Do Distributed Differentially-Private Protocols Require Oblivious Transfer?

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Abstract

We study the cryptographic complexity of two-party differentially-private protocols for a large natural class of boolean functionalities. Information theoretically, McGregor et al. [FOCS 2010] and Goyal et al. [Crypto 2013] demonstrated several functionalities for which the maximal possible accuracy in the distributed setting is significantly lower than that in the client-server setting. Goyal et al. [Crypto 2013] further showed that "highly accurate" protocols in the distributed setting for any non-trivial functionality in fact imply the existence of one-way functions. However, it has remained an open problem to characterize the exact cryptographic complexity of this class. In particular, we know that semi-honest oblivious transfer helps obtain optimally accurate distributed differential privacy. But we do not know whether the reverse is true.

We study the following question: *Does the existence of optimally accurate distributed differentially private protocols for any class of functionalities imply the existence of oblivious transfer?* We resolve this question in the affirmative for the class of boolean functionalities that contain an XOR embedded on adjacent inputs.

- We construct a protocol implementing oblivious transfer from *any* optimally accurate, distributed differentially private protocol for any functionality with a boolean XOR embedded on adjacent inputs.
- While the previous result holds for optimally accurate protocols for any privacy parameter $\epsilon > 0$, we also give a reduction from oblivious transfer to distributed differentially private protocols computing XOR, for a constant small range of non-optimal accuracies and a constant small range of values of privacy parameter ϵ.

At the heart of our techniques is an interesting connection between optimally-accurate two-party protocols for the XOR functionality and noisy channels, which were shown by Crépeau and Kilian [FOCS 1988] to be sufficient for oblivious transfer.

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1 Introduction

Differential privacy [Dwo06, DMNS06, DN04, DN03] has become one of the most well-studied and popular privacy notions in recent years[1]. It provides powerful input privacy guarantees to participants of a statistical query database. Informally a randomized function computed on a database is said to be differentially private, if the output distribution induced by the presence of a particular record is statistically close to the distributed induced when the record is absent. While maintaining *privacy* of participants, any differentially private algorithm must also guarantee some meaningful accuracy.

Consider a confidential dataset owned by a trusted server. The server must release the outcome of some statistic evaluated on the dataset, to an untrusted client. Even in this setting, where privacy is a concern only at the server's end, there is an evident tradeoff between privacy and accuracy. In fact, for any given privacy parameter ϵ, there is a maximum possible accuracy (which we call the *optimal accuracy*) such that any algorithm with better than optimal accuracy will fail to remain differentially private. Privacy-accuracy tradeoffs are reasonably well-understood in the client-server setting [DN03, DMT07, DY08, KRSU10]. There has also been a huge body of work in designing algorithms that achieve close to optimal accuracies for various functionalities and data mining tasks in the client-server setting.

The focus of this work is the distributed setting, where the database is jointly hosted by multiple mutually distrusting servers. This setting was first studied by Dwork et al. [DKM$^+$06]. As an illustrative example, consider two hospitals which together wish to compute the correlation between the occurrence of smoking and lung cancer by taking into account their combined patient records. In this setting, we require the servers to engage in a protocol, at the end of which the privacy of each record of both the servers is guaranteed without a significant loss in accuracy. Note that the privacy requirements must be met for both servers, *given their entire view of the protocol transcript*, not just the computed output; possibly necessitating an additional loss in accuracy (over and above the loss in the client-server setting).

The intuition that the distributed setting would necessitate a greater accuracy loss than the client-server setting has been proved to be correct in the information theoretic world for different classes of functions in various works. Beimel, Nissim and Omri [BNO08] showed accuracy limits for distributed differentially-private protocols for n parties each holding their own inputs. McGregor, Mironov, Pitassi, Reingold, Talwar and Vadhan [MMP$^+$10] showed large accuracy gaps in the two-party setting for several natural functionalities with n-bit inputs. Goyal, Mironov, Pandey and Sahai [GMPS13] demonstrated a constant gap between the maximal achievable accuracies in the client-server and distributed settings for any non-trivial boolean functionality.

In the computational setting this gap vanishes, if a semi-honest protocol for oblivious transfer exists. In this case, both servers can use secure multi-party computation [GMW87] to simulate the client-server differentially private function evaluation, thereby achieving optimally accurate output evaluated on the union of their databases. Although this assumption is sufficient, it is not clear whether this assumption is *necessary* as well.

Indeed, there has been a separate line of work, starting with Haitner, Omri and Zarosim [HOZ13] demonstrating black-box separations between one-way functions and distributed differentially private algorithms with optimal accuracies, for two-party n-bit functionalities. Khurana, Maji and Sahai [KMS14] showed a black-box separation between public-key encryption and distributed differentially private algorithms with optimal accuracies for two-party boolean functionalities. In fact, these separations also extend to a range of non-optimal accuracies that are information theoretically

[1] See [Dwo11] for a survey of results.

impossible to achieve in the distributed setting. These results provide evidence that some "strong" cryptographic assumption is likely to be necessary for optimally accurate distributed differentially private function evaluation.

Despite the above research, the following question has remained elusive:

"Does there exist any class of functionalities whose distributed differentially private evaluation with optimal accuracy, necessitates the existence of oblivious transfer?"

We prove that any protocol to compute the boolean XOR functionality in a distributed differentially private manner with optimal accuracy and overwhelming probability of agreement (on the output) between both parties, implies the existence of oblivious transfer. Our result also directly lends itself to *any* boolean functionality that contains an embedded XOR on two adjacent inputs. Roughly, a function f is said to contain an embedded XOR if and only if the ideal functionality for f can be used to compute the boolean XOR functionality in the semi-honest setting. We give a formal definition of what it means for a function to contain an embedded XOR, later in the paper.

Interestingly, in the setting of secure computation, the ideal XOR functionality is known to be trivial. This is because the output of the functionality combined with the input of any individual party reveals completely, the input of the other party. Thus, parties can simply send each other their inputs – this corresponds to a secure evaluation of the XOR functionality. However, an optimally accurate distributed differentially private (noisy) protocol for XOR is not trivial, in fact we show that it gives oblivious transfer. Furthermore, our proof of security crucially relies on the fact that an ideal (non-noisy) XOR is fully informative about the input of the other party.

Relationship between Differential Privacy and MPC. It is interesting to observe the "philosophical" differences between the requirements of differential privacy and secure computation:

- In (computationally) differentially-private protocols, "privacy comes first." We would like to first ensure privacy of each individual input and then with this constraint, would like to compute an output which is as accurate as possible.
- In secure computation, "accuracy comes first." We would like to release an accurate output to the function we are computing first and then with this constraint, would like to ensure privacy of the inputs to the extent possible. This leads to the notion of simulation: the transcript leaks no information about the inputs beyond what can be deduced from the output itself.

Nevertheless, as already mentioned, general secure computation methods immediately give a way to achieve the same (optimal) level of accuracy in distributed differentially-private protocols as the best achievable accuracy in the client-server setting. By relying completely on oblivious transfer for secure computation [Kil88], our results show that the reverse is true as well (at least for the differentially private evaluation of any two-party functionality with an embedded XOR).

1.1 Our Contribution

Before elaborating upon our results, we briefly summarize what is known so far about accuracy gaps in the distributed differentially private computation of boolean functionalities.

Alice and Bob with inputs x and y, respectively, wish to compute $f(x, y)$ in a differentially private manner in the distributed setting. An ϵ-differentially private protocol for some functionality f ensures that the probability of Alice's views conditioned on y and y' are $\lambda := e^{\epsilon}$ multiplicatively-close to each other, where y and y' represented as bit-strings differ only in one coordinate (i.e. they

are adjacent inputs). A protocol between them is α-accurate if for any x and y, the output of the protocol agrees with $f(x, y)$, with probability at least α.

For boolean functionalities, the optimal accuracy (in the client-server model) is $\alpha_\epsilon^* := \frac{\lambda}{\lambda+1}$, where $\lambda = e^\epsilon$. Goyal et al. [GMPS13] showed that in the information theoretic setting, $f = \mathsf{AND}$ can only be computed ϵ-differentially privately up to accuracy $\alpha_\epsilon^{(\mathsf{AND})} := \frac{\lambda(\lambda^2+\lambda+2)}{(\lambda+1)^3}$. Similarly, for $f = \mathsf{XOR}$ the maximal achievable accuracy in the information theoretic setting is $\alpha_\epsilon^{(\mathsf{XOR})} := \frac{\lambda^2+1}{(\lambda+1)^2}$. Note that $\alpha_\epsilon^{(\mathsf{XOR})} < \alpha_\epsilon^{(\mathsf{AND})} < \alpha_\epsilon^*$, for any finite $\epsilon > 0$.

We say that a function f contains an embedded XOR if there exist two inputs x_0, x_1, y_0, y_1 and outputs z_0, z_1 such that $f(x_1, y_b) = z_{\mathsf{XOR}(a,b)}$ for all $a, b \in \{0, 1\}$. Similarly, we can define an embedded AND (equivalently, an embedded OR). By observing that any boolean function f which is sensitive to both parties' inputs either contains an embedded XOR or AND*on adjacent inputs* [CK89], the maximal achievable accuracy becomes

$$\alpha_\epsilon^{(f)} := \begin{cases} \alpha_\epsilon^{(\mathsf{XOR})}, & \text{if } f \text{ contains an embedded } \mathsf{XOR} \text{ on adjacent inputs} \\ \alpha_\epsilon^{(\mathsf{AND})}, & \text{otherwise.} \end{cases} \tag{1}$$

Given a semi-honest secure protocol for oblivious transfer, the optimal accuracy α_ϵ is achievable for any boolean f. With respect to the necessity of cryptographic assumptions, Goyal et al. [GMPS13] showed that achieving any accuracy between α_ϵ and $\alpha_\epsilon^{(f)}$ for any function f in the distributed setting implies the existence of one-way functions. We strengthen their result to show that any two-party differentially private protocol that computes the XOR functionality in a differentially private manner with accuracy close to α_ϵ implies the existence of semi-honest secure oblivious transfer. Our result also extends to a weaker variant of differential privacy, namely *computational differential privacy* [MPRV09]. All our results hold for two-party functionalities where both parties obtain the same output with overwhelming probability. Our results can be summarized as follows.

Informal Theorem 1. Semi-honest oblivious transfer reduces to any two-party ϵ-DP protocol with accuracy $\rho(> 1/2)$ such that $\rho \geq \alpha_\epsilon = \frac{e^\epsilon}{1+e^\epsilon}$.

Informal Theorem 2. Semi-honest oblivious transfer reduces to any two-party ϵ_k-IND-CDP protocol with accuracy $\rho_k(> 1/2)$ such that $\rho_k \geq \alpha_{\epsilon_k} = \frac{e^{\epsilon_k}}{1+e^{\epsilon_k}}$.

Informal Theorem 3. $(\rho_k, \frac{\lambda_k}{m_k} - 1, \frac{\lambda}{m_k} - 1)$ *Weak* noisy channels [DKS99, Wul09] reduce to any two-party ϵ_k-IND-CDP protocol with accuracy $\rho_k(> 1/2)$ where $\rho_k \geq \alpha_{\epsilon_k} = \frac{e^{\epsilon_k}}{1+e^{\epsilon_k}}$ and $m_k = \rho_k/(1-\rho_k)$.

We prove the first two theorems via a reduction from (standard) noisy channels, which are known to imply semi-honest OT [CK88]. The first two can also be viewed as special cases of the third.

Furthermore, for a range of non-optimal accuracies we also show a reduction to weak noisy channels [DKS99, Wul09]. Invoking known reductions of oblivious transfer to weak binary symmetric channels, we obtain that for a small range of values of ϵ_k, $\alpha_{\epsilon_k}^{(f)} > \rho_k >> \alpha_{\epsilon_k}^*$, there exist constants $c_1, \left\{c_2 < \frac{e^{c_1}}{1+e^{c_1}}\right\}$ such that for all $\epsilon_k > c_1$ and $\rho_k > c_2$, any two-party ϵ_k-private ρ_k-accurate IND-CDP protocol implies the existence of oblivious transfer.

1.2 Related Work

Accuracy-privacy Tradeoffs in Differential Privacy The tradeoff between privacy and accuracy is quite central in designing differentially private algorithms. As mentioned before, in the client-server setting (where a single trusted server owns the entire database), the work of Dinur and

Nissim [DN03] first showed limitations for a wide class of private algorithms. These limitations were further explored in [DMT07, DY08, KRSU10].

The work of Dwork et al. [Dwo06, DMNS06] proposed generic techniques for differentially private function evaluation, based on adding noise as a function of the sensitivity of database queries. The optimality of such techniques was studied in various settings in [DNR+09, HT10, UV11]. Variants of these techniques were shown to be optimal for certain classes of queries by [GRS09, GS10], and were shown to be non-optimal for other classes by Brenner and Nissim [BN10].

As mentioned before, there has also been a significant amount of work characterizing the accuracy of two-party differentially-private protocols. McGregor et al. [MMP+10] first showed that information theoretically, a large accuracy loss is inherent to the distributed differentially private computation of functionalities such as the inner product and hamming distance over n-bit inputs. This was followed by the work of Goyal et al. [GMPS13] who showed large gaps in the client-server and two-party accuracies for the differentially-private computation of boolean functionalities. Finally, the works of Haitner et al. [HOZ13] and Khurana et al. [KMS14] showed that it is impossible to use one-way functions or even key-agreement in a black-box way to bridge any of these accuracy gaps. Our work subsumes these results for the case of XOR.

Cryptographic Completeness of Finite Two-Party Ideal Functionalities There has been a bulk of work on the complexity of two-party functionalities in the information theoretic setting [Kil88, CK89, Kus89, Kil91, Kil00, KKMO00, KMQR09, MPR09]. Chor and Kushilevitz [CK91] established that all Boolean functions either reduce to SFE or can be trivially simulated. In the computationally bounded setting, Maji et al. [MPR10] give a complete characterization of deterministic two-party finite functionalities while a series of works [MPR12, KMPS14, KKM+15] give an information-theoretic characterization of (randomized, fixed-role) two-party functionalities.

Note that all constant communication protocols for Boolean functionalities can be viewed as two-party ideal finite functionalities, and therefore characterized according to [MPR12, KMPS14, KKM+15]. Yet, our characterization extends to any polynomial-round *protocols* for differentially private computation with optimal accuracy, of certain classes of Boolean functionalities. This requires extra techniques to account for the entire transcript of protocol execution, which may leak information over and above the output of the ideal functionality.

1.3 Technical Overview

We consider the simple setting of distributed differentially private evaluation of boolean functions. Alice and Bob, with inputs x and y respectively, execute a protocol to compute a Boolean function $f(x, y)$. The protocol must preserve privacy (according to the differential privacy guarantee) of the input of each party. We know that any non-trivial Boolean function must embed an AND or an XOR minor on adjacent inputs. we focus on the XOR functionality; and our proof directly extends to any functionality with an embedded XOR on adjacent inputs. We also consider protocols with perfect agreement, that is, where Alice and Bob always get the same output at the end of the protocol (which is equivalent to saying that the output is part of the transcript). However, our proof also extends to protocols where parties agree on the output with overwhelming probability.[2]

Our main idea will be to use any protocol that implements the XOR functionality to construct an ideal noisy channel. An ideal noisy channel with flip probability $p < 1/2$ is a functionality that takes input a bit X from the sender, samples an independent bernoulli random variable (the 'error') E, where $E \sim \mathsf{Ber}(p)$, computes $\tilde{X} = (X \oplus E)$ and outputs it to the receiver.

[2]Note that if we relax the requirement of output agreement, then in fact there is a simple information theoretically secure protocol which would achieve optimal accuracy.

Consider an optimally accurate differentially private evaluation of the boolean XOR functionality, where both parties agree on the output with overwhelming probability. In this case, the output of the differentially private functionality can be interpreted as a "noisy" version of the correct output. In the optimally accurate setting, the probability that the output is correct is exactly $\alpha_\epsilon = \frac{e^\epsilon}{1+e^\epsilon}$. In other words, let Z denote the output of the protocol, then for all inputs X, Y; the output $Z = (X \oplus Y) \oplus E$, where E is a bernoulli random variable $E \sim \mathsf{Ber}(\frac{1}{1+e^\epsilon})$.

Our protocol to realize a noisy channel is simple: the sender (Alice) and receiver (Bob) sample independent random (private) input bits $X \xleftarrow{\$} \{0,1\}$ and $Y \xleftarrow{\$} \{0,1\}$. They invoke the differentially private protocol for XOR with inputs (X, Y) and obtain output Z, where $Z = (X \oplus Y) \oplus E$, and E is the error as defined above. The sender outputs X and the receiver outputs $Z \oplus Y (= X \oplus E)$. It is easy to see that this protocol *correctly* implements a noisy channel with noise $E \sim \mathsf{Ber}(\frac{1}{1+e^\epsilon})$. However observe that the underlying differentially private protocol for XOR may not be an ideal secure computation protocol for the noisy XOR functionality. In particular, the protocol transcript may leak information which is in addition to the official output. The only privacy guarantee we may rely upon comes from the differential privacy condition. Thus, it remains to prove that the above noisy channel is an "ideal" noisy channel.

In the computational setting, the ideal noisy channel functionality can be realized by a protocol with the following security properties [CK88]. Roughly, no efficient distinguisher on the sender's end, or on the receiver's end respectively, should be able to distinguish the cases when the error E was 0 from when E was 1. More formally, let $\mathcal{D}_R$ denote a distinguisher that obtains the entire view of the receiver, and $\mathcal{D}_S$ denote a distinguisher that obtains the entire view of the sender at the end of the protocol. Then, for any non-uniform PPT distinguisher $\mathcal{D}_R$, the following security guarantee must hold: $\Pr[\mathcal{D}_R = 1 | E = 0] - \Pr[\mathcal{D}_R = 1 | E = 1] = \mathsf{negl}(k)$ over the randomness of the protocol. Symmetrically, at the sender's end, for any non-uniform PPT distinguisher $\mathcal{D}_S$, $\Pr[\mathcal{D}_S = 1 | E = 0] - \Pr[\mathcal{D}_S = 1 | E = 1] = \mathsf{negl}(k)$ over the randomness of the protocol. Here $\mathsf{negl}(\cdot)$ denotes some function that is asymptotically smaller than the inverse of any polynomial function, and k denotes the security parameter.

The challenge now is to prove that no efficient distinguisher on the sender side, or on the receiver side, can distinguish the case when $E = 0$ from the case when $E = 1$. Here, we use the following properties of the optimally accurate differentially private XOR functionality.

- Because of optimal accuracy, the protocol output is correct with probability exactly $\frac{e^\epsilon}{1+e^\epsilon}$.
- The (ideal, non-noisy) XOR functionality is fully informative: its output along with any of the parties' inputs, can be used to correctly compute the input of the other party.

Since the protocol is optimally accurate, the protocol output is the same as the correct XOR of both parties' inputs (that is, $Z = X \oplus Y$) with probability exactly $\alpha_\epsilon = \frac{e^\epsilon}{1+e^\epsilon}$. Moreover, by the full-informative property of XOR, the correct output, together with the input of any party can be used to compute correctly the other party's input. In other words, for all X, Y, the noisy output Z of the differentially private protocol, together with the input Y, helps compute a guess for the other party's input that is correct with probability at least α_ϵ ($Z \oplus Y$ equals X with probability α_ϵ).

Note that if a party could guess the other party's input with probability any better than α_ϵ, this would directly violate differential privacy. Therefore, the output already allows computing the best possible guess (upto differential privacy limits) for the other party's input. Informally, this means that any extra information about the error (say, leaked from the transcript) could be used to obtain a better guess of the other party's input and directly violate differential privacy. To prove security of our noisy channel, we must formalize these arguments. This is done in Section 3 and forms the core of our proof of security.

2 Preliminaries

Notation. Let $\pi := \langle A, B \rangle$ be a two-party protocol. Let $\mathsf{view}_\pi^P(x, y)$ be the random variable which, in a random execution of π with inputs x, y for $P \in \{A, B\}$ respectively, consists of (x, R_P, trans), where R_P is the randomness used by party P and trans is the sequence of messages exchanged between the parties in the sampled execution.

Let out_P be the function applied by party P on $\mathsf{view}_\pi^P(x, y)$ to obtain the output for P, $\mathsf{out}_P(\mathsf{view}_\pi^P(x, y))$. We say that the protocol is *symmetric* if both parties receive the same output, i.e., for every x, y: $\mathsf{out}_A(\mathsf{view}_\pi^A(x, y)) = \mathsf{out}_B(\mathsf{view}_\pi^B(x, y))$. This is called the *official* output of the protocol, denoted by $\mathsf{out}_\pi(x, y)$. For the rest of this paper, we consider only symmetric protocols, however we note that our results can be easily extended to protocols in which both parties agree on the output with overwhelming probability.

In this paper, we work in the computational setting, and consider the family of protocols $\{\pi_k\}_{k\in\mathbb{N}}$, where k is the security parameter. Then, the view of party $P \in \{A, B\}$ is denoted by $\mathsf{view}_\pi^P(k, x, y)$.

2.1 Noisy Channels.

Informally, a noisy channel takes as input a bit b and outputs a bit $b' = b \oplus e$ where *error bit* $e \sim \mathsf{Ber}(1-\rho)$ is sampled independently, and $\oplus$ is the bitwise exclusive-or operation. The security requirement, roughly speaking, is that the error bit e remains "semantically secure" from all parties using the noisy channel. Somewhat counterintuitively, the flip probability of a ρ-noisy channel is $(1-\rho)$. This is done deliberately to match DP protocols.

For succinctness, we will directly define a (ρ, α, β)-Weak Binary Symmetric Channel. A noisy channel is a $(\rho, 0, 0)$-Weak Binary Symmetric Channel; it is defined again in Appendix A.

Definition 1 (Passive (ρ, α, β)-Weak Binary Symmetric Channel [DKS99, Wul09])**.** A protocol family $\{\pi_k := \langle S, R \rangle(1^k)\}_{k\in\mathbb{N}}$ implements a (ρ, α, β) Passive Weak Binary Symmetric Channel for $\rho > 1/2$ if S and R are PPT algorithms and the following holds.

The input of party S, called the sender is a bit $b \leftarrow \{0, 1\}$ and the input of the receiver R is nothing. At the end of the protocol, the receiver R obtains output bit $b' \in \{0, 1\}$, while the output of the sender S is nothing — denoted by $\perp$. A dishonest sender or dishonest receiver may receive additional information. Then, the following conditions must be satisfied:

Correctness: For all $k \in \mathbb{N}$, $|\Pr[b = b'|b] - \rho| \leq \mathsf{negl}(k)$. The bit flip probability is $(1-\rho)$.

Sender Security: Informally, the receiver R can distinguish the case when error $e := b' \oplus b$ is 0 from when it is 1 with probability not more than q. Formally, there exists a negligible function $\mathsf{negl} : \mathbb{N} \to \mathbb{R}$ such that for every non-uniform PPT algorithm ("distinguisher") $\mathcal{D}_R$, every polynomial $p(\cdot)$, every sufficiently large $k \in \mathbb{N}$, every advice string z_k of size at most $p(k)$, all $b' \in \{0, 1\}$ and *uniformly chosen* $b \xleftarrow{\$} \{0, 1\}$, we have:

$$\mathsf{Adv}_\pi^R(k) := \left|\left(\Pr[\mathcal{D}_R(z_k, \mathsf{view}_\pi^R(k, b)) = 1 | e = 0] \; - \; \Pr[\mathcal{D}_R(z_k, \mathsf{view}_\pi^R(k, b)) = 1 | e = 1]\right)\right| \leq \beta + \mathsf{negl}(k),$$

where $e = b' \oplus b$; $b' = \mathsf{out}(\mathsf{view}_\pi^R(k, b))$ and the probability is over the randomness of b, π and $\mathcal{D}_R$.

Receiver Security: This is defined symmetrically. Formally, there exists a negligible function $\mathsf{negl} : \mathbb{N} \to \mathbb{R}$ such that for every non-uniform PPT algorithm ("distinguisher") $\mathcal{D}_S$, every polynomial $p(\cdot)$, every sufficiently large $k \in \mathbb{N}$, every advice string z_k of size at most $p(k)$, and *uniformly chosen* $b \xleftarrow{\$} \{0, 1\}$ we have:

$$\mathsf{Adv}_\pi^S(k) := \left|\left(\Pr[\mathcal{D}_S(z_k, \mathsf{view}_\pi^S(k, b)) = 1 | e = 0] \; - \; \Pr[\mathcal{D}_S(z_k, \mathsf{view}_\pi^S(k, b)) = 1 | e = 1]\right)\right| \leq \alpha + \mathsf{negl}(k),$$

where e was defined above, and the probability is over the randomness of b, π and $\mathcal{D}_S$.

A protocol implementing the noisy channel defined here, is sufficient to implement the semi-honest OT functionality defined above. A reduction between these two primitives was first given by Crépeau and Kilian [CK88], and a sketch of their reduction can be found in Appendix A.

Furthermore, a protocol implementing the weak binary symmetric channel defined above is sufficient to implement the OT functionality. This reduction is from [Wul09], but the possibility results for obtaining OT from weak BSC hold over only a small range of values of these parameters. More specifically, we use the following corollary:

Corollary 1. *[Wul09] Let ρ, α, β be constants, and let $\bar{\epsilon} = \frac{(1-\rho)^2}{(1-\rho)^2+\rho^2}$. If at least one of the conditions $2\alpha + \beta + \bar{\epsilon} \leq 0.12$, or $\beta + \bar{\epsilon} < \frac{(1-\alpha)^4}{44}$, or $88\alpha + 44\bar{\epsilon} < (1-\beta)^2$, or $196\alpha + 98\beta + \frac{49}{2} < (1-2\bar{\epsilon})^2$ holds, then there exists a protocol that uses (ρ, α, β)-passive Weak BSC and efficiently implements OT secure in the semi-honest model.*

2.2 Differential Privacy

First, we present the formal definition of differential privacy [Dwo06] and its computational variant that we will use [MPRV09].

Definition 2 (ϵ-Differential Privacy)**.** We say that a randomized function $M : \{0,1\}^n \mapsto \mathcal{R}$, with a finite range $\mathcal{R}$, is an ϵ-differentially-private (ϵ-DP) mechanism for $\epsilon \geq 0$ if for every $(x, x') \in \{0,1\}^n \times \{0,1\}^n$ satisfying $|x - x'|_h = 1$ and every subset $S \subset \mathcal{R}$ we have that over the randomness of M:

$$\Pr[M(x) \in S] \leq e^{\epsilon} \times \Pr[M(x') \in S].$$

Definition 3 (ϵ-Indistinguishable-Computational Differential Privacy)**.** We say that an ensemble $\{M_k\}_{k\in\mathbb{N}}$ of randomized functions $M_k : \{0,1\}^n \mapsto \mathcal{R}_k$ with finite range $\mathcal{R}_k$, provides ϵ_k-IND-CDP if there exists a negligible function $\mathsf{negl} : \mathbb{N} \mapsto \mathbb{R}$ such that for every non-uniform PPT algorithm ("distinguisher") D, every polynomial $p(\cdot)$, every sufficiently large $k \in \mathbb{N}$, every $(x, x') \in \{0,1\}^n \times \{0,1\}^n$ satisfying $|x - x'|_h = 1$, and every advice string z_k of size at most $p(k)$ it holds that:

$$\Pr[D_k(M_k(x)) = 1] \leq e^{\epsilon_k} \times \Pr[D_k(M_k(x')) = 1] + \mathsf{negl}(k),$$

where we write $D_k(y)$ for $D(1^k, z_k, y)$ and the probability is taken over the randomness of mechanism M_k and distinguisher D_k.

Note that assuming dense sets, this is a strictly weaker definition than other (simulation-based) definitions of CDP [MPRV09]. Therefore, our reductions will automatically extend to other simulation-based definitions.

Next, we define what it means for a protocol to be differentially private over a subset of transcripts, and we recall the definitions of two-party differential privacy and accuracy of two-party DP protocols, from [MPRV09, GMPS13].

Definition 4 (Differential Privacy over a subset of transcripts)**.** We say that an ensemble $\{M_k\}_{k\in\mathbb{N}}$ of randomized functions $M_k : \{0,1\}^n \mapsto \mathcal{R}_k$ with finite range $\mathcal{R}_k$, provides ϵ_k-IND-CDP over some subset of executions $\mathbb{S}$ if there exists a negligible function $\mathsf{negl} : \mathbb{N} \mapsto \mathbb{R}$ such that for every non-uniform PPT algorithm ("distinguisher") D, every polynomial $p(\cdot)$, every sufficiently large $k \in \mathbb{N}$, every adjacent pair $(x, x') \in \{0,1\}^n \times \{0,1\}^n$, and every advice string z_k of size $\leq p(k)$:

$$\Pr[D_k(M_k(x)) = 1 \wedge (M_k(x) \in \mathbb{S}_k)] \leq e^{\epsilon_k} \times \Pr[D_k(M_k(x')) = 1 \wedge (M_k(x') \in \mathbb{S}_k)] + \mathsf{negl}(k)$$

where we write $D_k(y)$ for $D(1^k, z_k, y)$ and the probability is taken over the randomness of mechanism M_k and distinguisher D_k.

Definition 5 (Two-Party Differential Privacy)**.** Let $\pi : \langle A, B\rangle$ be a protocol where the inputs of A and B are in $\{0,1\}^n$. We say that π provides ϵ-DP if: (1) for every $x \in \{0,1\}^n$ the mechanism represented by the function $\mathsf{view}^A_\pi(x,\cdot)$ over the inputs $y \in \{0,1\}^n$ is ϵ-DP, and (2) for every $y \in \{0,1\}^n$ the mechanism represented by $\mathsf{view}^B_\pi(\cdot, y)$ over the inputs $x \in \{0,1\}^n$ is ϵ-DP.

In the two-party computational setting, ϵ_k-IND-CDP is defined analogously. Formally, let $\{\pi_k := \langle A, B\rangle(1^k)\}_{k\in\mathbb{N}}$ be an ensemble of interactive functions where the inputs of A and B are in $\{0,1\}^n$. We say that $\{\pi_k\}_{k\in\mathbb{N}}$ provides ϵ_k-IND-CDP if: (1) for every $x \in \{0,1\}^n$ the ensemble $\{\mathsf{view}^A_\pi(k, x, \cdot)\}_k$ provides ϵ_k-IND-CDP over the inputs $y \in \{0,1\}^n$, and (2) for every $y \in \{0,1\}^n$ the ensemble $\{\mathsf{view}^A_\pi(k, \cdot, y)\}_k$ provides ϵ_k-IND-CDP over the inputs $x \in \{0,1\}^n$.

Definition 6 (Accuracy in Differential Privacy [GMPS13])**.** The *accuracy* of a randomized Boolean mechanism $M : \{0,1\}^n \mapsto \{0,1\}$ with respect to a Boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ is defined as: $\mathsf{Acc}_f(M) = \min_x\{\Pr[M(x) = f(x)]\}$, where the probability is taken over the randomness of M.

The accuracy of a *symmetric* two-party protocol $\pi := \langle A, B\rangle$ w.r.t. $f : \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}$ is defined as the accuracy of the (Boolean) mechanism $\mathsf{out}_\pi : \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}$; where out_π returns the official output. Accuracy in the computational setting is defined analogously.

Optimal accuracy for Boolean functions. We list the following facts about the accuracy of DP mechanisms that compute a Boolean function:

1. For every Boolean mechanism $M : \{0,1\}^n \mapsto \{0,1\}$ and every Boolean function $f : \{0,1\}^n \mapsto \{0,1\}$, if M is ϵ-DP then: $\mathsf{Acc}_f(M) \leq \frac{\lambda}{1+\lambda}$ where $\lambda = e^\epsilon$.[3] We call the bound $\rho = \frac{\lambda}{1+\lambda}$, the *optimal* accuracy, which is achieved by setting $M(x) = f(x) \oplus e$ such that $\Pr[e = 0] = \frac{\lambda}{1+\lambda}$.

2. If M only satisfies ϵ-IND-CDP, then there exists a negligible function $\mathsf{negl}(\cdot)$ such that $\mathsf{Acc}_f(M) \leq \frac{\lambda}{1+\lambda} + \mathsf{negl}(k)$.

3. If a symmetric protocol ensemble $\{\pi_k\}_{k\in\mathbb{N}}$ provides ϵ-IND-CDP for a constant $\epsilon > 0$, then the accuracy of this ensemble w.r.t. the XOR function is at most $\frac{\lambda+\mathsf{negl}(k)}{1+\lambda} = \rho + \mathsf{negl}'(k)$ for constant ϵ. The accuracy ρ can be achieved using secure two-party computation [MPRV09].

3 Noisy Channels Reduce to Optimal Two-Party IND-CDP

Theorem 1. *If there exists a two-party ϵ_k-IND-CDP protocol with accuracy $\rho_k(> 1/2)$ such that $\rho_k \geq \frac{e^\epsilon_k}{1+e^{\epsilon_k}}$ with respect to the* exclusive-or *function for a constant $\epsilon_k > 0$, then there exists a protocol implementing the ρ_k-noisy-channel functionality.*

Proof. Let $\{\pi_k\}_k$ where $\pi_k = \langle A, B\rangle(1^k)$ be an ensemble of ϵ_k-IND-CDP protocols for computing the XOR function with accuracy $\rho_k \geq \frac{\lambda}{1+\lambda}$ where $\lambda = e^{\epsilon_k}$, and $\epsilon_k > 0$ is a constant. Note that since the protocol is ϵ_k-IND-CDP and $\epsilon_k > 0$, we have that $\rho_k \leq \frac{\lambda}{1+\lambda} + \mathsf{negl}(k)$ for some negligible function $\mathsf{negl}(k)$. We abuse notation for the rest of the proof and denote ϵ_k by ϵ, and ρ_k by ρ.

The following protocol ensemble $\{\pi_k := \langle S, R\rangle(1^k)\}_k$ implements a ρ-noisy-channel:

1. S receives bit x as input, and R has no input. R samples a random bit y and the parties execute the ϵ-IND-CDP protocol $\langle A(x), B(y)\rangle(1^k)$ and obtain the (same) bit z as official output of this protocol.

[3] Informally, if this is not the case, then there exists a distinguisher such that the ratio between the probability that it guesses the input correctly versus incorrectly is greater than e^ϵ, thereby violating ϵ-DP.

2. R outputs $\tilde{x} = z \oplus y$ and S outputs $\perp$.

The correctness of this protocol follows directly from the accuracy of the ϵ-IND-CDP protocol. We now show that it satisfies sender-security.

Sender security. Assume to the contrary, that the protocol does not satisfy sender-security. That is, there exists a non-uniform PPT distinguisher $\mathcal{D}_R$, a fixed polynomial $q(\cdot)$, and infinitely many values k for which (there exists a polynomial-sized advice string z_k such that) $\mathsf{Adv}^R_\pi(k) \geq 1/q(k)$. Fix one such k from now on and let:

$$p_k = \Pr[\mathcal{D}_R(z_k, \mathsf{view}^R_{\pi_k}) = 1 | e = 0] - \Pr[\mathcal{D}_R(\mathsf{view}^R_{\pi_k}) = 1 | e = 1]. \tag{2}$$

Note that $\mathsf{Adv}^R_\pi(k) = |p_k|$. Without loss of generality, let $p_k > 0$ for this k, and therefore by assumption $p_k \geq 1/q(k)$. We abuse notation and write $\mathcal{D}_R = 1$ to denote the event that $\mathcal{D}_R(1^k, z_k, \mathsf{view}^R_{\pi'_k}) = 1$.[4] Since $p_k \neq 0$ we must have that $0 < \Pr[\mathcal{D}_R = 1] < 1$.

Let e be the random variable denoting the *error bit* for the ϵ-IND-CDP protocol. That is, for the ϵ-IND-CDP protocol, $e = \tilde{x} \oplus x$. Since we are in the computational setting, the accuracy of the protocol may be different for each input, denoted by: $\rho_{00}, \rho_{01}, \rho_{10}, \rho_{11}$. However, they must all be within a negligible distance from each other and therefore lie within the interval $[\rho_1 - \mathsf{negl}(k), \rho_1 + \mathsf{negl}(k)]$. Since a correct output is equivalent to $e = 0$, and each input is selected with equal probability, $\Pr[e = 0]$ (which is equivalent to "average" accuracy) also lies in the same interval. We show that if p_k is noticeable then differential privacy is violated on the set of transcripts where $\mathcal{D}_R$ outputs 1.

Claim 1. $\Pr[e = 0 \wedge \mathcal{D}_R = 1] > e^\epsilon \times \Pr[e = 1 \wedge \mathcal{D}_R = 1] + \frac{p_k}{2}$.

Proof. Let $\Pr[e = 0] = \rho^*$, and $\mu(k)$ be a negligible function so that $\rho^* = \frac{\lambda}{1+\lambda} + \mu(k) > 1/2$.
In addition, $\Pr[e = 0]/\Pr[e = 1]$ is equal to $\rho^*/(1 - \rho^*) = \lambda + \mu'(k)$ for some negligible function μ'. Now, since $\Pr[\mathcal{D}_R = 1] \neq 0$, we can write (using Bayes' rule):

$$\begin{aligned}
&\Pr[e = 0 \wedge \mathcal{D}_R = 1] \\
&= \Pr[\mathcal{D}_R = 1 | e = 0] \times \Pr[e = 0] \\
&= \big(p_k + \Pr[\mathcal{D}_R = 1 | e = 1]\big) \times \Pr[e = 0] && \text{(By equation 2)} \\
&= \left(p_k + \frac{\Pr[e = 1 | \mathcal{D}_R = 1] \times \Pr[\mathcal{D}_R = 1]}{\Pr[e = 1]}\right) \times \Pr[e = 0] && \text{(Bayes' rule)} \\
&= p_k \cdot \Pr[e = 0] + \Pr[e = 1 \wedge \mathcal{D}_R = 1] \times \frac{\Pr[e = 0]}{\Pr[e = 1]}
\end{aligned}$$

Note that: $p_k \cdot \Pr[e = 0] = p\rho^* > p_k/2$, and $\frac{\Pr[e=0]}{\Pr[e=1]} = \frac{\rho^*}{1-\rho^*} = \lambda + \mu'(k) > \lambda$. Therefore, $\Pr[e = 0 \wedge \mathcal{D}_R = 1] > \frac{p_k}{2} + \lambda \cdot \Pr[e = 1 \wedge \mathcal{D}_R = 1]$. □

Claim 2. *If* $\Pr[e = 0 \wedge \mathcal{D}_R = 1] > e^\epsilon \times \Pr[e = 1 \wedge \mathcal{D}_R = 1] + \frac{p_k}{2}$ *is such that* $\frac{p_k}{2}$ *is non-negligible over the randomness of uniformly chosen sender input* $x = b$*, then the protocol ensemble* $\{\pi_k\}_k$ *does not preserve* ϵ*-IND-CDP on the PPT-checkable subset of transcripts satisfying* $\mathcal{D}_R = 1$.

[4] Note that the input of the sender in sampling view $\mathsf{view}^R_{\pi'_k}$ is uniformly chosen by definition of sender-security; and further, since k has been fixed, letting $\mathcal{D}_R := \mathcal{D}_R(1^k, z_k, \mathsf{view}^R_{\pi'_k})$ is unambiguous and well defined. Note that now, $p_k = \Pr[\mathcal{D}_R = 1 | e = 0] - \Pr[\mathcal{D}_R = 1 | e = 1]$.

Proof. From $\Pr[e = 0 \wedge \mathcal{D}_R = 1] > e^{\epsilon} \Pr[e = 1 \wedge \mathcal{D}_R = 1] + \frac{p_k}{2}$ it follows that $\Pr[\tilde{x} = x \wedge \mathcal{D}_R = 1] > e^{\epsilon} \Pr[\tilde{x} \neq x \wedge \mathcal{D}_R = 1] + \frac{p_k}{2}$, over the randomness of x where $\tilde{x}$ denotes the output of the receiver. Since x is *uniformly chosen* in $\{0, 1\}$,

$$\begin{aligned}&\Pr[\tilde{x} = 1 \wedge \mathcal{D}_R = 1 | x = 1] + \Pr[\tilde{x} = 0 \wedge \mathcal{D}_R = 1 | x = 0] \\ &> e^{\epsilon} \left(\Pr[\tilde{x} = 0 \wedge \mathcal{D}_R = 1 | x = 1]\right) + e^{\epsilon} \left(\Pr[\tilde{x} = 1 \wedge \mathcal{D}_R = 1 | x = 0]\right) + \frac{p_k}{2}\end{aligned}$$

Now, it is easy to observe that *either* of the following statements hold.

1. $\Pr[\tilde{x} = 1 \wedge \mathcal{D}_R = 1 | x = 1] > e^{\epsilon} \times \Pr[\tilde{x} = 1 \wedge \mathcal{D}_R = 1 | x = 0] + \frac{p_k}{4}$ OR,
2. $\Pr[\tilde{x} = 0 \wedge \mathcal{D}_R = 1 | x = 0] > e^{\epsilon} \times \Pr[\tilde{x} = 0 \wedge \mathcal{D}_R = 1 | x = 1] + \frac{p_k}{4}$

In either case, it is possible to claim the existence of a distinguisher. If p_k is noticeable and statement 1 holds, then there exists a distinguisher $D_k^{1'}$ with output equal to receiver output $\tilde{x}$, which violates IND-CDP over the PPT checkable subset corresponding to $\mathcal{D}_R = 1$. On the other hand, if p_k is noticeable and statement 2 is true, then there exists a distinguisher $D_k^{2'}$ with output equal to $1 - \tilde{x}$, which violates IND-CDP over the PPT checkable subset corresponding to $\mathcal{D}_R = 1$.

It follows from this claim that if p_k is noticeable, then the protocol ensemble $\{\pi_k\}_k$ does not preserve ϵ-IND-CDP on the PPT-checkable subset of transcripts on which $\mathcal{D}_R = 1$, because there exists distinguisher $D'_k \in \{D_k^{1'}, D_k^{2'}\}$ and a corresponding pair of inputs $(x^*, x^{*\prime}) \in (\{0, 1\} \times \{0, 1\})$ such that $\Pr[D'_k = 1 \wedge \mathcal{D}_R = 1 | x = x^*] > e^{\epsilon} \times \Pr[D'_k = 0 \wedge \mathcal{D}_R = 1 | x = x^{*\prime}] + \frac{p_k}{4}$. In other words, there exists a non-uniform distinguisher that violates ϵ-IND-CDP on this subset. This proves the claim. □

Lemma 1. *A two-party protocol ensemble that provides ϵ-IND-CDP over all executions also provides ϵ-IND-CDP over any PPT-checkable subset of executions.*

Proof. Assume to the contrary that there exists a two-party ϵ-IND-CDP protocol for which there is a non-uniform PPT distinguisher D'_k that violates ϵ-IND-CDP over some PPT-checkable subset of executions (denoted by $\mathbb{S}_k$). Let S_k denote the code of a PPT-checking algorithm that returns 1 if some execution $M_k(x) \in \mathbb{S}_k$, and 0 otherwise.

Then, we construct a non-uniform PPT distinguisher D_k (Figure 1) that accepts S_k, D'_k as advice z_k, and violates ϵ-IND-CDP for the protocol.

Figure 1: Algorithm for ϵ-IND-CDP Distinguisher D_k

1. Obtain inputs $M_k(x)$, S_k, D'_k
2. If $S_k(M_k(x)) = 1$, $D_k(M_k(x)) = D'_k(M_k(x))$
3. If $S_k(M_k(x)) \neq 1$, $D_k(M_k(x)) = 0$

We know that for some polynomial $p(\cdot)$, some sufficiently large $k \in \mathbb{N}$, some $(x^*, x^{*\prime}) \in \{0, 1\} \times \{0, 1\}$, some advice string z'_k of size at most $p(k)$ and all functions $\mathsf{negl} : \mathbb{N} \mapsto \mathbb{R}$, it holds that:

$$\Pr[D'_k(M_k(x^*)) = 1 \wedge (M_k(x^*) \in \mathbb{S}_k)] > e^{\epsilon_k} \Pr[D'_k(M_k(x^{*\prime})) = 1 \wedge (M_k(x^{*\prime}) \in \mathbb{S}_k)] + \mathsf{negl}(k)$$

where the probability is taken over the randomness of mechanism M_k and distinguisher D'_k, and $D'_k(y)$ represents $D'(1^k, z_k, y)$. Then, by a simple manipulation we have:

$$\begin{aligned}
\Pr[D_k(M_k(x^*)) = 1] &= \Pr[D_k(M_k(x^*)) = 1 \wedge (M_k(x^*) \in \mathbb{S}_k)] \\
&> e^{\epsilon} \Pr[D'_k(M_k(x^{*\prime})) = 1 \wedge (M_k(x^{*\prime}) \in \mathbb{S}_k)] + \mathsf{negl}(k) \\
&= e^{\epsilon} \Pr[D_k(M_k(x^{*\prime})) = 1 \wedge (M_k(x^{*\prime}) \in \mathbb{S}_k)] \\
&\quad + e^{\epsilon} \Pr[D_k(M_k(x^{*\prime})) = 1 \wedge (M_k(x^{*\prime}) \notin \mathbb{S}_k)] + \mathsf{negl}(k) \\
&= e^{\epsilon} \Pr[D_k(M_k(x^{*\prime})) = 1] + \mathsf{negl}(k)
\end{aligned}$$

Therefore, we have a non-uniform PPT distinguisher D_k such that for some polynomial $p(\cdot)$, some sufficiently large $k \in \mathbb{N}$, for the same $(x^*, x^{*\prime}) \in (\{0,1\} \times \{0,1\})$, some advice string z_k of size at most $p(k)$ and all functions $\mathsf{negl} : \mathbb{N} \mapsto \mathbb{R}$ it holds that:

$$\Pr[D'_k(M_k(x^*)) = 1] > e^{\epsilon_k} \times \Pr[D'_k(M_k(x^{*\prime})) = 1] + \mathsf{negl}(k)$$

This completes the proof of the lemma. □

Combining the claims above, we observe that if p_k is noticeable, then the protocol ensemble $\{\pi_k\}_k$ does not preserve ϵ-IND-CDP. This is a contradiction, therefore $p_k = \mathsf{negl}(k)$, and the noisy channel is sender-secure.

Receiver security. The output z of the ϵ-IND-CDP-protocol, obtained by both parties, is symmetric with respect to the input of each party. Moreover, since the inputs of both parties are chosen uniformly at random, the security of the receiver follows in a manner similar to sender security. This completes the proof of the theorem.
The following is corollary of Theorem 1 and Crépeau-Kilian's reduction [CK88] of OT to noisy channels (See section A):

Corollary 2. *If there exists a two-party ϵ_k-IND-CDP protocol with accuracy ρ_k such that $\rho_k \geq \frac{e^{\epsilon_k}}{1+e^{\epsilon_k}}$ with respect to the* exclusive-or *function for a constant $\epsilon_k > 0$, then there exists an ensemble of protocols implementing the semi-honest oblivious-transfer functionality in the computational setting.*

4 Noisy Channels Reduce to Non-Optimal Two-Party IND-CDP

Theorem 2. *If there exists a two-party ϵ_k-IND-CDP protocol with non-optimal accuracy $\rho_1 \leq \frac{e^{\epsilon_k}}{1+e^{\epsilon_k}}$ with respect to the* exclusive-or *function for a constant $\epsilon_k > 0$, then there exists a protocol implementing the $(\rho_1, \frac{\lambda}{m} - 1, \frac{\lambda}{m} - 1)$-passive weak binary symmetric channel functionality where $\rho_1 > 1/2$, $\lambda = e^{\epsilon_k}$ and $m = \frac{\rho_1}{1-\rho_1}$.*

Proof: Let $\{\pi_k\}_k$ where $\pi_k = \langle A, B \rangle(1^k)$ be an ϵ-IND-CDP protocol for computing the XOR function with accuracy $\rho \leq \frac{\lambda}{1+\lambda}$ where $\lambda = e^{\epsilon}$, and $\epsilon > 0$ is a constant. Again we shall abbreviate ϵ_k by ϵ. Note again that since the protocol is ϵ-IND-CDP and $\epsilon > 0$ is a constant, we have that $\rho_1 \leq \frac{\lambda}{1+\lambda} + \mathsf{negl}(k)$ for some negligible function $\mathsf{negl}(k)$.

The following protocol ensemble $\{\pi_k := \langle S, R \rangle(1^k)\}_k$ implements a $(\rho_1, \frac{\lambda}{m} - 1, \frac{\lambda}{m} - 1)$-passive weak binary symmetric channel:

1. S receives a bit x as input, and R has no input. R samples a bit y uniformly at random in $\{0,1\}$ and the parties execute the ϵ-IND-CDP protocol $\langle A(x), B(y) \rangle(1^k)$ and obtain the (same) bit z as official output this protocol.

2. R outputs $\tilde{x} = z \oplus y$ and S outputs $\perp$.

The correctness of this protocol follows directly from the accuracy of the ϵ-IND-CDP protocol. We now show that it satisfies the sender-security requirement.

Sender security. Assume to the contrary, that the protocol does not satisfy sender-security. That is, there exists a non-uniform PPT distinguisher $\mathcal{D}_R$, a fixed polynomial $q(\cdot)$, and infinitely many values k for which (there exists a polynomial-sized advice string z_k such that) $\mathsf{Adv}_R\pi'(k) \geq \left(\frac{\lambda}{m} - 1\right) + 1/q(k)$. Fix one such k from now on and let:

$$p_k = \left|\Pr[\mathcal{D}_R(z_k, \mathsf{view}^R_{\pi'_k}) = 1 | e = 0] - \Pr[\mathcal{D}_R(\mathsf{view}^R_{\pi'_k}) = 1 | e = 1]\right| - \left(\frac{\lambda}{m} - 1\right) \tag{3}$$

W.l.o.g., let $p_k > 0$ for this k, and therefore by assumption $p_k \geq 1/q(k)$. We also assume w.l.o.g. that $\left(\Pr[\mathcal{D}_R(z_k, \mathsf{view}^R_{\pi'_k}) = 1 | e = 0] - \Pr[\mathcal{D}_R(\mathsf{view}^R_{\pi'_k}) = 1 | e = 1]\right) > 0$, else we can use the distinguisher with output 0 to achieve the same effect. We abuse notation and use $\mathcal{D}_R = 1$ to denote the event that $\mathcal{D}_R(z_k, \mathsf{view}^R_{\pi'_k}) = 1$.[5] Note that since $p_k \neq 0$ we must have that $0 < \Pr[\mathcal{D}_R = 1] \leq 1$.

Let e be a random variable defining the *error bit* for the ϵ-IND-CDP protocol. That is, for the IND-CDP protocol, $e = \tilde{x} \oplus x$. Before going ahead, we make a few remarks. It holds like in the optimal XOR case, the accuracy of the protocol, possibly different for each input, lies within a negligible distance from $\frac{\lambda}{1+\lambda}$. Since a correct output is equivalent to $e = 0$, and each input is selected with equal probability, $\Pr[e = 0]$ (which is equivalent to "average" accuracy) also lies in the same interval. We show that if p_k is not negligible then differential privacy is violated on the set of transcripts where $\mathcal{D}_R$ outputs 1. That is,

Claim 3. $\Pr[e = 0 \wedge \mathcal{D}_R = 1] > e^\epsilon \times \Pr[e = 1 \wedge \mathcal{D}_R = 1] + \frac{p_k}{2}$.

Proof. Let $\Pr[e = 0] = \rho^*$, and $\mu(k)$ be a negligible function so that $\rho^* = \rho_1 + \mu(k) > 1/2$. In addition, $\Pr[e = 0]/\Pr[e = 1]$ is equal to $\rho^*/(1 - \rho^*) = m + \mu'(k)$ for $m = \rho_1/(1 - \rho_1)$ and some negligible function μ'.

Now, since $\Pr[\mathcal{D}_R = 1] \neq 0$, we can write (using Bayes' rule):

$$\begin{aligned}
&\Pr[e = 0 \wedge \mathcal{D}_R = 1] \\
&= \Pr[\mathcal{D}_R = 1 | e = 0] \times \Pr[e = 0] \\
&= \left(p_k + \frac{\lambda - m}{m} + \Pr[\mathcal{D}_R = 1 | e = 1]\right) \times \Pr[e = 0] && \text{(By equation 3)} \\
&= \left(p_k + \frac{\lambda - m}{m} + \frac{\Pr[e = 1 | \mathcal{D}_R = 1] \times \Pr[\mathcal{D}_R = 1]}{\Pr[e = 1]}\right) \times \Pr[e = 0] && \text{(Bayes' rule)} \\
&= p_k \cdot \Pr[e = 0] + \frac{\lambda - m}{m} \times \Pr[e = 0] + \Pr[e = 1 \wedge \mathcal{D}_R = 1] \times \frac{\Pr[e = 0]}{\Pr[e = 1]} \\
&= p_k \cdot \Pr[e = 0] + \Pr[e = 1 \wedge \mathcal{D}_R = 1] \times \frac{\Pr[e = 0]}{\Pr[e = 1]} \times \left(\frac{\lambda - m}{m} \cdot \frac{1}{\Pr[\mathcal{D}_R = 1 | e = 1]} + 1\right)
\end{aligned}$$

Note: $p_k \cdot \Pr[e = 0] = p\rho^* > p_k/2$, $\Pr[\mathcal{D}_R = 1 | e = 1] \leq 1$ and $\frac{\Pr[e=0]}{\Pr[e=1]} = \frac{\rho^*}{1-\rho^*} = m + \mu'(k) > m$. Therefore,

$$\Pr[e = 0 \wedge \mathcal{D}_R = 1] > \frac{p_k}{2} + \lambda \cdot \Pr[e = 1 \wedge \mathcal{D}_R = 1]$$

[5] Note that the input of the sender in sampling view $\mathsf{view}^R_{\pi'_k}$ is uniformly chosen by definition of sender-security; and further, since k has been fixed, letting $\mathcal{D}_R := \mathcal{D}_R(z_k, \mathsf{view}^R_{\pi'_k})$ is unambiguous and well defined. Alternatively, one can keep an indicator variable. Note that in this notation, $p_k = \Pr[\mathcal{D}_R = 1 | e = 0] - \Pr[\mathcal{D}_R = 1 | e = 1] - (\frac{\lambda}{m} - 1)$.

It follows from this claim, along with Claim 2 and Lemma 1 that if p_k is noticeable, then the protocol $\{\pi_k\}_k$ does not preserve ϵ-IND-CDP. This is a contradiction, therefore our noisy channel construction is sender-secure.

Receiver security. The output z of the ϵ-IND-CDP-protocol, obtained by both parties, is symmetric with respect to the input of each party. Moreover, since the inputs of both parties are chosen uniformly at random, receiver security follows in a manner similar to sender security. ∎

While our reduction to weak noisy channels holds for all parameters $\epsilon > 0$ and accuracies ρ, the range of parameters for which such channels give OT is small. The following corollary follows from Theorem 2, and Corollary 1 taken from [Wul09].

Corollary 3. *If there exists a two-party ϵ-IND-CDP protocol with non-optimal accuracy $\rho_k^1 \leq \frac{e^{\epsilon_k}}{1+e^{\epsilon_k}}$ with respect to the* exclusive-or *function for a constant $\epsilon > 0$, then there exist constants $c_1, \left\{c_2 < \frac{e^{c_1}}{1+e^{c_1}}\right\}$, such that for all $\epsilon_k > c_1$ and $\rho_k^1 > c_2$, there is a protocol implementing the semi-honest oblivious transfer functionality.*

5 Conclusion and Open Problems

5.1 Extension to Functionalities with an Embedded XOR

Recall that we say that a function f contains an embedded XOR on *adjacent* inputs if there exist adjacent inputs x_0, x_1, y_0, y_1 and outputs z_0, z_1 such that $f(x_1, y_b) = z_{\mathsf{XOR}(a,b)}$ for all $a, b \in \{0,1\}$. It is easy to observe that any finite functionality f with an embedded XOR, which can be computed with optimal accuracy restricted to its embedded XOR on adjacent inputs, can be used to obtain a differentially private optimally accurate XOR functionality over boolean inputs. Accuracy of XOR follows from the accuracy of the original functionality f, and privacy of XOR follows because differential privacy is a worst-case guarantee which must be maintained even when restricted to a single bit of the adjacent inputs. The resulting differentially private optimally accurate XOR protocol can then be used to obtain a secure noisy channel and therefore, perform oblivious transfer.

5.2 Conclusion

We give a partial characterization of differentially private protocols for functionalities, with optimal accuracies for any privacy parameter ϵ (and also close to optimal accuracies for a small range of privacy parameters). As our main result, we show that the differentially private evaluation, with optimal accuracy, of any functionality that contains an embedded XOR on adjacent inputs – can be used to obtain a semi-honest secure protocol for oblivious transfer. This result also extends to a small fraction of non-optimal accuracies for a small fraction of differential privacy parameters ϵ.

5.3 Open Problems

Characterizing All Functionalities. It remains an intriguing open problem to obtain a complete characterization of functionalities whose differentially private evaluation with optimal accuracy in a distributed setting, is cryptographically complete. It is interesting to obtain a complete characterization even for boolean functionalities, since the differentially private evaluation of any non-trivial functionality with optimal accuracy (such as the inner product and hamming distance functionalities considered by McGregor et al. [MMP+10]) implies the differentially private evaluation of a non-trivial boolean functionality with optimal accuracy.

Consider, the case of boolean AND. This functionality is interesting, because any non-trivial boolean functionality must contain embedded AND or XOR on adjacent inputs [CK89]. Therefore, for instance, showing that any (possibly polynomial round[6]) protocol that gives a differentially private protocol for the boolean AND functionality with optimal accuracy, is cryptographically complete – would imply the completeness of an optimally accurate distributed differentially private protocol for any non-trivial boolean functionality. However, unlike XOR, the AND functionality is not completely informative about the other party's input. In case the input of a party is 0, even a non-noisy output of the ideal AND functionality conveys absolutely no information about the input of the other party. In case the input is 1, the output allows to exactly compute the other party's input. Therefore, if a party has input 0, the differentially-private output would be completely useless for this party, while there could be additional leakage from the transcript (allowed by differential privacy) that we do not know how to use. Such functionalities seem to have interesting connections to weak versions of oblivious transfer, for which it is so far not known how to obtain oblivious transfer.

Characterizing non-optimal accuracies. From the works of McGregor et al. [MMP+10] and Goyal et al. [GMPS13] in the information theoretic setting, it is clear that for any privacy parameter ϵ, there is a constant gap in the maximal achievable accuracies of any ϵ differentially private protocol in the client-server and distributed settings.

Goyal et al. [GMPS13] additionally showed that any hope of bridging this gap would imply the existence of one-way functions. The black box separation results of [HOZ13, KMS14] also hold for differentially private protocols with any accuracy in this range. Yet, it is unclear whether protocols with accuracies in this range must imply the existence of oblivious transfer. We show that our techniques when extended to non-optimal accuracies give rise to weak noisy channels and weak versions of oblivious transfer, which for some constant range of ϵ and constant fraction of the gap, do imply full-fledged oblivious transfer. Yet, there is a large gap between the upper and lower bounds for weak oblivious transfer amplification, and since our reductions go via noisy channels – this gap lends itself to our setting.

It may be possible to close this gap via other techniques; however we also believe that this novel connection between noisy channels and distributed differentially private protocols for functionalities gives reason to revive (and continue) research on the characterization of weak noisy channels.

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A Preliminaries

Here, we complete the definitions for all primitives that we use - in particular, oblivious transfer and noisy channels. We consider only two-party protocols in the semi-honest setting where each party follows instructions honestly throughout execution, but may be curious to learn any additional information without deviating from the protocol.

A.1 Oblivious Transfer and Noisy Channels

There are several definitions of both noisy channels as well as oblivious transfer in literature (see, e.g., [Wul09]). We specify here the variants we will use.

Oblivious Transfer.

Definition 7. A protocol family $\{\pi_k := \langle S, R\rangle(1^k)\}_{k\in\mathbb{N}}$ is said to implement the $\binom{2}{1}$-*oblivious transfer* functionality, or simply oblivious transfer (OT), if S and R are PPT algorithms and the following holds. The input of party S, called the sender, is a pair of bits (m_0, m_1) and the input of party R, called the receiver, is a single bit σ. At the end of the protocol, the output of R is m_σ and that of S is *nothing*—denoted by the special symbol $\perp$. Moreover, the following conditions hold.

Sender security: For $\sigma \in \{0,1\}$, and $k \in \mathbb{N}$:

$$\left\{m_{1-\sigma}, \mathsf{view}_\pi^R(k,(m_0,m_1),\sigma) : (m_0,m_1) \xleftarrow{\$} \{0,1\}^2\right\}_{\sigma,k}$$
$$\approx_c \left\{u, \mathsf{view}_\pi^R(k,(m_0,m_1),\sigma) : (u,m_0,m_1) \xleftarrow{\$} \{0,1\}^3\right\}_{\sigma,k}$$

Receiver security: For $(m_0,m_1) \in \{0,1\}^2$ and $k \in \mathbb{N}$:

$$\left\{\sigma, \mathsf{view}_\pi^S(k,(m_0,m_1),\sigma) : \sigma \xleftarrow{\$} \{0,1\}\right\}_{(m_0,m_1),k} \approx_c$$
$$\left\{u, \mathsf{view}_\pi^S(k,(m_0,m_1),\sigma) : (u,\sigma) \xleftarrow{\$} \{0,1\}^2\right\}_{(m_0,m_1),k}$$

This is the definition of semi-honest OT presented in [Hai08], with a proof that such a (semi-honest) OT protocol suffices to obtain full-fledged, i.e., malicious OT in a black-box manner.

ρ-Noisy-Channel.

Now, we directly define what it means for a protocol to implement a ρ-noisy-channel.

Definition 8. A protocol family $\{\pi_k := \langle S, R\rangle(1^k)\}_{k\in\mathbb{N}}$ implements the *ρ-noisy-channel* functionality for $\rho > 1/2$, if for PPT algorithms S and R, the following holds. The input of party S, called the sender, is a bit b and the input of party R, called the receiver, is nothing. At the end of the protocol, the output of R is a bit, denoted by b' and that of S is nothing – denoted by $\perp$. Then, the following conditions must hold:

Correctness: For all $k \in \mathbb{N}$, we have that $|\Pr[b' = b|b] - \rho| \leq \mathsf{negl}(k)$.

Sender security: Informally, the receiver R cannot distinguish the case when error $e := b' \oplus b$ is 0 from the case when it is 1. Formally, there exists a negligible function $\mathsf{negl} : \mathbb{N} \to \mathbb{R}$ such that

for every non-uniform PPT algorithm ("distinguisher") $\mathcal{D}_R$, every polynomial $p(\cdot)$, every sufficiently large $k \in \mathbb{N}$, every advice string z_k of size at most $p(k)$, and *uniformly chosen* $b \xleftarrow{\$} \{0,1\}$ we have:

$$\mathsf{Adv}^R_\pi(k) := \left|\Pr[\mathcal{D}_R(z_k, \mathsf{view}^R_\pi(k,b)) = 1 | e = 0] - \Pr[\mathcal{D}_R(z_k, \mathsf{view}^R_\pi(k,b)) = 1 | e = 1]\right| \leq \mathsf{negl}(k)$$

where $e := b' \oplus b$; $b' = \mathsf{out}(\mathsf{view}^R_\pi(k,b))$ and the probability is over the randomness of b, π and $\mathcal{D}_R$.

Receiver security: This is defined symmetrically. Formally, there exists a negligible function $\mathsf{negl} : \mathbb{N} \to \mathbb{R}$ such that for every non-uniform PPT algorithm ("distinguisher") $\mathcal{D}_S$, every polynomial $p(\cdot)$, every sufficiently large $k \in \mathbb{N}$, every advice string z_k of size at most $p(k)$, and *uniformly chosen* $b \xleftarrow{\$} \{0,1\}$ we have:

$$\mathsf{Adv}^S_\pi(k) := \left|\Pr[\mathcal{D}_S(z_k, \mathsf{view}^S_\pi(k,b)) = 1 | e = 0] - \Pr[\mathcal{D}_S(z_k, \mathsf{view}^S_\pi(k,b)) = 1 | e = 1]\right| \leq \mathsf{negl}(k)$$

where e was defined above and the probability is over the randomness of b, π and $\mathcal{D}_S$.

Noisy channels are sufficient for OT. We remark that a protocol implementing the noisy channel defined above, is sufficient to implement the semi-honest OT functionality defined above. A reduction between these two primitives was first given by Crépeau and Kilian [CK88], and we provide a brief outline of their semi-honest protocol in the computational setting (with security parameter k).

- The sender sends multiple $\mathsf{poly}(k)$ uniformly chosen random bits to the receiver, by transmitting each bit twice over the noisy channel. The receiver classifies received bits in the following manner - if the same value is received in both transmissions, the bit is called a "good" bit. If a different bit is received each time, the bit is "bad".
- The receiver picks set I_s of k indices corresponding to "good" bits, and a set I_{1-s} of k indices corresponding to "bad" bits, respectively. He sends I_0, I_1 to the sender over a clear channel.
- For all $m \in \{0,1\}$, the sender sends the XOR of the bit b_m with each of the bits at indices specified by I_m. The receiver only considers bits received corresponding to indices in I_s and computes her guess for b_s as the majority of the XOR of these bits with the good bits.