Extended Criterion for Absence of Fixed Points

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Abstract

One of the criteria for substitutions used in block ciphers is the absence of fixed points. In this paper we show that this criterion must be extended taking into consideration a mixing key function. In practice, we give a description of AES when fixed points are reached. Additionally, it is shown that modulo addition has more advantages then XOR operation.

Keywords: S-box, Block Cipher, Fixed Point, AES

1 Introduction

Substitution boxes (S-boxes) map an n-bit input message to an m-bit output message. They provide confusion in symmetric algorithms. For different tasks S-boxes are used in various forms. In stream ciphers a substitution is represented usually as a vectorial Boolean function [1]. Permutations are a subclass of substitutions and are commonly used in block ciphers as lookup tables. Regardless of ciphers an S-box can be converted from one form to another one.

Substitutions must satisfy various criteria for providing high level of protection against different types of attacks [2]. A substitution satisfying all criteria is perfect. However, such substitutions do not exist up to date. Therefore, in practice, substitutions satisfying several important criteria are used. They are called optimal S-boxes. Optimality criteria vary from cipher to cipher. Generating permutations with optimal criteria is a quite difficult task, especially for a large n and m. The problem of generating a set of S-boxes with similar properties can be particularly solved by using EA- or CCZ-equivalence [3, 4].

One of criteria is absence of fixed points. It is used in many ciphers for increasing resistance against statistical attacks [5]. Designers of modern cryptographic primitives try to get rid of the fixed points. This is achieved by applying affine equivalence, which is a special case of EA-equivalence. The S-box of advanced encryption standard (AES) was constructed using this technique [5, 6]. However, the application of this method does not totally prevent the appearance of fixed points. In this paper we show an isomorphic (equivalent) form of AES when fixed points are reached.

Two ciphers E_i and E_j are isomorphic to each other if there exist invertible maps $\phi : x^i \mapsto x^j$, $\psi : y^i \mapsto y^j$ and $\chi : k^i \mapsto k^j$ such that $y^i = E_i(x^i, k^i)$ and $y^j = E_j(x^j, k^j)$ are equal for all x^i, k^i, x^j and k^j [7, 8]. Obviously, the cipher can have a lot of isomorphic basic transformations as well as full encryption procedures. The cipher BES is a well-known example of isomorphic AES [9]. Another example of isomorphic AES is the description of encryption procedure using system of equation of degree 2 [10]. We give one more description of AES which includes a substitution with a fixed point while almost all transformations are unmodified.

2 Preliminaries

Arbitrary substitution can be represented at least in three different forms: algebraic normal form (ANF), over field $\mathbb{F}_{2^n}$ and as a lookup table. Most of substitutions used in block ciphers have a table representation because of simplicity of description and understanding [11]. Meanwhile arbitrary S-box S can be always associated with a vectorial Boolean function F in $\mathbb{F}_{2^n}[x]$. If a substitution is a permutation then F is defined uniquely [12].

The natural way of representing F as a function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ is by its algebraic normal form:

$$\sum_{I \subseteq \{1,\dots,n\}} a_I \left(\prod_{i \in I} x_i \right), \qquad a_I \in \mathbb{F}_2^m,$$

the sum is being calculated in $\mathbb{F}_2^m$ [1]. The algebraic degree of F is the degree of its ANF. F is called affine if it has algebraic degree at most 1 and it is called linear if it is affine and $F(0) = 0$. A vectorial Boolean function given in table representation can be easily transformed to ANF form and vice versa.

Two functions $F, G : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ are called extended affine equivalent (EA-equivalent) if there exist an affine permutation A_1 of $\mathbb{F}_2^m$, an affine permutation A_2 of $\mathbb{F}_2^n$ and a linear function L_3 from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ such that

$$F(x) = A_1 \circ G \circ A_2(x) + L_3(x). \tag{1}$$

Clearly, A_1 and A_2 can be presented as $A_1(x) = L_1(x) + c_1$ and $A_2(x) = L_2(x) + c_2$ for some linear permutations L_1 and L_2 and some $c_1 \in \mathbb{F}_2^m$, $c_2 \in \mathbb{F}_2^n$. Two functions F and G are linear equivalent if equation (1) is hold for $L_3(x) = 0$, $c_1 = 0$, $c_2 = 0$. If the equation (1) is preserved only for $L_3(x) = 0$, then functions F and G are called affine equivalent [13].

In matrix form EA-equivalence is represented as follows

$$F(x) = M_1 \cdot G(M_2 \cdot x \oplus V_2) \oplus M_3 \cdot x \oplus V_1$$

where elements of $\{M_1, M_2, M_3, V_1, V_2\}$ have dimensions $\{m \times m, n \times n, m \times n, m, n\}$ [3].

An element $a \in \mathbb{F}_2^n$ is a fixed point of $F : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ if $F(a) = a$. The absence of fixed points criterion is defined as follows.

Proposition 1 *A substitution must not have fixed points, i.e.*

$$F(a) \neq a, \quad \forall a \in \mathbb{F}_2^n.$$

For any positive integers n and m, a function F from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ is called differentially δ-uniform if for every $a \in \mathbb{F}_2^n \setminus \{0\}$ and every $b \in \mathbb{F}_2^m$, the equation $F(x) + F(x + a) = b$ admits at most δ solutions [1]. Vectorial Boolean functions used as S-boxes in block ciphers must have low differential uniformity to allow high resistance to differential cryptanalysis [14].

The nonlinearity criterion is closely connected to the notion of Walsh transform which can be described as the function

$$\lambda(u, v) = \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x) + u \cdot x},$$

where "$\cdot$" denotes inner products in $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$ respectively [1]. A substitution has an optimal resistance to linear cryptanalysis if the maximum absolute value of Walsh coefficients is small [15]. Substitutions with the smallest value of $\lambda(u, v)$ exist for odd n only.

These two criteria are major while selecting substitutions for new ciphers. However, there are many others criteria like propagation criterion, absolute indicator, correlation immunity, strict avalanche criterion, etc [1, 2, 16]. It has been still not proven the importance of the criteria for block ciphers. For example, the substitution used in AES does not satisfy most of them [17]. Moreover, no theoretical or practical attacks were proposed on modern block ciphers based on these criteria.

Let $E : \{0,1\}^l \times \{0,1\}^k \mapsto \{0,1\}^l$ be a function taking a key K of length k bits and input message (plaintext) M of length l bits and return output message (ciphertext) $E(M, K)$. For each key K let $E_K : \{0,1\}^l \mapsto \{0,1\}^l$ be a function defined by $E_K(M) = E(M, K)$. Then E is a block cipher if E_K and E_K^{-1} are efficiently computable and E_K is a permutation for every K.

Most of the modern block ciphers are iterative (Fig. 1). Usually a round function is run multiple times with different parameters (round keys). An arbitrary iterative block cipher can be mathematically described as follows

$$E_K(M) = PW_{k_{r+1}} \circ \prod_{i=2}^{r} (R_{k_i}) \circ IW_{k_1}(M),$$

where R is a round procedure, IW is a prewhitening procedure and PW is a postwhitening procedure. In Fig. 1 a key schedule is an algorithm that takes a master key K as input and produces the subkeys $k_1, k_2, \ldots, k_{r+1}$ for all stages of encryption algorithm.

A mixing key procedure of a block cipher is an algorithm which injects a round key into an encryption procedure. In the majority of the modern block ciphers, the mixing key function is implemented as exclusive or (XOR) operation because of low-cost implementations.

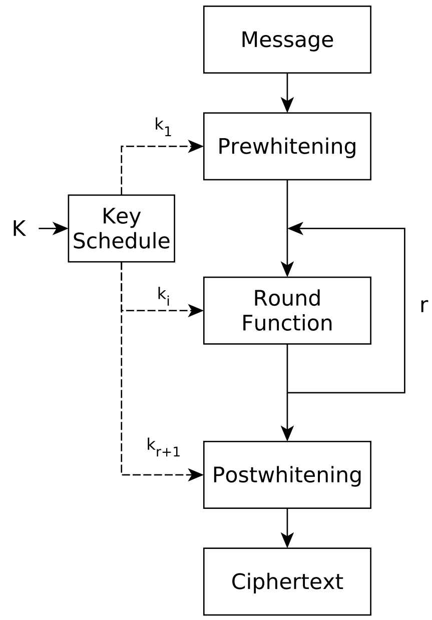


Figure 1: General Structure of An Iterative Block Cipher

3 A Brief Description of AES

AES is a substitution permutation network (SPN) block cipher that supports a fixed block size of 128 bits and a key size of 128, 192 or 256 bits [6]. The number of rounds depends on the key size and is equal to 10, 12 or 14, respectively. The round function consists of four functions: AddRoundKey (σ_k), SubBytes (γ), ShiftRows (π) and MixColumns (θ).

The entire encryption algorithm is described as follows (Fig. 2)

$$E_K(M) = \sigma_{k_{r+1}} \circ \pi \circ \gamma \circ \prod_{i=2}^{r} (\sigma_{k_i} \circ \theta \circ \pi \circ \gamma) \circ \sigma_{k_1}(M).$$

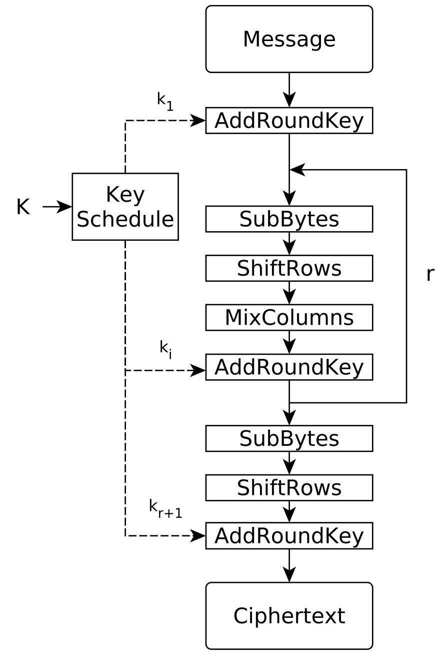


Figure 2: Encryption Algorithm of AES

The SubBytes transformation processes the state of the cipher using a nonlinear byte substitution table that operates on each of the state bytes independently [6]. The S-box of AES was generated by finding the inverse element in the field $\mathbb{F}_{2^8}$ followed by applying affine polynomial. In terms of equation (1) the transformation has the form

$$F(x) = A_1(x^{-1}) = L_1(x^{-1}) + c_1.$$

The substitution table generated by vectorial Boolean function $F : \mathbb{F}_{2^8} \mapsto \mathbb{F}_{2^8}$ satisfies the following criteria

- the maximum value of non-trivial XOR difference transformation probability is 2^{-6},
- the maximum absolute value of linear approximation probability bias is 2^{-4},
- the minimum algebraic degree of the component functions is 7 [5, 18].

It should be noticed that the chosen polynomial x^{-1} allows to describe the S-box and the entire cipher by overdefined system of equations of degree 2 [19]. But in the same time it gives resistant to differential, linear and many other cryptanalytical methods. In addition to the general properties, the constant of the AES S-box has been chosen in such way that it has no fixed points [5].

The MixColumns transformation takes all the columns of the state and mixes their data (independently of one another) to produce new columns [6]. This transformation can be represented in different ways. One of them is the matrix multiplication. For an input state x and 4×4 matrix M the output state y of the transformation is described as

$$y = M \cdot x.$$

The matrix with maximum distance separable (MDS) property is used in AES. In terms of Rijndael the MDS property associates with a branch number (β)

$$\beta = \min_{x \neq 0}(W(x) + W(y)),$$

where $W(z)$ is the byte weight of a vector z.

From the definition of MDS matrix, it is known that the maximum differential branch number of m by m matrix is $m+1$ [11, 20]. Hence, MDS matrices have the perfect diffusion property for byte-oriented ciphers.

Multiplication in a field $\mathbb{F}_{2^n}$ is a linear transformation with respect to XOR, so it preserves the linear property [9]

$$\theta(x + y) = \theta(y) + \theta(y).$$

The ShiftRows transformation processes the state by cyclically shifting the last three rows of the state by different offsets [6]. More precisely, row i is moved to the left by i byte positions for $0 \leq i \leq 3$. The ShiftRows is also a linear function that preserves $\pi(x + y) = \pi(y) + \pi(y)$ property.

Both MixColumns and ShiftRows transformations help to ensure that the number of active S-boxes is large even after few rounds [5]. These functions are the basis of protection offered by the AES against differential and linear cryptanalysis.

AddRoundKey transformation is the mixing key function in which a round key is added to the state using XOR operation. The length of a round key is equal to the size of the state. XOR operation of two n-bit length vectors a and b can be performed bit by bit n times. Therefore, AddRoundKey operation of AES can be done independently of each byte.

4 A New Cipher Isomorphic to AES

There exist several examples of ciphers isomorphic to AES. For example, the big encryption system (BES) describes AES over $\mathbb{F}_{2^8}$ [9]. On the other hand, the cipher AES can be also represented as the system of multivariate equations of the 2^{nd} degree over $\mathbb{F}_2$ [19]. These two examples are based on the algebraic features of the substitution. However, there is another approach based on linear properties of the basic functions (MixColumns and ShiftRows).

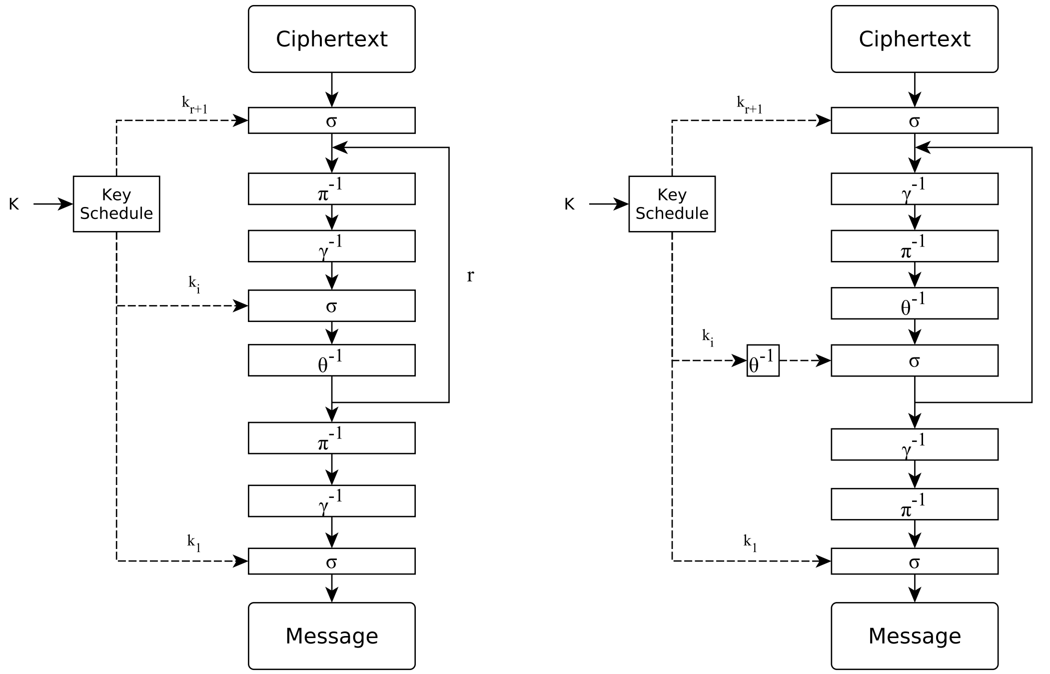
The cipher AES is based on Rijndael that was proposed by Daemen and Rijmen to AES process [21]. Authors have used design simplicity principle, which led to performance improvement and code compactness properties of the cipher on a wide range of platforms. For increasing decryption performance of software implementation they have used precomputed lookup tables and the linear properties of the basic functions.

The original decryption algorithm for arbitrary ciphertext C mathematically can be represented as follows (Fig. 3(a)) [6]

$$D_K(C) = \sigma_{k_1} \circ \gamma^{-1} \circ \pi^{-1} \circ \prod_{i=2}^{r} (\theta^{-1} \circ \sigma_{k_{r-i+2}} \circ \gamma^{-1} \circ \pi^{-1}) \circ \sigma_{k_{r+1}}(C).$$

For using precomputed tables it is necessary to transform the decryption round function to the similar one of encryption algorithm. Since functions γ^{-1} and π^{-1} are computed independently they have the commutative property $\gamma^{-1} \circ \pi^{-1} = \pi^{-1} \circ \gamma^{-1}$ [5, 9]. In Section 3 it was stated that functions θ^{-1} and σ are linear w.r.t. XOR, hence

$$\theta^{-1} \circ \sigma_{k_{r-i+2}} = \sigma_{\theta^{-1}(k_{r-i+2})} \circ \theta^{-1}$$



(a) Original

(b) Algorithm for the Fast Software Implementation

Figure 3: Decryption Algorithm of AES

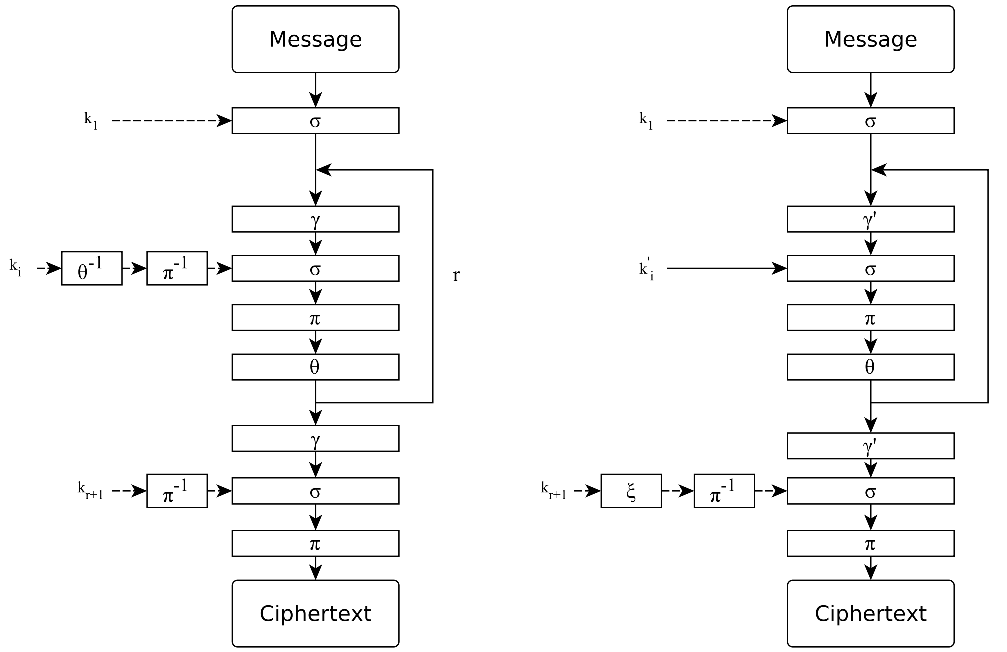
Thus, the whole decryption algorithm has the form (Fig. 3(b))

$$D_K(C) = \sigma_{k_1} \circ \pi^{-1} \circ \gamma^{-1} \circ \prod_{i=2}^{r} (\sigma_{\theta^{-1}(k_{r-i+2})} \circ \theta^{-1} \circ \pi^{-1} \circ \gamma^{-1}) \circ \sigma_{k_{r+1}}(C).$$

Usage of such elementary transformations helps to achieve a significant acceleration of the decryption procedure due to the isomorphic properties of the basic functions [5].

Obviously, the same technique can be applied to the encryption algorithm. However, the task is to find a representation of the cipher in which properties of a new substitution will differ from the original one. For simplicity of description, let us assume that the round keys are independent of each other. Then the encryption procedure takes a form (Fig. 4(a))

$$E_K(M) = \pi \circ \sigma_{\pi^{-1}(k_{r+1})} \circ \gamma \circ \prod_{i=2}^{r} (\theta \circ \pi \circ \sigma_{\pi^{-1} \circ \theta^{-1}(k_i)} \circ \gamma) \circ \sigma_{k_1}(M).$$



(a) Modified Encryption Algorithm

(b) Isomorphic Encryption Algorithm with a Fixed Point

Figure 4: Isomorphism of AES

The last equation shows that the final ShiftRows operation is redundant in terms of resistance to attacks. As it was stated above the availability of this function is necessary for fast implementation of the decryption procedure.

Arbitrary permutation S can be represented as a vectorial Boolean function $F : \mathbb{F}_{2^n} \mapsto \mathbb{F}_{2^n}$ which has the form [3]

$$F(x) = F'(x) + F(0).$$

Since the characteristic of the field is 2, the constant can be moved to the round keys. Let ξ be a function in which the constant $F(0)$ is XORed with all bytes of a state. If the round keys $\pi^{-1} \circ \theta^{-1} \circ \xi(k_i)$ are denoted by k'_i then encryption procedure takes the form (Fig. 4(b))

$$E_K(M) = \pi \circ \sigma_{\pi^{-1} \circ \xi(k_{r+1})} \circ \gamma' \circ \prod_{i=2}^{r} (\theta \circ \pi \circ \sigma_{k'_i} \circ \gamma') \circ \sigma_{k_1}(M),$$

where γ' is the SubBytes function which consists of the substitution of the form $F(x) = L(x^{-1})$.

Fig. 4(b) shows that the structure of the cipher remains unchanged. Clearly, if an adversary finds a round key for modified cipher she also automatically obtains corresponding round key of the original cipher because of the linear dependence of the keys k_i and k'_i. However, the new substitution $F(x) = L(x^{-1})$ has the fixed point in $x = 0$. Consequently, the substitution of AES doesn't satisfy the absence of fixed points criterion.

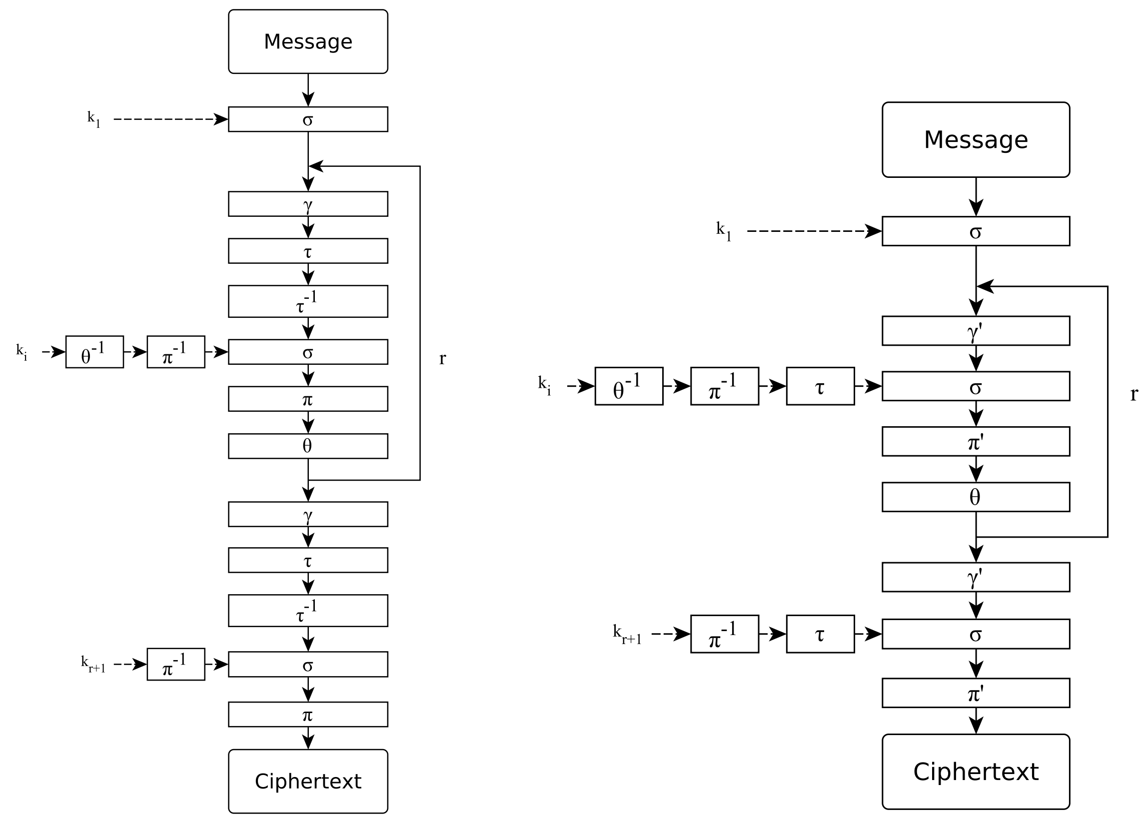
Described features of the cipher appears from the fact that the operation XOR is linear with respect to MixColumns and ShiftRows. If one replaces AddRoundKey with some nonlinear function (i.e. based on addition modulo 2^n), then it will be impossible to find an isomorphic cipher of such a form. From this point of view a mixed key function based on modulo addition is cryptographically stronger than a function based on XOR operation .

Furthermore, fixed points are directly connected with cyclic properties of substitutions. Inserting an invertible linear function (τ) into the encryption procedure gives a new isomorphic cipher (Fig. 5(a)). Herewith, the linearized polynomial can be added to the round key and the inverse function can be a part of the new substitution (Fig. 5(b)). The cyclic properties of the new substitution will depend on the selected function τ.

Thereby, the cyclic and the absence of fixed points properties of a substitution can be controlled by adversary in the case of a linear mixing key function. Thus, a new criterion for substitutions follows from the description above.

Proposition 2 *Substitutions S_1, S_2, ..., S_n used in a confusion layer must belong to different classes of equivalence.*

Clearly, if substitutions are in the same class (i.e. EA-equivalent) then the adversary can find an isomorphic cipher which consists of one substitution and modified linear layer. Consequently, there will be no advantages to use multiple substitutions. The criterion has to be considered both in the design of new ciphers and in the analysis of existing ones [22, 23]. Since CCZ-equivalence is the most general case of known equivalence, it makes



(a) Initial Cipher (b) Isomorphic Cipher

Figure 5: Modified AES with an Invertible Linear Function

sense to check whether substitutions belong to different CCZ-equivalence classes.

5 Conclusions

It was shown that the absence of fixed points criterion works only in case if S-box is considered as a separate function. There are isomorphic representations of ciphers in which this criterion is not met. The new method of AES description allows to reconsider some of criteria for substitutions from the practical point of view. This may lead to a weakening of the cipher strength.

Since an invertible linear function can be added to encryption procedure, the adversary can control both the cyclic and absence of fixed points properties of substitutions. It was shown that mixing key function based on modulo addition is more resistant with respect to the absence of fixed points criterion than function based on XOR operation.

Isomorphism of ciphers adds additional restrictions on using multiple substitutions. The proposed criterion can be used to reduce the probability of finding the weakest one.

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