A More Explicit Formula for Linear Probabilities of Modular Addition Modulo a Power of Two

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Abstract: Linear approximations of modular addition modulo a power of two was studied by Wallen in 2003. He presented an efficient algorithm for computing linear probabilities of modular addition. In 2013 Schulte-Geers investigated the problem from another viewpoint and derived a somewhat explicit formula for these probabilities. In this note we give a closed formula for linear probabilities of modular addition modulo a power of two, based on what Schulte-Geers presented: our closed formula gives a better insight on these probabilities and more information can be extracted from it.

Key Words: Modular addition modulo a power of two, Linear probability, Symmetric cipher, Linear cryptanalysis

1. Introduction

Linear cryptanalysis is a strong tool in cryptanalysis of symmetric ciphers. In [1] linear approximations of modular addition modulo a power of two is investigated and an efficient algorithm for computing these probabilities is given. A somewhat explicit formula for linear probabilities of this operator is also given in [2]. In this note, we propose a closed formula for linear probabilities of modular addition modulo a power of two based on the algorithm presented in [2]. Our closed formula exhibits a better insight for these probabilities and more information can be derived from it.

In this note, we use the following notations:

$w(x)$: Hamming weight of a binary vector $x = (x_{n-1}, \dots, x_0)$,

$\cdot$: Standard dot product,

$\oplus$: Bitwise XOR operator,

$|B|$: Number of symbols in a block B,

$\bar{\alpha}$: Complement of a bit α,

o-block: A block of symbols 1,2 or 4,

e-block: A block of symbols 3,5 or 6,

0-block: A block of symbol 0,

7-block: A block of symbol 7,

$[cond]$: 1 if $cond = true$ and 0 otherwise.

2. A Closed Formula for Linear Probabilities of Modular Addition

Suppose that the input masks $(a_{n-1}, \dots, a_0)$ and $(b_{n-1}, \dots, b_0)$ and the output mask $(c_{n-1}, \dots, c_0)$ is given. We wish to compute

$$\left| P(a \cdot x \oplus b \cdot y = c \cdot r) - \frac{1}{2} \right|, \tag{1}$$

where

$$r = x + y \mod 2^n,$$

$x = (x_{n-1}, \dots, x_0)$, $y = (y_{n-1}, \dots, y_0)$ and $r = (r_{n-1}, \dots, r_0)$. To compute (1), we recall the algorithm presented in [2]: put

$$s_i = a_{n-1-i} \oplus b_{n-1-i} \oplus c_{n-1-i}, \qquad 0 \le i < n.$$

Now put $z_0 = 0$ and

$$z_{i+1} = z_i \oplus s_i, \qquad 1 \le i < n-1.$$

The bias (1) is zero if there exists an $0 \le i < n$ such that $z_i = 0$ holds and $a_i = b_i = c_i$ does not hold. Otherwise, we have

$$\left| P(a \cdot x \oplus b \cdot y = c \cdot r) - \frac{1}{2} \right| = 2^{-(w(z)+1)}, \quad z = (z_{n-1}, \dots, z_0).$$

We can reformulate the above algorithm in this form: put

$$S_i = a_{n-1-i} + 2b_{n-1-i} + 4c_{n-1-i}, \qquad 0 \leq i < n.$$

So we have a sequence $S_0, \dots, S_{n-1}$ of symbols in $\{0, \dots, 7\}$. Is not hard to see that (1) can be computed by means of the (informal) automata of Picture 1. We begin by state 0 in the automata and traverse the diagram symbol by symbol. If we meet "halt" then (1) is equal to zero, and otherwise (1) is equal to 2^{-w}. We illustrate our algorithm through some examples:

Example 1. Let $n = 9$ and

$$(a_8, \dots, a_0) = (0,1,1,0,1,1,1,0,0),$$

$$(b_8, \dots, b_0) = (0,1,1,0,1,1,0,0,0),$$

$$(c_8, \dots, c_0) = (0,1,1,0,1,0,1,0,1).$$

Then we have

$$S_0 \dots S_8 = 077073504.$$

Traversing the diagram, we get the bias 2^{-5}.

Example 2. Let $n = 11$ and

$$(a_{10}, \dots, a_0) = (0,0,1,1,1,0,1,1,0,0,1),$$

$$(b_{10}, \dots, b_0) = (0,0,1,1,1,0,0,0,1,1,1),$$

$$(c_{10}, \dots, c_0) = (0,0,1,1,1,0,0,1,0,1,1).$$

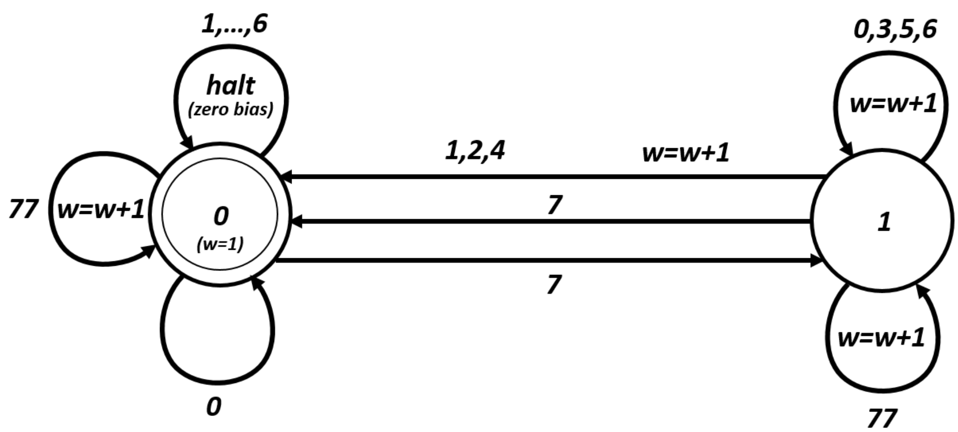
Then we have

$$S_0 \dots S_{10} = 00777015267.$$

Traversing the diagram, we get the bias 0.

In the appendix we have presented a pseudo-code for computing (1). It can be easily checked that the algorithm is very fast.

With the aid of Picture (1) which is by itself derived from [2], the proof of following theorem is straightforward:



Picture 1

Theorem 1. Notations as before, let

$$S_0, \dots, S_{n-1} = B_1 \dots B_m.$$

Here, B_i's, $1 \leq i \leq m$, are o-blocks, e-blocks, 0-blocks or 7-blocks. Define $\alpha_1 = 0$ and for $1 < i \leq m$

$$\alpha_i = \begin{cases} 1 & \# \{B_j : 1 \leq j < i, B_j \text{ } is \text{ } 7 - block \text{ } of \text{ } odd \text{ } length\} + \# \{B_j : 1 \leq j < i, B_j \text{ } is \text{ } o - block\} \text{ } is \text{ } odd, \\ 0 & \# \{B_j : 1 \leq j < i, B_j \text{ } is \text{ } 7 - block \text{ } of \text{ } odd \text{ } length\} + \# \{B_j : 1 \leq j < i, B_j \text{ } is \text{ } o - block\} \text{ } is \text{ } even. \end{cases}$$

Then (1) is equal to

$$\frac{q}{2^w},$$

where

$$q = \prod_{i=1}^{m} (1 - \bar{\alpha}_i [B_i \text{ } is \text{ } o - block \text{ } or \text{ } e - block]),$$

and

$$w = 1 + \sum_{B_i \text{ } is \text{ } o-block \text{ } or \text{ } e-block} |B_i| + \sum_{B_i \text{ } is \text{ } 7-block} \frac{\lfloor |B_i| \rfloor}{2} + \sum_{B_i \text{ } is \text{ } 0-block} \alpha_i |B_i|.$$

We state some of the direct consequences of Theorem 1 here:

- If (1) is not zero, then we cannot see a symbol in $\{1,2,4\}$ followed by some blocks which are not 7-blocks followed by a symbol in $\{1, \dots 6\}$: as an special case, there cannot be a symbol in $\{1,2,4\}$ before a symbol in $\{1, \dots ,6\}$.

- If (1) is not zero, then it is less than or equal to $2^{-(d+1)}$ where d is the total number of symbols in $\{1, \dots ,6\}$.

- If (1) is not zero, then there are (at least) $3^f 4^g - 1$ other sequences with the same probability, where

$$f = \sum_{B_i \text{ is } o-block \text{ or } e-block} |B_i|,$$

$$g = \sum_{B_i \text{ is } 0-block} \alpha_i |B_i|.$$

- If (1) is zero, then there are (at least) $3^f 4^g - 1$ other sequences with zero bias, where

$$f = \sum_{B_i \text{ is } o-block \text{ or } e-block} |B_i|,$$

$$g = \sum_{B_i \text{ is } 0-block} |B_i|.$$

References

[1] Johan Wallén: Linear Approximations of Addition Modulo 2^n. FSE 2003: 261-273

[2] Ernst Schulte-Geers: On CCZ-equivalence of addition mod 2^n. Des. Codes Cryptography 66(1-3): 111-127 (2013)

Appendix

Input: S[0],…,S[n-1]

Output: halt (zero bias) or w (value of the exponent)

```
i=0, s=0, w=1

while (i<n) do
        index=i
        j=0
        if (S[index]=7)
           while (S[i]=7)
                  j=j+1
                  i=i+1
           end (while)
           if (j is odd) s=1-s
           w = w + (j div 2)
        else if (S[index]=0)
                 i=i+1
                 if (s=1) w=w+1
        else if (S[index] is in {1,2,4})
                if (s=0) halt
                s=1-s
                w=w+1
                i=i+1
         else if (S[index] is in {3,5,6})
               if (s=0) halt
               else
                   w=w+1
                   i=i+1
               end (if)
        end (if)
 end (while)
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