment. For example, in a white-box environment the adversary can use tools such as decompilers and debuggers to reverse engineer the implementation of the cipher, and to read and alter values of intermediate results of the cipher during its execution. A typical example of an application in which a cipher is implemented in a white-box environment is a content protection system in which a client is executed on the main processor of a PC, a tablet, a mobile device, or a set-top box.

In 2004, Billet *et al.* [3] presented an attack on the white-box AES implementation of Chow *et al.* with a work factor of 2^{30}. In 2012, Tolhuizen [10] proposed an improvement to the most time-consuming phase of the BGE attack, reducing the work factor of this phase to 2^{19}. If the improvement of Tolhuizen is implemented, then the work factor of the BGE attack is dominated by the other phases of the BGE attack, and equals 2^{29}. This paper presents several improvements to the other phases of the BGE attack, and shows that the work factor of the BGE attack is reduced to 2^{22} when Tolhuizen's improvement and the improvements presented in this paper are implemented.

The BGE attack triggered the design of new white-box AES implementations, such as the ones proposed by Xiao and Lai in 2009 [11] and by Karroumi in 2010 [6]. In [9], De Mulder *et al.* presented a cryptanalysis of Xiao and Lai's white-box AES implementation, showing that this implementation is insecure.

In [6], Karroumi uses the concept of dual ciphers [1,2,4] and the white-box techniques of Chow *et al.* to design a new white-box AES implementation. In [6], Karroumi argues that the additional secrecy introduced by the dual cipher increases the work factor of the BGE attack to 2^{93}. This paper shows that the white-box AES implementations of Chow *et al.* and Karroumi are the same. As a consequence, Karroumi's white-box AES implementation is vulnerable to the attack it was designed to resist.

Organization of this paper. Section 2 describes aspects of AES, the white-box AES implementation of Chow *et al.*, and the BGE attack that are relevant to this paper. The improvements of the BGE attack and their work factor are presenteded in Sect. 3. The cryptanalysis of Karroumi's white-box AES implementation is presented in Sect. 4, and Sect. 5 contains the conclusions.

2 Preliminaries

2.1 AES

AES [7] is a key-iterated block cipher operating on 16-byte blocks. This paper assumes throughout and without loss of generality that the AES variant in [7] with a 128-bit key is used. AES consists of 10 rounds and has 11 round keys which are derived from the secret AES key using the AES key scheduling algorithm. Each AES round and the operations within a round update a 16-byte state; the initial and final state are the AES plaintext and ciphertext, respectively. AES can be described elegantly by interpreting the bytes of the state as elements of the finite field $\mathbf{F}_{256}$, and by defining AES operations as mappings over this field (see also [7]). As the final round is not relevant for the discussion in this paper, only the first 9 rounds are considered in the following text. Each round r with $1 \leq r \leq 9$ comprises four operations:

`ShiftRows:` a permutation on the indices of the 16 bytes of the state;

`AddRoundKey:` a byte-wise addition of 16 round key bytes $k_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ and the 16-byte state;

`SubBytes`: applies the AES S-box, denoted by S, to every byte of the 16-byte state;

`MixColumns`: a linear operation on $\mathbf{F}_{256}^{16}$. The `MixColumns` operation is represented by a 4×4 matrix `MC` over $\mathbf{F}_{256}$; the linear operation applies 4 instances of this matrix in parallel to the 16-byte state. The 16 coefficients of `MC` are denoted by mc_{ij} for $0 \leq i, j \leq 3$.

In literature, the boundaries between rounds are defined in different ways. In this paper, `ShiftRows` and `MixColumns` are the first and final operations within a round, respectively. That is, the order of the operations within a round is identical to the order used to describe the operations above. For details about AES, refer to [7].

AES subrounds. The mappings in the following definition will be used to describe the white-box AES implementations and the attacks on the implementations. In the following text, the finite field representation as defined in [7] is referred to as the AES polynomial representation, and $\oplus$ and $\otimes$ denote the addition and multiplication operations in this representation, respectively.

Definition 1. *Let $x_i, y_i \in \mathbf{F}_{256}$ for $0 \leq i \leq 3$ be represented using the AES polynomial representation. The mapping $AES^{(r,j)} : \mathbf{F}_{256}^4 \rightarrow \mathbf{F}_{256}^4$ for $1 \leq r \leq 9$ and $0 \leq j \leq 3$, called an AES subround, is defined by $(y_0, y_1, y_2, y_3) = AES^{(r,j)}(x_0, x_1, x_2, x_3)$ with*

$$y_i = mc_{i0} \otimes S\big(x_0 \oplus k_0^{(r,j)}\big) \oplus mc_{i1} \otimes S\big(x_1 \oplus k_1^{(r,j)}\big) \oplus \\ mc_{i2} \otimes S\big(x_2 \oplus k_2^{(r,j)}\big) \oplus mc_{i3} \otimes S\big(x_3 \oplus k_3^{(r,j)}\big) \ ,$$

for $0 \leq i \leq 3$.

Observe that an AES subround consists of the key additions, the S-box operations and the `MixColumns` operations in an AES round that are associated with a single `MixColumns` matrix operation, and that one AES round comprises four AES subrounds. The subrounds are indexed by j in Def. 1, and this paper assumes throughout that the four subrounds in a round are numbered left to right. The bytes $k_i^{(r,j)}$ for $0 \leq i, j \leq 3$ are the 16 bytes of the AES round key of round r.

2.2 Chow *et al.*'s White-Box AES Implementation and the BGE Attack

This section describes aspects of Chow *et al.*'s white-box AES implementation [5] and the BGE attack [3] that are relevant to this paper. For an in-depth tutorial on how Chow *et al.*'s white-box AES implementation is constructed, refer to [8].

Encoded AES subrounds. In the following text, $P_i^{(r,j)}$ and $Q_i^{(r,j)}$ for $0 \leq i \leq 3$ denote bijective mappings on the vector space $\mathbf{F}_2^8$, referred to as *encodings* in white-box cryptography. The encodings are generated randomly and are kept secret in a white-box implementation (for details about encodings, refer to [5,8]). A vector of four mappings, such as $\big(P_0^{(r,j)}, P_1^{(r,j)}, P_2^{(r,j)}, P_3^{(r,j)}\big)$ or $\big(Q_0^{(r,j)}, Q_1^{(r,j)}, Q_2^{(r,j)}, Q_3^{(r,j)}\big)$, denotes the mapping defined by applying the i-th element of the vector to its i-th input byte for $0 \leq i \leq 3$. For $a \in \mathbf{F}_2^n$ the mapping $\oplus_a : \mathbf{F}_2^n \to \mathbf{F}_2^n$ denotes the addition with a. With slight abuse of notation, an input to $AES^{(r,j)}$ is considered to be an element of $\mathbf{F}_{256}^4$ using the AES polynomial representation in the following definition, and an output of $AES^{(r,j)}$ is considered to be an element of $(\mathbf{F}_2^8)^4$. Further, in the following definition the permutations $\Pi_i^{(r,j)} : (\mathbf{F}_2^8)^4 \to (\mathbf{F}_2^8)^4$ $(i = 1, 2)$ for $1 \leq r \leq 9$ and $0 \leq j \leq 3$ permute the order of the input bytes and output bytes of an AES subround, respectively, and $\pi^{(r)} : \{0,1,2,3\} \to \{0,1,2,3\}$ for $1 \leq r \leq 9$ permutes the order of the four AES subrounds within an AES round. These permutations are randomly chosen and kept secret in a white-box implementation.

Definition 2. *The mapping* $AES_{enc}^{(r,j)} : (\mathbf{F}_2^8)^4 \to (\mathbf{F}_2^8)^4$ *for* $1 \leq r \leq 9$ *and* $0 \leq j \leq 3$*, called an encoded AES subround, is defined by*

$$AES_{enc}^{(r,j)} = (Q_0^{(r,j)}, Q_1^{(r,j)}, Q_2^{(r,j)}, Q_3^{(r,j)}) \circ \overline{AES}^{(r,j)} \circ (P_0^{(r,j)}, P_1^{(r,j)}, P_2^{(r,j)}, P_3^{(r,j)}) \ ,$$

where the mapping $\overline{AES}^{(r,j)}$ *is defined by*

$$\Pi_2^{(r,j)} \circ AES^{(r,\pi^{(r)}(j))} \circ \Pi_1^{(r,j)} = \mathtt{MC}^{(r,j)} \circ (S,S,S,S) \circ \oplus_{[\bar{k}_i^{(r,j)}]_{0 \leq i \leq 3}} \ ,$$

$$\begin{array}{lll} \textit{with} & [\bar{k}_i^{(r,j)}]_{0 \leq i \leq 3} = (\Pi_1^{(r,j)})^{-1}\big([k_i^{(r,\pi^{(r)}(j))}]_{0 \leq i \leq 3}\big) \\ \textit{and} & \mathtt{MC}^{(r,j)} \quad\quad = \Pi_2^{(r,j)} \circ \mathtt{MC} \circ \Pi_1^{(r,j)} \ . \end{array}$$

In [3], Billet *et al.* showed in their cryptanalysis of Chow *et al.*'s white-box AES implementation [5] that for rounds $1 \leq r \leq 9$, it is possible to compose certain white-box tables in such a way that an adversary has access to the encoded AES subrounds of each round.

Although the permutations in Def. 2 are not explicitly specified in [5], they are implicitly included in the 32-bit and 128-bit wide $\mathbf{F}_2$-linear encodings (referred to as *mixing bijections* in [5]); in other words, in a practical white-box AES implementation each wide mixing bijection in [5] is cancelled up to an unknown permutation on the indices of the involved bytes. This assumption was also made in the BGE attack [3].

In Chow *et al.*'s white-box AES implementation, the output encodings $Q_i^{(r-1,j)}$ and input encodings $P_i^{(r,j)}$ for $0 \leq i, j \leq 3$ of successive AES rounds are pairwise annihilating to maintain the functionality of AES. The data-flow of the white-box implementation between successive AES rounds $r-1$ and r determines the 16 pairs of output/input encodings which are pairwise annihilating.

BGE attack. As indicated above, the adversary has access to the encoded AES subrounds $AES_{\text{enc}}^{(r,j)}$ for $1 \leq r \leq 9$ and $0 \leq j \leq 3$. Next, the BGE attack [3] comprises the following three phases: Phases 1 and 2 retrieve the bytes of the AES round key associated with round r for some r with $2 \leq r \leq 9$, and Phase 3 determines the correct order of the round key bytes and extracts the AES key.

Phase 1 retrieves the encodings $Q_i^{(r,j)}$ $(0 \leq i \leq 3)$ up to an affine part for each encoded AES subround j $(0 \leq j \leq 3)$. Because of the pairwise annihilating property of the encodings between successive rounds, the encodings $P_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ can be retrieved up to an affine part by applying the same technique to the encoded AES subrounds of the previous round.

Phase 2 assumes that all encodings of an encoded AES round are affine mappings (as the other parts have been retrieved in Phase 1). Phase 2 first retrieves the affine encodings $Q_i^{(r,j)}$ $(0 \leq i \leq 3)$ for each encoded AES subround j $(0 \leq j \leq 3)$. During this process, the key-dependent affine mappings $\widetilde{P}_i^{(r,j)}(x) = P_i^{(r,j)}(x) \oplus \bar{k}_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ are obtained as well. As in Phase 1, the affine encodings $P_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ are retrieved by applying the same technique to the encoded AES subrounds of the previous round. This enables the adversary to compute the round key bytes $\bar{k}_i^{(r,j)} = \widetilde{P}_i^{(r,j)}(0) \oplus P_i^{(r,j)}(0)$ for $0 \leq i, j \leq 3$.

Phase 3 retrieves the round key bytes of round $r+1$ as discussed above in Phases 1 and 2, and uses the fact that the round key bytes of rounds r and $r+1$ are related to each other via both the data-flow of the white-box implementation and the AES key scheduling algorithm to retrieve the AES round key. Finally, assuming that the AES variant with a 128-bit key is used, the adversary can use the property of the AES key scheduling algorithm that the AES key can be computed if one of the round keys is known.

Work factor of the BGE attack. In [3], the authors claim that the work factor associated with the three phases of the BGE attack is around 2^{30}. As a result, the white-box AES implementation of Chow *et al.* is insecure. For detailed information about the BGE attack, refer to [3].

3 Reducing the Work Factor of the BGE Attack

In 2012, Tolhuizen [10] presented an improvement of the first phase of the BGE attack. If the improvement of Tolhuizen is implemented, then the work factor of the BGE attack is dominated by the second phase. In this section we present several improvements to the other phases of the BGE attack:

1. A method to reduce the expected work factor of Phase 2 of the BGE attack;
2. An efficient method to retrieve the round key bytes of round $r+1$ after the round key bytes of round r are extracted;
3. An efficient method to determine the correct order of the round key bytes, given the round key bytes of two consecutive rounds.

As the work factors of Phases 1 and 2 of the BGE attack are reduced by Tolhuizen's improvement and the first improvement above, respectively, it is now important to have an efficient method for Phase 3 of the BGE attack as well, as otherwise the work factor of this phase could dominate the overall work factor. The second and third improvements above comprise such a method for Phase 3. It will be shown that Tolhuizen's improvement to Phase 1 of the BGE attack and the above improvements to the other phases reduce the work factor of the BGE attack to 2^{22}. The improved BGE attack comprises the following four (instead of three) phases:

Phases 1 and 2: retrieve the round key bytes $\bar{k}_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ associated with round r $(2 \leq r \leq 8)$.

The first two phases are the ones of the BGE attack [3] using Tolhuizen's improvement, and retrieve the round key bytes $\bar{k}_i^{(r,j)}$ for $0 \leq i, j \leq 3$ associated with round r for some r with $2 \leq r \leq 8$.

Work factor of Phase 1. Tolhuizen's improvement [10] reduces the work factor of Phase 1 to around $2 \cdot 4 \cdot 4 \cdot (35 \cdot 2^8) < 2^{19}$. The first three factors (i.e., $2 \cdot 4 \cdot 4$) denote the number of encodings involved in Phase 1, i.e., four encodings for each of the four subrounds for each of the two consecutive rounds. The fourth factor (i.e., $35 \cdot 2^8$) denotes the work factor required to retrieve one encoding up to an affine part using Tolhuizen's method.

Work factor of Phase 2. The expected work factor F of the second phase as described in [3] equals approximately $2 \cdot 4 \cdot 4 \cdot 2^{15} \cdot 2^8 = 2^{28}$, and is measured in the number of evaluations of mappings on $\mathbf{F}_2^8$. The evaluations are required to determine if a mapping on $\mathbf{F}_2^8$ is affine. The mappings f that need to be tested for being affine are listed in [3, Proposition 3]. Each f is associated with a secret encoding $P_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ of a round r. As Phase 2 needs to be applied to two consecutive rounds, this involves a total of $2 \cdot 4 \cdot 4$ mappings (which corresponds to the first three factors in F). The mappings f are permutations on $\mathbf{F}_2^8$ and have the structure

$$f = S^{-1} \circ Q_{(c,d)}^{-1} \circ Q \circ S \circ \oplus_k \circ P \ , \tag{1}$$

where S denotes the AES S-box mapping (viewed as a permutation on $\mathbf{F}_2^8$), k denotes a key byte, P and Q denote bijective affine mappings on $\mathbf{F}_2^8$, and $Q_{(c,d)}^{-1}$ denotes a bijective affine mapping on $\mathbf{F}_2^8$ for each pair $(c, d) \in \mathbf{F}_{256}^2$. Furthermore, $Q_{(c,d)}^{-1} = Q^{-1}$ for one specific pair $(c, d) \in \mathbf{F}_{256}^2$. An affine-test is performed for each possible pair $(c, d) \in \mathbf{F}_{256}^2$ until the corresponding mapping f is affine. The expected number of pairs for which the test is performed equals approximately 2^{15}, which is the fourth factor in F. The fifth factor in F, i.e., 2^8, is associated with the test used in [3].

Instead of the test used in [3], which requires 2^n evaluations to determine if $f : \mathbf{F}_2^n \to \mathbf{F}_2^n$ is affine, we use the following algorithm to reduce the expected

number of evaluations. If e_i $(1 \leq i \leq n)$ denotes the i-th unit vector in $\mathbf{F}_2^n$, then the algorithm first verifies if the equation

$$f(e_1 \oplus e_2) = f(0) \oplus f(e_1) \oplus f(e_2) \tag{2}$$

holds true. If this equation does not hold true, then the algorithm terminates with "f is not affine". Observe that the algorithm requires 4 evaluations of f in this case. If Eq. 2 holds true, then the algorithm applies the method used in [3] to determine if f is affine (with the only difference that f is not re-evaluated for the four input values $0, e_1, e_2$ and $e_1 \oplus e_2$). In this case 2^n evaluations of f are required.

To show the correctness of this algorithm, it is sufficient to show that an affine mapping always satisfies Eq. 2. If f is affine, then $f(x) = A(x) \oplus b$ for some $A \in \mathbf{F}_2^{n \times n}$ and some $b \in \mathbf{F}_2^n$. It follows that $f(0) \oplus f(e_1) \oplus f(e_2) = b \oplus A(e_1) \oplus b \oplus A(e_2) \oplus b = A(e_1 \oplus e_2) \oplus b = f(e_1 \oplus e_2)$.

Lemma 1. *If f is a random permutation on $\mathbf{F}_2^n$ and if $E(n)$ denotes the expected number of evaluations of f required by the algorithm described above, then $E(n) < 5$.*

Proof. Let $p(n)$ denote the probability that Eq. 2 holds true for a random permutation. To determine $p(n)$, note that $f(0), f(e_1), f(e_2)$ and $f(e_1 \oplus e_2)$ are four distinct elements of $\mathbf{F}_2^n$ if f is a permutation. From this it follows that $f(0) \oplus f(e_1) \oplus f(e_2)$ and $f(e_1 \oplus e_2)$ are both elements of $\mathbf{F}_2^n \setminus \{f(0), f(e_1), f(e_2)\}$. Further, as f is a random permutation, $f(e_1 \oplus e_2)$ is a random element of this set. Hence, $p(n) = 1/(2^n - 3)$ and $E(n) = 4(1-p) + 2^n p = 4 + (2^n - 4)/(2^n - 3) < 5$. $\square$

Under the assumption that f in Eq. 1 behaves as a random permutation on $\mathbf{F}_2^8$ for every incorrect guess for (c, d), the expected work factor of the affine-test is reduced from 2^8 to approximately 5 evaluations if f is not affine and the work factor is 2^8 if f is affine. This implies that the fifth factor in F is reduced to approximately 5. That is, the expected work factor of Phase 2 of the BGE attack is now approximately $2 \cdot 4 \cdot 4 \cdot 2^{15} \cdot 5 \approx 2^{22}$.

Phase 3: retrieve the round key bytes $\bar{k}_i^{(r+1,j)}$ $(0 \leq i, j \leq 3)$ associated with round $r + 1$.

As mentioned in the description of the BGE attack in Sect. 2.2, [3] obtains the round key bytes of round $r + 1$ by applying Phases 1 and 2 to round $r + 1$ as well. Here, we present a more efficient method based on the affine-test described above. The method comprises the following three steps for each encoded AES subround j $(0 \leq j \leq 3)$ associated with round $r + 1$ to retrieve the round key bytes $\bar{k}_i^{(r+1,j)}$ $(0 \leq i, j \leq 3)$:

Step 1 applies Phase 1 (using Tolhuizen's improvement) to round $r + 1$ in order to retrieve the encodings $Q_i^{(r+1,j)}$ $(0 \leq i \leq 3)$ up to an affine part.

Step 2 first removes the non-affine part of the output encodings as recovered in Step 1 from the encoded AES subround. Next, Step 2 removes the input encodings $P_i^{(r+1,j)}$ $(0 \leq i \leq 3)$ from the encoded AES subround (observe that the inverses of these input encodings were obtained in Phases 1 and 2). The resulting mapping $f^{(r+1,j)} : (\mathbf{F}_2^8)^4 \rightarrow (\mathbf{F}_2^8)^4$ is given by

$$f^{(r+1,j)} = \big(\hat{Q}_0^{(r+1,j)}, \hat{Q}_1^{(r+1,j)}, \hat{Q}_2^{(r+1,j)}, \hat{Q}_3^{(r+1,j)}\big) \circ \overline{AES}^{(r+1,j)} \ ,$$

where $\hat{Q}_i^{(r+1,j)}$ $(0 \leq i \leq 3)$ are affine output encodings.

Step 3 retrieves the round key bytes $\bar{k}_i^{(r+1,j)}$ $(0 \leq i \leq 3)$. To find a key byte, say $\bar{k}_0^{(r+1,j)}$, fix the other three input bytes to $f^{(r+1,j)}$ (e.g., to zero), search over all possible 2^8 values of the key byte k and verify if

$$g_k(x) = f^{(r+1,j)}\big(k \oplus S^{-1}(x), 0, 0, 0\big)$$

is affine using the test described above. In case $g_k(x)$ is affine, then $\bar{k}_0^{(r+1,j)} = k$. Repeat this for $\bar{k}_i^{(r+1,j)}$ $(i = 1, 2, 3)$.

The correctness of Step 3 uses the fact that the mapping $S\big(c \oplus S^{-1}(x)\big)$ is non-affine for all non-zero values of c. This has already been proven in [3, proof of Proposition 3].

Work factor of Phase 3. The work factor of Step 3 equals $4 \cdot 4 \cdot 2^7 \cdot 5 \approx 2^{13}$, where $4 \cdot 4$ denotes the number of round key bytes, 2^7 denotes the expected number of key values for which the affine-test is performed and 5 denotes the expected number of evaluations of the affine-test if g_k is not affine. The work factor of Step 1 is $4 \cdot 4 \cdot (35 \cdot 2^8) < 2^{18}$, where the first two factors denote the number of output encodings involved in Step 1. As a result, the work factor of Phase 3 is dominated by Step 1 and is less than 2^{18}.

Phase 4: determine the correct order of the round key bytes and extract the secret AES key.

After Phases 1 - 3, the values of the round key bytes of two consecutive rounds r and $r+1$ are known. However, for each round, the order of the round key bytes of each subround and the order of the four subrounds are still unknown. Notice that there are still $(4!)^5 \approx 2^{23}$ possibilities for the round key if only the bytes of that round key are considered. In [3], it is indicated how the correct order can be determined given the "shuffled" round key bytes of rounds r and $r+1$. However, [3] does not contain an explicit description of such a method. As the work factor of the first three phases equals 2^{22}, it is desirable to have a method to determine the correct order of the round key bytes with a work factor that is less than 2^{22}. Below we present such a method, comprising the following three steps:

Step 1 retrieves $\mathtt{MC}^{(r,j)}$ associated with each subround j $(0 \leq j \leq 3)$ of round r. Recall that the encodings $P_i^{(r,j)}$ and $Q_i^{(r,j)}$ $(0 \leq i, j \leq 3)$ were obtained in Phases

1 and 2. Together with the knowledge of the round key bytes $\bar{k}_i^{(r,j)}$ $(0 \leq i, j \leq 3)$, compute

$$\texttt{MC}^{(r,j)} = \left(Q_0^{(r,j)}, Q_1^{(r,j)}, Q_2^{(r,j)}, Q_3^{(r,j)}\right)^{-1} \circ AES_{enc}^{(r,j)} \circ \left(P_0^{(r,j)}, P_1^{(r,j)}, P_2^{(r,j)}, P_3^{(r,j)}\right)^{-1} \circ \oplus_{[\bar{k}_i^{(r,j)}]_{0 \leq i \leq 3}} \circ (S,S,S,S)^{-1} \ ,$$

for $j = 0, 1, 2, 3$.

Step 2 : for each $\texttt{MC}^{(r,j)}$ $(0 \leq j \leq 3)$, compute permutations $\Pi_1, \Pi_2 : (\mathbf{F}_2^8)^4 \rightarrow (\mathbf{F}_2^8)^4$ such that

$$\texttt{MC}^{(r,j)} = \Pi_2 \circ \texttt{MC} \circ \Pi_1 \ . \tag{3}$$

Let $(\Pi^{(1)}, \Pi^{(2)})$ denote the pairs of permutations for which MC remains invariant, i.e., $\texttt{MC} = \Pi^{(2)} \circ \texttt{MC} \circ \Pi^{(1)}$. It is easily verified that there are exactly four such pairs. The four permutations $\Pi^{(1)}$ are the four different circular shifts on the indices of a 4-byte vector, and $\Pi^{(2)} = (\Pi^{(1)})^{-1}$ for each of these pairs. This implies that there are also exactly four different pairs of permutations satisfying Eq. 3, given by

$$\left(\Pi^{(1)} \circ \Pi_1 \ , \ \Pi_2 \circ \Pi^{(2)}\right) \ . \tag{4}$$

As a consequence, finding one pair of permutation matrices satisfying Eq. 3 suffices to find the remaining three as well. Notice that exactly one of these four pairs of permutations equals the pair $(\Pi_1^{(r,j)}, \Pi_2^{(r,j)})$ of the encoded subround (see also Def. 2); in other words, one of these pairs is the correct pair.

After this, the order of the round key bytes associated with each subround is known up to an uncertainty of four possibilities (circular shifts). Observe that the order of the four subrounds is still unknown.

Step 3 determines the correct order of the round key bytes. For each of the possible orderings of the four AES subrounds of round r and the round key bytes within these subrounds (as determined in Step 2), obtain a candidate for the $(r+1)^{th}$ round key using the following two methods: (i) the AES key scheduling algorithm and (ii) the data-flow of the white-box AES implementation between the encoded subrounds of rounds r and $r+1$. Notice that once an order of the round key bytes of round r is selected, the order of the round key bytes of round $r+1$ can be determined using the corresponding pair of permutations of each of the subrounds of round r (see also Eq. 4) and the data-flow of the white-box implementation. With overwhelming probability, only one ordering of round key bytes of round r results in the same $(r+1)^{th}$ round key; this ordering corresponds to the correct round key of round r. Finally, use the property of the AES key scheduling algorithm that the AES key can be computed if one of the round keys is known.

Work factor of Phase 4. A naive approach yields an expected work factor of $(4!)^2 \approx 2^9$ for Step 2 by searching over all possible pairs of permutations. Step 2 reduces the number of possible orderings of the round key bytes from 2^{23} to $4^4 \cdot 4! < 2^{13}$ (where the first and second factor denote the possible orderings of

round key bytes within each subround and of the four subrounds, respectively), which equals the work factor of Step 3. As a result, the overall work factor of Phase 4 is dominated by the work factor of Step 3 and hence is less than 2^{13}.

Conclusion

The work factor of the improved BGE attack is dominated by the work factor of the second phase and equals 2^{22}.

Note that the uncertainty in the order of the round key bytes results in the need to retrieve key bytes of two consecutive rounds. This affects the work factor of the original BGE attack. In the improved BGE attack this is no longer the case, as the work factors of the phases that determine the correct order (i.e. Phases 3 and 4) are negligible compared to the work factor of Phase 2. A consequence of Tolhuizen's improvement is that the use of non-affine white-box encodings has a negligible impact on the overall work factor of the improved BGE attack.

4 Karroumi's White-Box AES Implementation

Karroumi's method to generate a white-box AES implementation [6] can be divided into two phases; *Phase 1* generates a dual AES cipher from a key-instantiated AES cipher, and *Phase 2* applies the white-box techniques presented by Chow *et al.* to the dual AES cipher. Below, aspects of these phases that are relevant to this paper are described.

Phase 1: Dual AES cipher

In this section we give a description of the set of dual AES ciphers used by Karroumi in [6]. First, we define a dual AES subround. The following notation is used: $m_\alpha : \mathbf{F}_{256} \to \mathbf{F}_{256}$ with $\alpha \in \mathbf{F}_{256}^*$ is defined by $m_\alpha(x) = \alpha \otimes x$, and $f_t : \mathbf{F}_{256} \to \mathbf{F}_{256}$ defined by $f_t(x) = x^{2^t}$ for $0 \leq t \leq 7$ are the automorphisms of $\mathbf{F}_{256}$ over $\mathbf{F}_2$. Further, $R_l : \mathbf{F}_{256} \to \mathbf{F}_{256}$ are the isomorphisms mapping elements in the AES polynomial representation to field elements in one of the polynomial representations of $\mathbf{F}_{256}$. There are 30 irreducible polynomials of degree 8 over $\mathbf{F}_2$, each one resulting in a unique polynomial representation of $\mathbf{F}_{256}$ (one of these representations being the AES polynomial representation), hence in total there are 30 distinct isomorphisms R_l ($1 \leq l \leq 30$). The addition and multiplication operations in the polynomial representation associated with R_l are denoted by $\oplus_l$ and $\otimes_l$, respectively ($\oplus_l$ and $\otimes_l$ being equal to $\oplus$ and $\otimes$ for exactly one value of l with $1 \leq l \leq 30$). Finally, the definition of a dual AES subround uses a set of mappings, denoted by $\mathcal{T}$, and defined by

$$\mathcal{T} = \{R_l \circ m_\alpha \circ f_t \mid 1 \leq l \leq 30, \alpha \in \mathbf{F}_{256}^* \text{ and } 0 \leq t \leq 7\} \ .$$

Observe that an element of $\mathcal{T}$ maps elements in the AES polynomial representation to elements in one of the 30 polynomial representations of $\mathbf{F}_{256}$.

Definition 3. *Let $\Delta_{r,j} \in \mathcal{T}$ with $\Delta_{r,j} = R_l \circ m_\alpha \circ f_t$ for some triple (l, α, t) with $1 \leq l \leq 30, \alpha \in \mathbf{F}_{256}^*$ and $0 \leq t \leq 7$, and let $\delta_{r,j} = R_l \circ f_t$. Further, let $v_i, w_i \in \mathbf{F}_{256}$ for $0 \leq i \leq 3$ be represented using the polynomial representation associated with R_l. The mapping $AES^{(r,j,\Delta_{r,j})} : \mathbf{F}_{256}^4 \to \mathbf{F}_{256}^4$ for $1 \leq r \leq 9$ and $0 \leq j \leq 3$, called a dual AES subround, is defined by $(w_0, w_1, w_2, w_3) = AES^{(r,j,\Delta_{r,j})}(v_0, v_1, v_2, v_3)$ with*

$$\begin{aligned} w_i = {} & \delta_{r,j}(mc_{i0}) \otimes_l \Delta_{r,j} \circ S \circ \Delta_{r,j}^{-1}\big(v_0 \oplus_l \Delta_{r,j}(k_0^{(r,j)})\big) \\ & \oplus_l \delta_{r,j}(mc_{i1}) \otimes_l \Delta_{r,j} \circ S \circ \Delta_{r,j}^{-1}\big(v_1 \oplus_l \Delta_{r,j}(k_1^{(r,j)})\big) \\ & \oplus_l \delta_{r,j}(mc_{i2}) \otimes_l \Delta_{r,j} \circ S \circ \Delta_{r,j}^{-1}\big(v_2 \oplus_l \Delta_{r,j}(k_2^{(r,j)})\big) \\ & \oplus_l \delta_{r,j}(mc_{i3}) \otimes_l \Delta_{r,j} \circ S \circ \Delta_{r,j}^{-1}\big(v_3 \oplus_l \Delta_{r,j}(k_3^{(r,j)})\big) \ , \end{aligned}$$

for $0 \leq i \leq 3$.

The following lemma presents a property that is required to show that a dual AES cipher maintains the functionality of AES. As the lemma is also used in the cryptanalysis in this paper, and as a formal proof of this property is omitted in [4] and [6], we include a proof as well.

Lemma 2. *If $\Delta_{r,j} \in \mathcal{T}$, then*

$$AES^{(r,j,\Delta_{r,j})} \circ (\Delta_{r,j}, \Delta_{r,j}, \Delta_{r,j}, \Delta_{r,j}) = (\Delta_{r,j}, \Delta_{r,j}, \Delta_{r,j}, \Delta_{r,j}) \circ AES^{(r,j)} \ ,$$

for $1 \leq r \leq 9$ and $0 \leq j \leq 3$.

Proof. Let x_i for $0 \leq i \leq 3$ be elements of $\mathbf{F}_{256}$ using the AES polynomial representation, let w_i for $0 \leq i \leq 3$ be elements of $\mathbf{F}_{256}$ using the polynomial representation associated with R_l (assuming that $\Delta_{r,j} = R_l \circ m_\alpha \circ f_t$), and let

$$(w_0, w_1, w_2, w_3) = \mathrm{AES}^{(r,j,\Delta_{r,j})} \circ (\Delta_{r,j}, \Delta_{r,j}, \Delta_{r,j}, \Delta_{r,j})(x_0, x_1, x_2, x_3) \ .$$

Substituting $v_i = \Delta_{r,j}(x_i)$ for $0 \leq i \leq 3$ in the equation in Def. 3 yields

$$w_i = \bigoplus_{z=0}^{3}{}_l \, \delta_{r,j}(mc_{iz}) \otimes_l \Delta_{r,j} \circ S \circ \Delta_{r,j}^{-1}\big(\Delta_{r,j}(x_z) \oplus_l \Delta_{r,j}(k_z^{(r,j)})\big) \ ,$$

for $0 \leq i \leq 3$. Next, observe that $\Delta_{r,j}(a) \oplus_l \Delta_{r,j}(b) = R_l \circ m_\alpha \circ f_t(a) \oplus_l R_l \circ m_\alpha \circ f_t(b) = R_l(m_\alpha \circ f_t(a) \oplus m_\alpha \circ f_t(b)) = R_l(m_\alpha(f_t(a) \oplus f_t(b)) = R_l(m_\alpha(f_t(a \oplus b))) = \Delta_{r,j}(a \oplus b)$ for all $a, b \in \mathbf{F}_{256}$ and all $\alpha \in \mathbf{F}_{256}^*$; the second equality holds true since R_l is an isomorphism, the third equality holds true as $\alpha(a \oplus b) = \alpha(a) \oplus \alpha(b)$ for all $a, b \in \mathbf{F}_{256}$ and the fourth equality holds true since f_t is an automorphism. It follows that

$$w_i = \bigoplus_{z=0}^{3}{}_l \, \delta_{r,j}(mc_{iz}) \otimes_l \Delta_{r,j} \circ S\big(x_z \oplus k_z^{(r,j)}\big) \ ,$$

for $0 \leq i \leq 3$. Next, note that $\delta_{r,j}(a) \otimes_l \Delta_{r,j}(b) = R_l \circ f_t(a) \otimes_l R_l \circ m_\alpha \circ f_t(b) = R_l(f_t(a) \otimes m_\alpha \circ f_t(b)) = R_l(m_\alpha(f_t(a \otimes b))) = \Delta_{r,j}(a \otimes b)$ for all $a, b \in \mathbf{F}_{256}$; the second equality holds true since R_l is an isomorphism and the third equality uses the fact that $a^{2^t} \otimes \alpha b^{2^t} = \alpha(ab)^{2^t}$ for all $a, b \in \mathbf{F}_{256}$ and all $\alpha \in \mathbf{F}_{256}^*$. It follows that

$$w_i = \bigoplus_{z=0}^{3}{}_l \, \Delta_{r,j}\Big(mc_{iz} \otimes S\big(x_z \oplus k_z^{(r,j)}\big)\Big) \ ,$$

for $0 \leq i \leq 3$. From this, $\Delta_{r,j}(a) \oplus_l \Delta_{r,j}(b) = \Delta_{r,j}(a \oplus b)$ for all $a, b \in \mathbf{F}_{256}$, and the definition of y_i in Def. 1, it follows that $w_i = \Delta_{r,j}(y_i)$ for $0 \leq i \leq 3$. □

Now, Karroumi [6] obtains a dual AES cipher as follows:

Step 1 assigns a randomly chosen $\Delta_{r,j} \in \mathcal{T}$ to each AES subround $AES^{(r,j)}$ ($1 \leq r \leq 9$ and $0 \leq j \leq 3$). Based on $\Delta_{r,j}$, the corresponding dual AES subround $AES^{(r,j,\Delta_{r,j})}$ is implemented as specified by Def. 3. The mappings $\Delta_{r,j}$ and $\delta_{r,j}$ (and the implementation of the dual cipher) are kept secret.

Step 2 ensures that the functionality of AES is maintained by including an additional operation (referred to as `ChangeDualState`) between `ShiftRows` and `AddRoundKey` operations of round r for $1 \leq r \leq 9$. If the inverse `ShiftRows` operation is defined by the mapping $sr(i, j) = (j + i) \bmod 4$ for $0 \leq i, j \leq 3$, then the `ChangeDualState` operation of round r applies the mapping $C_i^{(r,j)} : \mathbf{F}_{256} \to \mathbf{F}_{256}$ to the byte of the state associated with the i-th input byte of $\text{AES}^{(r,j,\Delta_{r,j})}$ for $0 \leq i, j \leq 3$, defined by $C_i^{(1,j)} = \Delta_{1,j}$ and $C_i^{(r,j)} = \Delta_{r,j} \circ \Delta_{r-1,\text{sr}(i,j)}^{-1}$ if $2 \leq r \leq 9$. Observe that for $2 \leq r \leq 9$, $C_i^{(r,j)}$ maps elements from $\mathbf{F}_{256}$ using the polynomial representation associated with $\Delta_{r-1,\text{sr}(i,j)}$ to elements of $\mathbf{F}_{256}$ using the polynomial representation associated with $\Delta_{r,j}$.

Karroumi presents two different but equivalent methods (from a security point of view) in [6] to perform the `ChangeDualState` operation, and specifies the white-box AES implementation using one of these methods. In this paper we use the specification as in [6]; the cryptanalysis can easily be adapted if the other method is used.

Phase 2: Apply the techniques of Chow *et al.*

The following description of Karroumi's white-box AES implementation is equivalent to the description in [6]:

Step 1 applies the techniques of Chow *et al.* to write the dual AES cipher (with a fixed key) obtained in Phase 1 as a series of lookup tables. In particular, the dual AES key addition operations and the dual S-box operations are merged into key-dependent bijective mappings $T_i^{(r,j,\Delta_{r,j})}$ for $0 \leq i, j \leq 3$ and $1 \leq r \leq 9$. These mappings are referred to as dual T-boxes and are defined by

$$T_i^{(r,j,\Delta_{r,j})} = \Delta_{r,j} \circ S \circ \Delta_{r,j}^{-1} \circ \oplus_{\Delta_{r,j}(k_i^{(r,j)})} \circ C_i^{(r,j)} \ ,$$

where each dual T-box mapping is implemented as a table mapping 8 input bits to 8 output bits. Recall that $C_i^{(r,j)}$ are the mappings defining the `ChangeDualState` operation. Next, write the other part of the dual AES cipher as a series of lookup tables as indicated by Chow *et al.* in [5]. The number and types of tables (including the tables representing the dual T-boxes) and the data-flow between tables are the same as in the lookup table implementation of AES in [5]. The only difference is that the values of the table entries of the dual AES implementation are likely to be different from the values of the corresponding entries in the AES implementation in [5] due to the dual version of the AES operations.

Step 2 applies the white-box encoding techniques of Chow *et al.* in [5] to this lookup table implementation of dual AES. As these white-box encoding techniques do not depend on the values of the table entries, the number and types of white-box tables, and the data-flow of Karroumi's white-box AES implementation are the same as in the white-box AES implementation of Chow *et al.* in [5].

In [6], Karroumi argues that the secrecy of the mappings $\Delta_{r,j}$, randomly selected from the set $\mathcal{T}$ and used to generate the dual cipher, increases the work factor of the BGE attack to 2^{93}.

4.1 Cryptanalysis

This section shows that Karroumi's white-box AES implementation [6] is insecure. Recall that Karroumi's white-box AES implementation uses the same number and types of white-box tables, and that the data-flow of the implementation is the same as in Chow *et al.*'s white-box AES implementation in [5]. As a result, the techniques of Billet *et al.* can be applied directly to compose lookup tables in Karroumi's implementation to obtain access to the encoded dual AES subrounds (instead of the encoded AES subrounds in case of Chow *et al.*'s implementation) for rounds $1 \leq r \leq 9$. In the following definition, $A_i^{(r,j)}$ and $B_i^{(r,j)}$ for $0 \leq i \leq 3$ denote bijective mappings (or encodings) on the vector space $\mathbf{F}_2^8$. Further, with slight abuse of notation, an output of $A_i^{(r,j)}$ is considered to be an element of $\mathbf{F}_{256}$ using the polynomial representation associated with the mapping R_l as defined by $\Delta_{r-1,\mathrm{sr}(i',j')}$, and an output of $AES^{(r,j,\Delta_{r,j})}$ is considered to be an element of $(\mathbf{F}_2^8)^4$. In the following definition, $\Pi_1^{(r,j)}, \Pi_2^{(r,j)}$ and $\pi^{(r)}$ are the permutations as used in Def. 2.

Definition 4. *The mapping* $AES_{enc}^{(r,j,\Delta_{r,j})} : (\mathbf{F}_2^8)^4 \rightarrow (\mathbf{F}_2^8)^4$ *for* $1 \leq r \leq 9$ *and* $0 \leq j \leq 3$*, called an encoded dual AES subround, is defined by*

$$AES_{enc}^{(r,j,\Delta_{r,j})} = (B_0^{(r,j)}, B_1^{(r,j)}, B_2^{(r,j)}, B_3^{(r,j)}) \circ \overline{AES}^{(r,j,\Delta_{r,j})} \circ (A_0^{(r,j)}, A_1^{(r,j)}, A_2^{(r,j)}, A_3^{(r,j)}) \ , \quad (5)$$

where the mapping $\overline{AES}^{(r,j,\Delta_{r,j})}$ *is defined by*

$$\Pi_2^{(r,j)} \circ AES^{(r,j',\Delta_{r,j'})} \circ (C_0^{(r,j')}, C_1^{(r,j')}, C_2^{(r,j')}, C_3^{(r,j')}) \circ \Pi_1^{(r,j)} \ , \quad (6)$$

with $j' = \pi^{(r)}(j)$.

The next lemma shows that an encoded dual AES subround can be represented by an encoded AES subround using the same key bytes:

Lemma 3. *An encoded dual AES subround* $AES_{enc}^{(r,j,\Delta_{r,j})}$ *is an encoded AES subround* $AES_{enc}^{(r,j)}$ *as in Def. 2 with*

$$P_i^{(1,j)} = A_i^{(1,j)} \quad \textit{and} \quad P_i^{(r,j)} = \Delta_{r-1,\mathrm{sr}(i',j')}^{-1} \circ A_i^{(r,j)} \quad \textit{if} \quad 2 \leq r \leq 9 \ ,$$

and

$$Q_i^{(r,j)} = B_i^{(r,j)} \circ \Delta_{r,j'} \ ,$$

for $0 \leq i, j \leq 3$ *and* $1 \leq r \leq 9$, *with* $i' = (\pi_1^{(r,j)})^{-1}(i)$ *and* $j' = \pi^{(r)}(j)$ *where* $(\pi_1^{(r,j)})^{-1}$ *denotes the permutation on the indices of a 4-byte vector as a result of the application of* $\left(\Pi_1^{(r,j)}\right)^{-1}$.

Proof. The proof is given for the case $2 \leq r \leq 9$; similar reasoning applies to the case $r = 1$. From the definition of the `ChangeDualState` operation (see Step 2 of Phase 1 of Karroumi's implementation) it follows that

$$\begin{aligned}(C_0^{(r,j')}, C_1^{(r,j')}, C_2^{(r,j')}, C_3^{(r,j')}) = (\Delta_{r,j'}, \Delta_{r,j'}, \Delta_{r,j'}, \Delta_{r,j'}) \circ \\ (\Delta_{r-1,\mathrm{sr}(0,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(1,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(2,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(3,j')}^{-1}) \quad \text{if } 2 \leq r \leq 9 \ ,\end{aligned}$$

for $0 \leq j \leq 3$. Substituting the above expression for the `ChangeDualState` operation in Eq. 6 and applying Lemma 2 gives

$$\begin{aligned}\overline{AES}^{(r,j,\Delta_{r,j})} = \Pi_2^{(r,j)} \circ (\Delta_{r,j'}, \Delta_{r,j'}, \Delta_{r,j'}, \Delta_{r,j'}) \circ AES^{(r,j')} \circ \\ (\Delta_{r-1,\mathrm{sr}(0,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(1,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(2,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(3,j')}^{-1}) \circ \Pi_1^{(r,j)} \ .\end{aligned}$$

Observe that $\Pi_2^{(r,j)}$ and $(\Delta_{r,j'}, \Delta_{r,j'}, \Delta_{r,j'}, \Delta_{r,j'})$ commute and thus can be swapped. By applying the equation

$$\begin{aligned}(\Delta_{r-1,\mathrm{sr}(0,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(1,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(2,j')}^{-1}, \Delta_{r-1,\mathrm{sr}(3,j')}^{-1}) \circ \Pi_1^{(r,j)} = \\ \Pi_1^{(r,j)} \circ (\Delta_{r-1,\mathrm{sr}(0',j')}^{-1}, \Delta_{r-1,\mathrm{sr}(1',j')}^{-1}, \Delta_{r-1,\mathrm{sr}(2',j')}^{-1}, \Delta_{r-1,\mathrm{sr}(3',j')}^{-1}) \ ,\end{aligned}$$

where $i' = (\pi_1^{(r,j)})^{-1}(i)$ for $i = 0, 1, 2, 3$ where $(\pi_1^{(r,j)})^{-1}$ denotes the permutation on the indices of a 4-byte vector as a result of the application of $\left(\Pi_1^{(r,j)}\right)^{-1}$, one gets the result of Lemma 3. □

From the discussion above it follows that Karroumi's white-box AES implementation and the white-box AES implementation of Chow *et al.* are the same. As a consequence, Karroumi's white-box AES implementation is vulnerable to the attack it was designed to resist.

5 Conclusion

The BGE attack on the white-box AES implementation of Chow *et al.* extracts the AES key from such an implementation with a work factor of 2^{30}. Taking Tolhuizen's improvement to the most time-consuming phase of the BGE attack as the starting point, Sect. 3 presented several improvements to the other phases of the BGE attack. It was shown that the overall work factor of the BGE attack is reduced to 2^{22} when all improvements are implemented. Unlike the original BGE attack, the use of non-affine white-box encodings and the randomization in the order of the bytes of the intermediate results in AES have a negligible contribution to the overall work factor of the improved BGE attack.

Karroumi's white-box AES implementation was designed to withstand the BGE attack. Section 4 showed that the white-box AES implementations of Chow *et al.* and Karroumi are the same. As a result, the BGE attack can be applied directly to extract the key from Karroumi's white-box AES implementation, implying that this implementation is insecure.

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