A practical post-quantum public-key cryptosystem based on spLWE

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Abstract. The Learning with Errors (LWE) problem has been widely used as a hardness assumption to construct public-key primitives. In this paper, we propose an efficient instantiation of a public-key encryption scheme based on LWE with a sparse secret, named as spLWE. We first construct an IND-CPA public-key encryption and convert it to an IND-CCA scheme in the quantum random oracle model by applying the modified Fujisaki-Okamoto conversion. To guarantee security of our base problem, we provide a polynomial time reduction to spLWE from the standard LWE with a uniformly chosen secret and Gaussian errors. We consider modified attacks for spLWE which exploit its sparsity of secret, to derive more suitable parameters. We finally estimate performance of our scheme: our implementation shows that our IND-CPA scheme takes 81 μ seconds and 21 μ seconds respectively for encryption and decryption with the parameters which have 128-quantum bit security.

Keywords: practical, post-quantum, IND-CCA, public-key encryption, sparse secret LWE, quantum random oracle model

1 Introduction

Recently, practical instantiations of post-quantum primitives have received more attention after announcements of NIST and NSA about quantum resistant cryptographic standards with the advance of quantum computers [18, 27]. The Learning with Errors (LWE) problem is one of promising hard problems for efficient quantum resistant cryptographic schemes. This problem was introduced by Regev [37] with a hardness proof of (quantum) reduction from worst-case lattice problems (SIVP, GapSVP). This work showed that LWE also has a self-random reducibility which results in a lot of constructions of cryptographic primitives (e.g. [22, 12, 32]).

In order to use LWE-based public-key encryptions in practical fields, like internet or IoT environments, improvement on speed and bandwidth is an important issue to be solved. However, their practical use have been limited since most of LWE-based public-key encryptions have relatively large size of parameters and slow speed. As one solution, Learning with Errors over the ring (RLWE) was introduced in [32]. RLWE brings more efficiency than LWE, but hardness of RLWE is not fully understood yet. Reductions related to LWE and its variants are not directly applied to RLWE, and RLWE could be weakened due to its additional ring structure.

In this paper, we introduce a practical post-quantum public-key encryption cryptosystem based on spLWE which is a variant LWE with a sparse secret vector. Based on spLWE, we propose a fast IND-CPA public-key encryption scheme and convert it to an IND-CCA version in quantum random oracle model by applying the modified Fujisaki-Okamoto conversion of Unruh [41]. To obtain better efficiency, we adapt the RLWE-based key encapsulation mechanism (KEM) in [35] to spLWE setting, and use the modified KEM as a component in the construction of our scheme. We obtain smaller ciphertext size by encrypting a message with a shared key of the KEM, differently from the previous LWE-based schemes [37, 38].

We generalize the reduction [11] from LWE with a uniformly chosen secret vector to LWE with a binary secret vector. As a result, we prove that spLWE can be reduced to the standard LWE, which means that the hardness of spLWE can also be based on the worst-case lattice problems. Moreover, we propose extended LWE attacks which exploit sparsity of secret vectors and derive concrete parameters based on those attacks. We present the performance of our

encryption scheme by implementing our scheme for the selected parameters. In 128-quantum bit security, the IND-CPA version of our encryption takes about $81\mu s$ and the IND-CCA version of our encryption takes $83\mu s$ for 128-bit message (PC with CPU 2.6GHz Intel Core i5 without parallelization).

1.1 Contributions

We propose an efficient public-key encryption based on spLWE and prove hardness of spLWE. We give concrete parameters for quantum security and implementation results of our scheme. Our main contributions are as follows:

1. We construct a spLWE-based public-key encryption scheme with two versions of security. One is IND-CPA secure and the other is IND-CCA secure under quantum random oracle model. In our construction, the ciphertext size of an IND-CPA encryption scheme for ℓ-bit message is $(n \log q + 2\ell)$-bit. This is smaller than that of the known LWE-based public-key encryptions [37, 38] which have $(n \log q + \ell \log q)$-bit ciphertext size.
2. We extend the range of LWE variants which has the provable hardness. It has been known that there exists a security reduction for the binary LWE [11]. We generalize this result to derive a reduction from LWE to spLWE. This enables us to consider sparse non-binary secret which is more suitable to improve our scheme.
3. We investigate attacks for spLWEby extending all known attacks of LWE which can be improved by exploiting the sparsity of secret. From the attacks, we give concrete parameters for both of classical and quantum security, and implement our schemes for some parameters. One of our implementation result gives $81\mu s$ (IND-CPA version), $135\mu s$ (IND-CCA version) encryption speed for 128-bit message, and 453 byte (IND-CPA version), 683 byte (IND-CCA version) ciphertext size with 128-quantum bit security.

1.2 Related Works

Practical instantiations and implementation results about post-quantum primitives in lattice based cryptography have been reported mostly in RLWE case rather than LWE one (e.g. [31], [39], [16], etc). In particular, Peikert [35] presented efficient and practical attice-based protocols for key transport, and encryption on RLWE that are suitable for proposed Internet standards and other open protocols. More recently, LWE-based key transport protocol was considered and evaluated its performance in [10]. Unlike the common belief about inefficiency of LWE, the literatures [10, 15] demonstrated that their LWE-based (not RLWE) key-exchange and signature scheme have sufficiently good performance compared to RLWE one.

In case of LWE-based public-key encryptions [37, 23, 36, 38, 33], it is hard to find concrete proposals. Many people only focus on efficiency or asymptotic security improvement. In efficiency aspect, Galbraith [20] proposed variants of LWE where the entries of random matrix are chosen to be small or binary to reduce the size of public-key. However, there was no complete proposal including attacks, and parameters for practical usage.

2 Preliminaries

Notations. In this paper, we use upper-case bold letters to denote matrices, and lower-case bold letters for column vectors. For a distribution $\mathcal{D}$, $a \leftarrow \mathcal{D}$ denotes choosing an element according to the distribution of $\mathcal{D}$, and $\boldsymbol{a} \leftarrow \mathcal{D}^m$ means that each components of $\boldsymbol{a}$ is sampled independently from $\mathcal{D}$. For a given set $\mathcal{A}$, $\mathcal{U}(\mathcal{A})$ means a uniform distribution on the set $\mathcal{A}$, and $a \leftarrow \mathcal{A}$ denotes choosing an element according to the uniform distribution on $\mathcal{A}$. We denote by $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z} = \{0, 1 \cdots, q-1\}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the additive group of real numbers modulo 1, and

$\mathbb{T}_q$ the a subgroup of $\mathbb{T}$ having order q, consisting of $\{0, \frac{1}{q}, \cdots, \frac{q-1}{q}\}$. The $\langle\ ,\ \rangle$ means the inner product of two vectors. A function $f(\lambda)$ is called *negligible* if $f(\lambda) = o(\lambda^{-c})$ for any $c > 0$, i.e., f decrease faster than any inverse polynomial.

2.1 Security Definitions

Definition 1 (γ-spread, [41]). *A public-key encryption is γ-spread if for every public-key generated by* Keygen *algorithm and every message $\boldsymbol{m}$,*

$$\max_{\boldsymbol{y}} Pr[\boldsymbol{y} \leftarrow \mathsf{Enc}_{pk}(\boldsymbol{m})] \leq \frac{1}{2^{\gamma}}.$$

In particular, we say that a public-key encryption is well-spread if $\gamma = \omega(\log(\lambda))$.

Definition 2 (One-way secure). *A public-key encryption is One-Way secure if no (quantum) polynomial time algorithm (adversary) $\mathcal{A}$ can find a message $\boldsymbol{m}$ from $\mathsf{Enc}_{pk}(\boldsymbol{m})$, given only public-key except with probability at most negl(λ).*

2.2 Key Encapsulation Mechanism

A *key encapsulation mechanism* (in short, KEM) is a key exchange algorithm to transmit an ephemeral key to a receiver with the receiver's public key. It differs from encryption scheme where a sender can choose a message. The sender cannot intend to make a specific ephemeral key. A KEM with ciphertext space $\mathcal{C}$ and key space $\mathcal{K}$ consists of probabilistic polynomial time algorithms Setup, Keygen, Encap(may be randomized), Decap(should be deterministic).

- Keygen outputs a public encapsulation key pk and secret decapsulation key sk.
- Encap takes an encapsulation key pk and outputs a ciphertext $c \in \mathcal{C}$ and a key $k \in \mathcal{K}$.
- Decap takes a decapsulation key sk and a ciphertext c, and outputs some $k \in \mathcal{K} \cup \{\perp\}$, where $\perp$ denotes decapsulation failure.

2.3 Lattice and Lattice Reduction Algorithm

A *lattice* $L \subseteq \mathbb{R}^m$ is a set of integer linear combinations of a $\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ which is a subset of independent column vectors in $\mathbb{R}^m$,

$$L = \{\sum_{i=1}^{n} a_i \boldsymbol{b}_i : a_i \in \mathbb{Z}\}$$

The set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, and its matrix form $\mathbf{B}$ are called a basis, and basis matrix of L respectively. Two bases matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ describe the same lattice, if and only if $\mathbf{B}_2 = \mathbf{B}_1\mathbf{U}$, where $\mathbf{U}$ is a unimodular matrix, i.e. $\det(U) = \pm 1$, $U \in \mathbb{Z}^{m \times m}$. Dimension of a lattice is defined as cardinality of a basis, i.e. $n = \dim(L)$. If $n = m$, we call lattice L to a full rank lattice. A sublattice is a subset $L' \subset L$ which is also a lattice. We define determinant (volume) of L by

$$\det(L) := \sqrt{\det(\boldsymbol{B}^T \boldsymbol{B})}$$

A length of the shortest vector in a lattice $L(\mathbf{B})$ is denoted by $\lambda_1(L(\mathbf{B}))$. More generally, the *i-th successive minima* $\lambda_i(L)$ is defined as the smallest radius r such that $\dim(\mathrm{span}(L \cap B(r))) \geq i$ where $B(r)$ is a n dimensional ball with radius r. There exist several bounds and estimations for the length of the shortest vector in a lattice.

- Minkowski's first theorem: $\lambda_1(L(\mathbf{B})) \leq \sqrt{n}(\det L(\mathbf{B}))^{1/n}$

- Gaussian heuristic: $\lambda_1(L(\mathbf{B})) \approx \sqrt{\frac{n}{2\pi e}}\det(L(\mathbf{B}))^{1/n}$ for random lattice L.

The *dual lattice* of L, denoted $\bar{L}$, is defined to be $\bar{L} = \{\boldsymbol{x} \in \mathbb{R}^n : \forall \boldsymbol{v} \in L, \langle \boldsymbol{x}, \boldsymbol{v}\rangle \in \mathbb{Z}\}$. We recall the Gram-Schmidt orthogonalization that is closely related with lattice basis reduction. The Gram-Schmidt algorithm computes orthogonal vectors $\{\mathbf{b}_1^*, \ldots, \mathbf{b}_m^*\}$ iteratively as follows:

$$\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j<i} \mu_{i,j}\mathbf{b}_j^* \text{ where } \mu_{i,j} = \frac{\mathbf{b}_i \cdot \mathbf{b}_j^*}{\mathbf{b}_j^* \cdot \mathbf{b}_j^*}.$$

The goal of lattice (basis) reduction algorithms is to find a good basis for a given lattice. A basis is considered good, when the basis vectors are almost orthogonal and correspond approximately to the successive minima of the lattice. Performance of lattice reduction algorithms is evaluated by the *root Hermite factor* δ_0 defined by

$$\delta_0 = (||\boldsymbol{v}||/\det(L)^{1/n})^{1/n}$$

where $\boldsymbol{v}$ is the shortest vector of the reduced output basis.

2.4 Discrete Gaussian Distribution

For given $s > 0$, a *discrete gaussian distribution* over a lattice L is defined as $D_{L,s}(x) = \rho_s(x)/\rho_s(L)$ for any $x \in L$, where

$$\rho_s(x) = \exp(-\pi\|x\|^2/s^2) \text{ and } \rho_s(L) := \sum_{x \in L} \rho_s(x).$$

We note that the standard deviation $\sigma = s/\sqrt{2\pi}$. For a lattice L, the *smoothing parameter* $\eta_\epsilon(L)$ is defined by the smallest real number $s' > 0$ such that $\rho_{1/s'}(\bar{L} \setminus \{\mathbf{0}\}) \leq \epsilon$. We collect some useful lemmas related to a discrete gaussian distribution and the smoothing parameter.

Lemma 1 ([8], Lemma 2.4). *For any real $s > 0$ and $T > 0$, and any $\boldsymbol{x} \in \mathbb{R}^n$, we have*

$$Pr[|\langle \boldsymbol{x}, D_{\mathbb{Z},s}^n\rangle| \geq T \cdot s\|\boldsymbol{x}\|] < 2\exp(-\pi \cdot T^2).$$

Lemma 2 ([37], Corollary 3.10). *Let L be an n-dimensional lattice, let $\boldsymbol{u}, \boldsymbol{z} \in \mathbb{R}^n$ be arbitrary vectors, and let r, α be positive real numbers. Assume that $(1/r^2 + (\|\boldsymbol{z}/\alpha\|)^2)^{-1/2} \geq \eta_\epsilon(\Lambda)$ for some $\epsilon < 1/2$. Then the distribution of $\langle \boldsymbol{z}, \boldsymbol{v}\rangle + e$ where $\boldsymbol{v} \leftarrow D_{L+\boldsymbol{u}}$ and $e \leftarrow D_\alpha$ is within statistical distance 4ϵ of D_β for $\beta = \sqrt{(r\|\boldsymbol{z}\|)^2 + \alpha^2}$.*

Lemma 3 ([23], Lemma 3.1). *For any $\epsilon > 0$ and an n-dimensional lattice Λ with basis matrix $\boldsymbol{B}$, the smoothing parameter $\eta_\epsilon(\Lambda) \leq \|\tilde{\boldsymbol{B}}\| \ln(2n(1+1/\epsilon))/\pi$ where $\|\tilde{\boldsymbol{B}}\|$ denotes the length of the longest column vector of $\tilde{\boldsymbol{B}}$ which is the Gram-Schmidt orthogonalization of $\boldsymbol{B}$.*

2.5 Learning with Errors

For integers $n, q \geq 1$, a vector $\boldsymbol{s} \in \mathbb{Z}_q^n$, and a distribution ϕ on $\mathbb{R}$, let $A_{q,\boldsymbol{s},\phi}$ be the distribution of the pairs $(\boldsymbol{a}, b = \langle \boldsymbol{a}, \boldsymbol{s}\rangle + e) \in \mathbb{T}_q^n \times \mathbb{T}$, where $\boldsymbol{a} \leftarrow \mathbb{T}_q^n$ and $e \leftarrow \phi$.

Definition 3 (Learning with Errors (LWE)). *For integers $n, q \geq 1$, an error distribution ϕ over R, and a distribution $\mathcal{D}$ over $\mathbb{Z}_q^n$, $\mathsf{LWE}_{n,q,\phi}(\mathcal{D})$, is to distinguish (given arbitrarily many independent samples) the uniform distribution over $\mathbb{T}_q^n \times \mathbb{T}$ from $A_{q,\boldsymbol{s},\phi}$ with a fixed sample $\boldsymbol{s} \leftarrow \mathcal{D}$.*

We note that a search variant of LWE is the problem of recovering $\boldsymbol{s}$ from $(\mathbf{a}, b) = (\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e) \in \mathbb{T}_q^n \times \mathbb{T}$ sampled according to $A_{q,\boldsymbol{s},\phi}$, and these are also equivalently defined on $\mathbb{Z}_q^n \times \mathbb{Z}_q$ rather than $\mathbb{T}_q^n \times \mathbb{T}$ for discrete (Gaussian) error distributions over $\mathbb{Z}_q$. Let $\mathsf{LWE}_{n,m,q,\phi}(\mathcal{D})$ denotes the case when the number of samples are bounded by $m \in \mathbb{N}$. We simply denote $\mathsf{LWE}_{n,q,\phi}$ when the secret distribution $\mathcal{D}$ is $U(\mathbb{Z}_q^n)$. In many cases, ϕ is a (discrete) Gaussian distribution so we simply denote by $\mathsf{LWE}_{n,m,q,s}$ instead of $\mathsf{LWE}_{n,m,q,\phi}$. Especially, we denote binLWE by the LWE problem whose secret vector is sampled from uniform distribution over $\{0,1\}^n$. For a set $X_{n,\rho,\theta}$ which consists of the vectors $\boldsymbol{s} \in \mathbb{Z}^n$ whose nonzero components are in $\{\pm 1, \pm 2, \pm 4, \cdots, \pm\rho\}$, and the number of nonzero components is θ, we write $\mathsf{spLWE}_{n,m,q,s,\rho,\theta}$ as the problem $\mathsf{LWE}_{n,m,q,s}(\mathcal{U}(X_{n,\rho,\theta}))$.

The following lemma will be used to derive some parameters from the modified attacks in section 4, and appendix.

Lemma 4 ([38]). *Given $\mathsf{LWE}_{n,m,q,s}$ samples and a vector $\boldsymbol{v}$ of length $\|\boldsymbol{v}\|$ in the lattice $L = \{\boldsymbol{w} \in \mathbb{Z}_q^m : \boldsymbol{w}^T \boldsymbol{A} \equiv 0 \mod q\}$, the advantage of distinguishing $\langle \boldsymbol{v}, \boldsymbol{e} \rangle$ from uniform random is close to $\exp(-\pi(\|\boldsymbol{v}\| s/q)^2)$.*

We give some variants of LWE and some notion, which were introduced in [11] to show the reduction between binLWE and LWE.

Definition 4 ("first-is-errorless" LWE). *For integers $n, q \geq 1$ and an error distribution ϕ over $\mathbb{R}$, the "first-is-errorless" variant of the LWE problem is to distinguish between the following two scenarios. In the first, the first sample is uniform over $\mathbb{T}_q^n \times \mathbb{T}_q$ and the rest are uniform over $\mathbb{T}_q^n \times \mathbb{T}$. In the second, there is an unknown uniformly distributed $\boldsymbol{s} \in \{0, \ldots, q-1\}^n$, the first sample we get is from $A_{q,\boldsymbol{s},\{0\}}$ (where $\{0\}$ denotes the distribution that is deterministically zero) and the rest are from $A_{q,\boldsymbol{s},\phi}$.*

Definition 5 (extLWE problem). *For integers $n, m, q, t \geq 1$, a set $X \subseteq \mathbb{Z}^m$, and a distribution χ over $\frac{1}{q}\mathbb{Z}^m$, the $\mathsf{extLWE}_{n,m,q,\chi,X}$ is as follows. The algorithm gets to choose $\boldsymbol{x} \in X$ and then receives a tuple $(\boldsymbol{A}, (\boldsymbol{b}_i)_{i\in[t]}, (\langle \boldsymbol{e}_i, \boldsymbol{x} \rangle)_{i\in[t]}) \in \mathbb{T}_q^{n\times m} \times (\mathbb{T}_q^m)^t \times (\frac{1}{q}\mathbb{Z})^t$. Its goal is to distinguish between the following two cases. In the first, $A \in \mathbb{T}_q^{n\times m}$ is chosen uniformly, $\boldsymbol{e}_i \in \frac{1}{q}\mathbb{Z}^m$ are chosen from χ, and $\boldsymbol{b}_i = \boldsymbol{A}^T \boldsymbol{s}_i + \boldsymbol{e}_i \mod 1$ where $\boldsymbol{s}_i \in \{0, \ldots, q-1\}^n$ are chosen uniformly. The second case is identical, except that the $\boldsymbol{b}_i$ are chosen uniformly in $\mathbb{T}_q^m$ independently of everything else.*

Definition 6 (Quality of a set). *A set $X \subset \mathbb{Z}^m$ is said of quality ξ if given any $\boldsymbol{x} \in X$, we can efficiently find a unimodular matrix $U \in \mathbb{Z}^{m\times m}$ such that if $U' \in \mathbb{Z}^{m\times(m-1)}$ is the matrix obtained from U by removing its leftmost column then all of the columns of U' are orthogonal to $\boldsymbol{z}$ and its largest singular value is at most ξ.*

We give a lemma to show a reduction to spLWE from the standard LWE in section 4.1.

Lemma 5. *The quality of a set $X \subseteq \{0, \pm 1, \pm 2, \ldots, \pm\rho\}^m$, $\rho = 2^l$ is bounded by $1 + \sqrt{\rho}$.*

Proof. Let $\mathbf{x} \in X$ and without loss of generality, we assume leftmost k components of $\mathbf{x}$ are nonzero, remainings are zero, and $|\mathbf{x}_i| \leq |\mathbf{x}_{i+1}|$ for nonzero components after reordering. We have $\mathbf{x}_{i+1} = \pm 2^{t_i} \mathbf{x}_i$ for some $t_i \leq l$. Now consider the upper bidiagonal matrix U whose diagonal is all 1s and whose diagonal above the main diagonal is $\mathbf{y} \in \mathbb{Z}^{m-1}$ such that $\mathbf{x}_{i+1} - [\mathbf{y}]_j \mathbf{x}_i = 0$ for $1 \leq j \leq k-1$, and rightmost $(m-k)$ components of $\mathbf{y}$ are 0. Since $\mathbf{x}_{i+1} = \pm 2^{t_i} \mathbf{x}_i$ it follows that $[\mathbf{y}]_j$ is 2^{t_j} or -2^{t_j}. Then U is clearly unimodular($\det(U) = 1$) and all the columns except the first one are orthogonal to $\mathbf{x}$. Moreover, by the triangle inequality, we can bound the norm(the largest singular value) of U by the sum of that of the diagonal 1 matrix and the off-diagonal matrix of which clearly have norm at most $\sqrt{\rho}$. □

3 Our spLWE-based public-key Encryption

In this section, we introduce a public key encryption scheme whose security is based on spLWE, whose ciphertext size is smaller than previous works [37, 38]. We use noisy subset sum in our encryption algorithm which is proposed in previous LWE-based encryption scheme [38], but our message encoding is different from it: we first construct a key encapsulation mechanism based on LWE, and conceal a message with an ephemeral key shared by KEM.

We propose two versions of one encryption scheme based on the spLWE-based KEM, where one is IND-CPA secure and the other is an IND-CCA conversion of IND-CPA one by the transformation proposed in [41]. We note that these different types of the scheme can be applied to various circumstances.

3.1 Our Key Encapsulation Mechanism

We use a *reconciliation* technique in [35] which is a main tool to construct our spLWE-based KEM. In our KEM, the sender generate a random number $v \in \mathbb{Z}_{2q}$ for some even integer $q > 0$, and send $\langle v \rangle_2$ where $\langle v \rangle_2 := [\lfloor \frac{2}{q} \cdot v \rfloor]_2 \in \mathbb{Z}_2$ to share $\lfloor v \rceil_2 := [\lfloor \frac{1}{q} \cdot v \rceil]_2 \in \mathbb{Z}_2$ securely. For all vectors $\boldsymbol{v} \in \mathbb{Z}_{2q}^k$, $\langle \boldsymbol{v} \rangle_2$ and $\lfloor \boldsymbol{v} \rceil_2$ are naturally defined by applying $\langle \rangle_2$ and $\lfloor \rceil_2$ component-wise, respectively. The receiver recovers $\lfloor v \rceil_2$ from $\langle v \rangle_2$ and sk using a special function named as *rec*. The reconciliation function *rec* is defined as follows.

Definition 7. *For disjoint intervals* $I_0 := \left\{0, 1, \cdots, \lfloor \frac{q}{2} \rceil - 1\right\}$, $I_1 := \left\{-\lfloor \frac{q}{2} \rceil, \cdots, -2, -1\right\}$ *and* $E = \left[-\frac{q}{4}, \frac{q}{4}\right) \cap \mathbb{Z}$, *we define*

$$rec : \mathbb{Z}_{2q} \times \mathbb{Z}_2 \to \mathbb{Z}_2 \text{ where } rec(w, b) := \begin{cases} 0 & \text{if } w \in I_b + E \mod 2q, \\ 1 & \text{otherwise.} \end{cases}$$

It is naturally extended to a vector-input function $\boldsymbol{rec} : \mathbb{Z}_{2q}^k \times \mathbb{Z}_2^k \to \mathbb{Z}_2^k$ *by applying rec component-wise.*

Following lemmas show that $\langle v \rangle_2$ reveals no information about $\lfloor v \rceil_2$, and *rec* decapsulates $\lfloor v \rceil_2$ correctly when it is provided with a proper approximation of v.

Lemma 6. *If* $v \in \mathbb{Z}_{2q}$ *is uniformly random, then* $\lfloor v \rceil_2$ *is uniformly random given* $\langle v \rangle_2$.

Proof. Suppose that $\langle v \rangle_2 = b \in \mathbb{Z}_2$. It implies that v is uniform over $I_b \cup (q + I_b)$. If $v \in I_b$, then $\lfloor v \rceil_2 = 0$, and if $v \in (q + I_b)$, then $\lfloor v \rceil_2 = 1$. Therefore $\lfloor v \rceil_2$ is uniformly random over $\{0, 1\}$ given $\langle v \rangle_2$. □

Lemma 7. *For* $v, w \in \mathbb{Z}_{2q}$, *if* $|v - w| < q/4$, *then* $rec(w, \langle v \rangle_2) = \lfloor v \rceil_2$.

Proof. Let $\langle v \rangle_2 = b \in \mathbb{Z}_2$, so $v \in I_b \cup (q + I_b)$. Then $\lfloor v \rceil_2 = 0$ if and only if $v \in I_b$. Since $(I_b + E) - E = I_b + (-\frac{q}{2}, \frac{q}{2})$ and $(q + I_b)$ are disjoint (mod $2q$), we know that $v \in I_b$ if and only if $w \in I_b + E$. □

The main idea of our KEM is sharing MSB of $\mathbf{u}^T\mathbf{A}\mathbf{s} + error$ between two parties as in [35]. Here we describe our spLWE-based KEM for k-bit sharing as follows.

- KEM.Params(λ): generate a bit-length of shared key k, a bit-length of seed y and spLWE parameters $n, m, q, s, \rho, \theta, s', \rho', \theta'$ with λ-bit security. Publish all parameters by pp.

- KEM.Keygen(pp): sample $seed_A \leftarrow \{0,1\}^y$, $\mathbf{A} \leftarrow Gen(seed_A)$, $\mathbf{E} \leftarrow D_{\mathbb{Z},s}^{m \times k}$ and $\mathbf{S} \leftarrow \mathcal{U}(X_{n,\rho,\theta})^k$, and compute $\mathbf{B} = \mathbf{A}\mathbf{S} + \mathbf{E} \in \mathbb{Z}_q^{m \times k}$. For a secret key $\mathsf{sk} = \mathbf{S}$, publish a corresponding public key $\mathsf{pk} = (seed_A, \mathbf{B})$.

– $\mathsf{KEM.Encap}(\mathsf{pk},\mathsf{pp})$: sample $\boldsymbol{u} \leftarrow X_{m,\rho',\theta'}$, $(\boldsymbol{e}_1, \boldsymbol{e}_2) \leftarrow D^k_{\mathbb{Z},s'} \times D^n_{\mathbb{Z},s'}$ and $\boldsymbol{e}_3 \in \{0,1\}^k$. Let $\boldsymbol{v} = \boldsymbol{u}^T\mathbf{B}+\boldsymbol{e}_1 \in \mathbb{Z}_q^k$ and $\bar{\boldsymbol{v}} = 2\boldsymbol{v}+\boldsymbol{e}_3 \in \mathbb{Z}_{2q}^k$. Compute $\boldsymbol{c_1} = \langle \bar{\boldsymbol{v}} \rangle_2 \in \mathbb{Z}_2^k$ and $\boldsymbol{c_2} = \boldsymbol{u}^T\mathbf{A}+\boldsymbol{e_2} \in \mathbb{Z}_q^n$ from $\mathbf{A} \leftarrow Gen(seed_A)$. Send a ciphertext $\boldsymbol{c} = (\boldsymbol{c_1}, \boldsymbol{c_2}) \in \mathbb{Z}_2^k \times \mathbb{Z}_q^n$ to the receiver, and store an ephemeral secret key $\boldsymbol{\mu} = \lfloor \bar{\boldsymbol{v}} \rceil_2 \in \mathbb{Z}_2^k$.

– $\mathsf{KEM.Decap}(\boldsymbol{c},\mathsf{sk})$: If q is odd, compute $\boldsymbol{w} = 2\boldsymbol{c_2}^T\boldsymbol{S} \in \mathbb{Z}_q^k$, and ouput $\boldsymbol{\mu} = \boldsymbol{rec}(\boldsymbol{w}, \boldsymbol{c_1})$.

We remark that if q is even, the *doubling* process in the encapsulation phase, i.e. converting $\boldsymbol{v} = \boldsymbol{u}^T\mathbf{B} + \boldsymbol{e}_1$ to $\bar{\boldsymbol{v}} = 2\boldsymbol{v} + \boldsymbol{e}_3$, is not required.

3.2 Our KEM-based Encryption Scheme

We now construct a public key encryption scheme based on the spLWE-based KEM in previous section. When message slot increases by one, ciphertext spaces of our scheme grow only one or two bits, which is more more efficient than known LWE based encryption schemes [37, 38], where the growth is about $\log q$ bits.

$\mathbf{PKE_1}$ (IND-CPA) : With a key exchange mechanism which shares ℓ-bit length key, it is well-known that one can convert it to a public key encryption of ℓ-bit length message having same security with the key exchange mechanism. This conversion only include XOR operations after generating an ephemeral key, We note that the ciphertext space is given $\mathbb{Z}_q^n \times \mathbb{Z}_2^{2\ell}$, which is very efficient to other LWE-based schemes ciphertext space $\mathbb{Z}_q^{n+\ell}$.

PKE_1 is described as follows.

– $\mathrm{PKE}_1.\mathsf{Params}(\lambda)$: let ℓ be a message length, and run $\mathsf{KEM.Params}(\lambda)$ with $k = \ell$. Publish all parameters by pp.

– $\mathrm{PKE}_1.\mathsf{Keygen}(\mathsf{pp})$: output a key pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KEM.Keygen}(\mathsf{pp})$.

– $\mathrm{PKE}_1.\mathsf{Enc}(\mathsf{pk}, \boldsymbol{m}, \mathsf{pp})$: for $\boldsymbol{c}, \boldsymbol{\mu} \leftarrow \mathsf{KEM.Encap}(\mathsf{pk},\mathsf{pp})$, let $\boldsymbol{c'} = \boldsymbol{m} \oplus \boldsymbol{\mu}$ and output a ciphertext $(\boldsymbol{c}, \boldsymbol{c'})$.

– $\mathrm{PKE}_1.\mathsf{Dec}((\boldsymbol{c}, \boldsymbol{c'}), \mathsf{sk})$: for $\boldsymbol{\mu} = \mathsf{KEM.Decap}(\boldsymbol{c}, \mathsf{sk})$, output $\boldsymbol{m} = \boldsymbol{c'} \oplus \boldsymbol{\mu}$.

$\mathbf{PKE_2}$ (IND-CCA) : We can apply the transformation suggested in [41], which gives more security for existing public key encryption scheme. As a trade-off for security, this scheme requires a more complex construction than PKE_1, but note that this also use light operations such as XOR or hashing which are not serious tasks for implementation.

We specially denote the encryption phase of PKE_1 by $\mathrm{PKE}_1.\mathsf{Enc}(\mathsf{pk}, \boldsymbol{m}, \mathsf{pp}; \boldsymbol{r})$ to emphasize that a random bit-string $\boldsymbol{r}$ is used for random sampling. Here $\mathrm{PKE}_1.\mathsf{Enc}(\mathsf{pk}, \boldsymbol{m}, \mathsf{pp}; \boldsymbol{r})$ becomes deterministic.

We also require quantumly secure hash functions $G : \{0,1\}^{k_1+\ell} \to \{0,1\}^*$, $H : \{0,1\}^{k_1} \to \{0,1\}^{k_2}$ and $H' : \{0,1\}^{k_1} \to \{0,1\}^{k_3}$, where k_i will be determined later. With these parameters, our scheme has a ciphertext space $\mathbb{Z}_q^n \times \mathbb{Z}_2^{k_1+k_2+k_3+\ell}$, which also increases slowly with the growth of message slot.

PKE_2 is described as follows.

– $\mathrm{PKE}_2.\mathsf{Params}(\lambda)$: let ℓ be a message length and $k_i > 0$ be integers such that hash functions $G : \{0,1\}^{k_1+\ell} \to \{0,1\}^*$, $H : \{0,1\}^{k_1} \to \{0,1\}^{k_2}$ and $H' : \{0,1\}^{k_1} \to \{0,1\}^{k_3}$ have λ-bit security. Let pp be an output of $\mathsf{KEM.Params}(\lambda)$ with $k = k_1$. Publish ℓ, pp and k_i.

– $\mathrm{PKE}_2.\mathsf{Keygen}(\mathsf{pp})$: output a key pair $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KEM.Keygen}(k_1)$.

– $\mathrm{PKE}_2.\mathsf{Enc}(\mathsf{pk}, \boldsymbol{m}, \mathsf{pp}, , \ell, k_i)$: randomly choose $\boldsymbol{\omega} \leftarrow \{0,1\}^{k_1}$, and let $\boldsymbol{c_m} = H(\boldsymbol{\omega}) \oplus \boldsymbol{m}$. Compute $\boldsymbol{c}_h = H'(\boldsymbol{\omega})$ and $(\boldsymbol{c}, \boldsymbol{c'}) \leftarrow \mathrm{PKE}_1.\mathsf{Enc}(\mathsf{pk}, \boldsymbol{\omega}; G(\boldsymbol{\omega}||\boldsymbol{c_m}))$. Output a ciphertext $(\boldsymbol{c}, \boldsymbol{c'}, \boldsymbol{c}_h, \boldsymbol{c_m})$.

– $\mathrm{PKE}_2.\mathsf{Dec}((\boldsymbol{c}, \boldsymbol{c'}, \boldsymbol{c}_h, \boldsymbol{c_m}), \mathsf{sk}, \mathsf{pp}, \ell, k_i)$: compute $\boldsymbol{\omega} = \mathrm{PKE}_1.\mathsf{Dec}((\boldsymbol{c}, \boldsymbol{c'}), \mathsf{sk})$ and $\boldsymbol{m} = H(\boldsymbol{\omega}) \oplus \boldsymbol{c_m}$. Check whether $(\boldsymbol{c}, \boldsymbol{c'}) = \mathrm{PKE}_1.\mathsf{Enc}(\mathsf{pk}, \boldsymbol{\omega}; G(\boldsymbol{w}||\boldsymbol{c_m}))$ and $\boldsymbol{c}_h = H'(\boldsymbol{\omega})$. If so, output $\boldsymbol{m}$, otherwise output $\perp$.

Remark that $\boldsymbol{w} \in \{0,1\}^{k_1}$ and $H(\boldsymbol{w}) \in \{0,1\}^{k_2}$ are used as symmetric keys for XOR operations. It is known that that in classical security model, the one-time pad key length should be at least λ, and it should be increased twice against a quantum adversary. In contrast, $H'(\boldsymbol{w}) \in \{0,1\}^{k_3}$ is used as a hash value, and hence k_3 should be larger than 2λ and 3λ to avoid the birthday attack, in classical and quantum security models, respectively. To sum up, our ciphertext space is at least $\mathbb{Z}_q^n \times \mathbb{Z}_2^{4\lambda+\ell}$ ($\mathbb{Z}_q^n \times \mathbb{Z}_2^{8\lambda+\ell}$), in a classical (quantum) security model.

3.3 Security

In this section, we show (IND-CPA, IND-CCA) security of our encryption scheme (PKE_1, PKE_2). Security of our encryption scheme is reduced to security of KEM, and security of KEM comes from hardness of spLWE. Consequently, under the hardness of spLWE, PKE_1 can be reached to IND-CPA security, and PKE_2 achieves further quantumly IND-CCA security with the random oracle assumption. Here is a statement for security of KEM.

Theorem 1. *Assuming the hardness of* $\mathsf{spLWE}_{n,m,q,s,\rho,\theta}$, *and* $\mathsf{spLWE}_{n,m,q,s',\rho',\theta'}$, *our KEM is IND-CPA secure.*

Proof. (Sketch) By Lemma 3, one cannot extract any information about $\boldsymbol{\mu} = \lfloor \boldsymbol{v} \rceil_2$ with $\boldsymbol{c_1}$. Moreover, even if one can know some information of $\boldsymbol{v}$, the distribution of $(\boldsymbol{c_2}, \boldsymbol{v})$ can be regarded as LWE instances as :

$$(\boldsymbol{c_2}, \boldsymbol{v}) = (\boldsymbol{u}^T \cdot \mathbf{A} + \boldsymbol{e_2}, \boldsymbol{u}^T \cdot \mathbf{B} + \boldsymbol{e_1}) = (\mathbf{C}, \mathbf{C} \cdot \mathbf{S} + \boldsymbol{e'})$$

for $\mathbf{C} = \boldsymbol{u}^{\mathbf{T}} \cdot \mathbf{A} + \boldsymbol{e_2}$ and for some $\boldsymbol{e'}$. Thus hardness of spLWE insures that the distribution of $(\boldsymbol{c_2}, \boldsymbol{v})$ is indistinguishable from the uniform distribution over $\mathbb{Z}_q^n \times \mathbb{Z}_q^k$. □

We refer [35] for the detailed IND-CPA game-based proof, where the only difference is that we assume the hardness of spLWE, not RLWE.

It is well-known in many cryptographic texts that PKE_1 has same security level with KEM. Hence security of PKE_1 is given straightforward from the previous theorem. Moreover, the transformation of [41] gives quantumly IND-CCA security for PKE_2, when it is converted from an IND-CPA secure PKE with random oracle modeled hashes. When put together the above statements, we can establish the following security theorem.

Theorem 2. *Assuming the hardness of* $\mathsf{spLWE}_{n,m,q,s,\rho,\theta}$, $\mathsf{spLWE}_{n,m,q,s',\rho',\theta'}$, *$PKE_1$ is IND-CPA secure, and PKE_2 is quantumly IND-CCA secure with further assumption that the function G, H and H' are modeled as random oracles.*

Proof. (Sketch) We only need to show that PKE_2 is IND-CCA secure. The transformation of [41] actually make an IND-CCA secure public key encryption from a public key encryption which is *well-spread* and *one-way*, and we briefly explain why (IND-CPA) PKE_1 is well-spread and one-way.

– Well-spreadness: Note that a ciphertext of PKE_1 is of the form

$$(\boldsymbol{c}_1, \boldsymbol{c}_2) = \left(\langle \mathbf{2}(\boldsymbol{u}^T B + \boldsymbol{e}_1) + \boldsymbol{e}_3 \rangle_2, \boldsymbol{u}^T A + \boldsymbol{e}_2\right),$$

where $\boldsymbol{u} \leftarrow X_{m,\rho',\theta'}, (\boldsymbol{e}_1, \boldsymbol{e}_2) \leftarrow D^k_{\mathbb{Z},s'} \times D^n_{\mathbb{Z},s'}$. From hardness of spLWE, distributions of $\boldsymbol{u}^T B + \boldsymbol{e}_1 \in \mathbb{Z}_q^k$ and $\boldsymbol{u}^T A + \boldsymbol{e}_2 \in \mathbb{Z}_q^n$ are statistically close to uniform distributions over $\mathbb{Z}_q^k$ and $\mathbb{Z}_q^n$, and then PKE_1 is well-spread.

– One-wayness: With an oracle $\mathcal{O}$ finding $\boldsymbol{m}$ from $\mathrm{PKE}_1.\mathrm{Enc}(\mathsf{pk}, \boldsymbol{m})$ for any pk with probability ϵ, an adversary equipped with $\mathcal{O}$ wins the IND-CPA game for PKE_1 with advantage bigger than $\frac{\epsilon}{2}$: After given $\mathrm{PKE}_1.\mathrm{Enc}(\mathsf{pk}, \boldsymbol{m}_b)$, the adversary outputs the answer of $\mathcal{O}$. It can be easily shown that the advantage is bigger than $\frac{\epsilon}{2}$.

3.4 Correctness

Similar to the security case, correctness of our (IND-CPA, IND-CCA) encryption scheme depends on that of our spLWE-based KEM. We remark that generally, one can obtain some correctness condition for all LWE variants by examining a bound of error term in the proof below. Here, we assume $s = s', \rho = \rho'$, and $\theta = \theta'$, which is used for our parameter instantiation.

Theorem 3. *Let $n, m, \sigma, \rho, \theta$ be parameters in $\mathsf{spLWE}_{n,m,q,\sigma,\rho,\theta}$, and ℓ be the shared key length in KEM. For a per-symbol error probability γ, the KEM decapsulates correctly if*

$$q \geq 8s\rho\sqrt{\frac{2\theta}{\pi}\ln(2/\gamma)}.$$

Proof. As shown in the description of $\mathsf{KEM.Decap}$, the ephemeral key is decapsulated correctly if $|\bar{\boldsymbol{v}} - \boldsymbol{w}| < q/4$ by lemma 7. Since $\bar{\boldsymbol{v}} = 2\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{S} + 2\boldsymbol{u}^T \boldsymbol{E} + 2\boldsymbol{e}_1 + \boldsymbol{e}_3$, and $\boldsymbol{w} = 2\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{S} + 2\boldsymbol{e}_2\boldsymbol{S}$, it is rephrased by

$$|2\boldsymbol{u}^T \cdot \mathbf{E} - 2\boldsymbol{e_1} \cdot \mathbf{S} + 2\boldsymbol{e_2} + \boldsymbol{e_3}| < q/4,$$

which is equivalent to

$$|2\langle \boldsymbol{u}, [\mathbf{E}]^j \rangle + 2\langle -\boldsymbol{e_1}, [\mathbf{S}]^j \rangle + 2[\boldsymbol{e_2}]_j + [\boldsymbol{e_3}]_j| < q/4, 1 \leq j \leq \ell$$

where $\boldsymbol{u} \leftarrow X_{m,\rho',\theta'}, [\mathbf{S}]^j \leftarrow X_{n,\rho,\theta}, [\mathbf{E}]^j \leftarrow D^m_{\mathbb{Z},s}, \boldsymbol{e}_1 \leftarrow D^n_{\mathbb{Z},s'}, [\boldsymbol{e_2}]_j \leftarrow D_{\mathbb{Z},s'}, [\boldsymbol{e_3}]_j \leftarrow \{0,1\}$. For simplicity, we ignore the small term $2[\boldsymbol{e_2}]_j + [\boldsymbol{e_3}]_j$. (This is compensated in our final choice of parameters.) By applying lemma 1 to a $(m+n)$ dimensional vector $\boldsymbol{x} = (\boldsymbol{u}, [\boldsymbol{S}]^j)$ and the bound $Ts\|\boldsymbol{x}\| = q/8$, we have the per-symbol error probability γ,

$$\gamma = 2\exp(-\pi(\frac{q}{8s\rho\sqrt{(2\theta)}})^2)$$

from $T = \frac{q}{8s\rho\sqrt{2\theta}}$. From the above equation, we get the bound on q as the statement.

4 The Hardness of spLWE

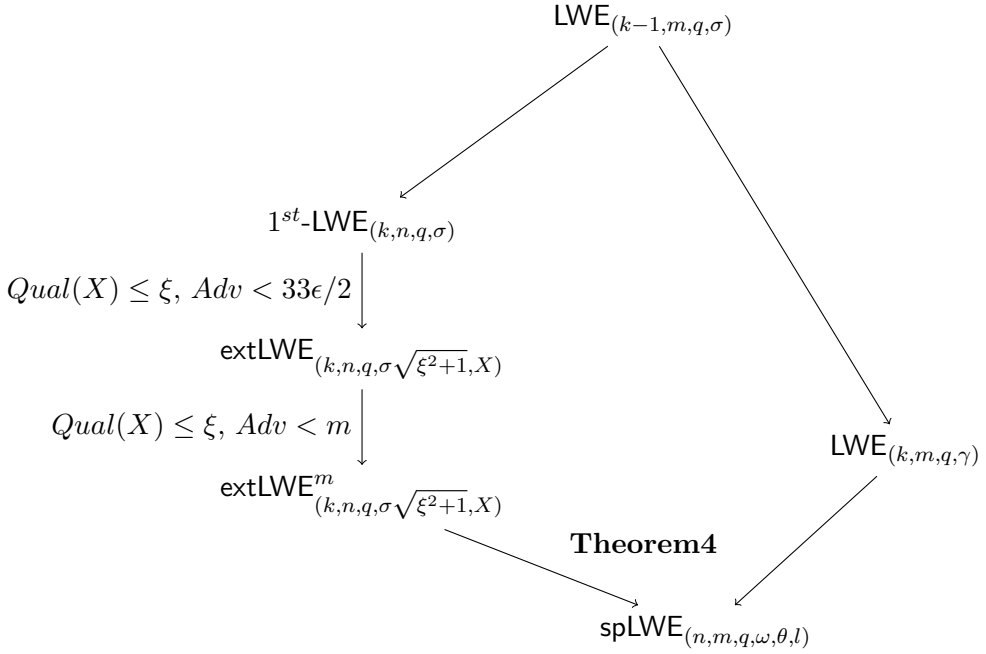
In this section, we show spLWE is as hard to solve as worst-case lattice problems. For that, we prove a reduction from standard LWE to spLWE by generalizing the reduction [11]. We also present modified attacks for spLWE which exploit the sparsity of a secret from all known attacks for LWE, binLWE[7, 13].

4.1 Reduction from standard LWE to spLWE

To show our reduction about spLWE, we need extLWE^m problem whose security was proved in [11]. They showed that for a set X of quality ξ, there exists a reduction from $\mathsf{LWE}_{n-1,m,q,\sigma}$ to $\mathsf{extLWE}^t_{(n,m,q,\sqrt{\sigma^2\xi^2+r^2},X)}$. Based on reduction from LWE to extLWE in [11], we prove a reduction of spLWE as in diagram below. Here ω, γ, σ are constant satisfying

$$\omega \leq \sigma\sqrt{2n\rho^2(2+2\sqrt{\rho}+\rho)},\ \gamma = \rho\sigma\sqrt{\theta(1+2\sqrt{\rho}+\rho)},\ \sigma \geq (\ln(2n(1+1/\epsilon))/\pi)^{1/2}/q.$$

Because $\mathsf{Qual}(X_{n,\rho,\theta}) < 1+\sqrt{\rho}$ by lemma 5, $\mathsf{extLWE}_{k,n,q,\sigma\sqrt{(1+\sqrt{\rho})^2+1},X_{n,\rho,\theta}}$ is secure based on $\mathsf{LWE}_{k-1,n,q,\sigma}$. Following theorem shows that $\mathsf{spLWE}_{n,m,q,\omega,\rho,\theta}$ problem is secure based on $\mathsf{LWE}_{k,m,q,\gamma}$ and $\mathsf{extLWE}_{n,m,q,\sigma\sqrt{(1+\sqrt{\rho})^2+1},X}$ for some $\omega, \gamma > 0$. This shows if $\binom{n}{\theta} \cdot (2l+2)^\theta > k\log q + 2\log(1/\delta)$, there is reduction between $\mathsf{spLWE}_{n,m,q,\omega,\rho,\theta}$ and $\mathsf{LWE}_{k-1,m,q,\sigma}$ problems.



Theorem 4. *Let k, n, m, $q \in \mathbb{N}$, $\epsilon \in (0,1/2)$, and δ,ω,α,$\gamma > 0$ such that*

$$\alpha \geq \sqrt{2\ln(2n(1+1/\epsilon))/\pi}/q,\ \beta = \alpha\sqrt{(1+\sqrt{\rho})^2+1},$$

$$\omega = \rho\beta\sqrt{2\theta},\ \gamma = \rho\beta\sqrt{\theta},\ \binom{n}{\theta}\cdot(2l+2)^\theta \geq k\log q + 2\log(1/\delta).$$

There exist (two) reductions to $\mathsf{spLWE}_{n,m,q,\omega,\rho,\theta}$ from $\mathsf{extLWE}^m_{k,n,q,\beta,X}$, $\mathsf{LWE}_{k,m,q,\gamma}$. Advantage of $\mathcal{A}$ for $\mathsf{LWE}_{n,m,q,\leq\alpha}(\mathcal{D})$ is bounded as follow:

$$Adv[\mathcal{A}] \leq 2Adv[\mathcal{C}_1] + Adv[\mathcal{C}_2] + 4m\epsilon + \delta$$

for the algorithms (distinguishers) of $\mathsf{extLWE}^m_{k,n,q,\beta,X}$, $\mathsf{LWE}_{k,m,q,\gamma}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Proof. The proof follows by a sequence of distribution to use hybrid argument as in [11]. We consider the following six distributions:

H_0:= $\{(\mathbf{A}, \mathbf{b} = \mathbf{A}^T\mathbf{x}+\mathbf{e}) \mid \mathbf{A} \leftarrow \mathbb{T}_q^{n\times m}, \mathbf{x} \leftarrow X, \mathbf{e} \leftarrow D^m_{\alpha'} \text{ for } \alpha' = \sqrt{\beta^2\|\mathbf{x}\|^2+\gamma^2} \leq \rho\beta\sqrt{2\theta} = \omega\}$.

H_1:=$\{(\mathbf{A}, \mathbf{A}^T\mathbf{x} - \mathbf{N}^T\mathbf{x} + \hat{\mathbf{e}} \mod 1) \mid \mathbf{A} \leftarrow \mathbb{T}_q^{n\times m}, \mathbf{x} \leftarrow X, \mathbf{N} \leftarrow D^{n\times m}_{\frac{1}{q}\mathbb{Z},\beta}, \hat{\mathbf{e}} \leftarrow D^m_\gamma\}$.

H_2:= $\{(q\mathbf{C}^T\mathbf{B}+\mathbf{N}, q\mathbf{B}^T\mathbf{C}\mathbf{x}+\hat{\mathbf{e}}) \mid \mathbf{B}\leftarrow\mathbb{T}_q^{k\times m}, \mathbf{C}\leftarrow\mathbb{T}_q^{k\times n},\ \mathbf{x}\leftarrow X, \mathbf{N}\leftarrow D^{n\times m}_{\frac{1}{q}\mathbb{Z},\beta}, \hat{\mathbf{e}}\leftarrow D^m_\gamma\}$.

H_3:= $\{(q\mathbf{C}^T\mathbf{B}+\mathbf{N}, \mathbf{B}^T\mathbf{s}+\hat{\mathbf{e}}) \mid \mathrm{s}\leftarrow\mathbb{Z}_q^k,\ \mathbf{B}\leftarrow\mathbb{T}_q^{k\times m}, \mathbf{C}\leftarrow\mathbb{T}_q^{k\times n},\ \mathbf{N}\leftarrow D^{n\times m}_{\frac{1}{q}\mathbb{Z},\beta}, \hat{\mathbf{e}}\leftarrow D^m_\gamma\}$.

H_4:= $\{(q\mathbf{C}^T\mathbf{B}+\mathbf{N}, \mathbf{u}) \mid \mathbf{u}\leftarrow\mathbb{T}^m,\ \mathbf{B}\leftarrow\mathbb{T}_q^{k\times m}, \mathbf{C}\leftarrow\mathbb{T}_q^{k\times n},\ \mathbf{N}\leftarrow D^{n\times m}_{\frac{1}{q}\mathbb{Z},\beta}\}$.

H_5:= $\{(\mathbf{A}, \mathbf{u}) \mid \mathbf{A}\leftarrow\mathbb{T}_q^{n\times m}, \mathbf{u}\leftarrow\mathbb{T}^m\}$.

Let $\mathcal{B}_i$ be the distinguisher for the distributions between H_i and H_{i+1} for $0\le i\le 4$. There exist some efficient transformations from the distributions $(\mathbf{C}, \mathbf{A}, \mathbf{N}^T\mathbf{z}), (\mathbf{C}, \hat{\mathbf{A}}, \mathbf{N}^T\mathbf{z})$ to H_1, H_2, from $(\mathbf{B}, \mathbf{B}^T\mathbf{s}+\hat{\mathbf{e}}), (\mathbf{B}, \mathbf{u})$ to H_3, H_4, and from $(\mathbf{C}, \hat{\mathbf{A}}), (\mathbf{C}, \mathbf{A})$ to H_4, H_5. In fact, the samples $(\mathbf{C}, \hat{\mathbf{A}}, \mathbf{N}^T\mathbf{z})$, $(\mathbf{B}, \mathbf{B}^T\mathbf{s}+\hat{\mathbf{e}})$, and $(\mathbf{C}, \hat{\mathbf{A}})$ are $\mathsf{extLWE}^m_{k,n,q,\beta,X}$, $\mathsf{LWE}_{k,m,q,\gamma}$, $\mathsf{extLWE}^m_{k,n,q,\beta,\{0^n\}}$ samples respectively. The others are uniform distribution samples in the corresponding domain. It follows that $Adv[\mathcal{B}_1]$, $Adv[\mathcal{B}_3]$, $Adv[\mathcal{B}_4]$ are bounded by the distinguishing advantages of $\mathsf{extLWE}^m_{k,n,q,\beta,X}$, $\mathsf{LWE}_{k,m,q,\gamma}$, $\mathsf{extLWE}^m_{k,n,q,\beta,\{0^n\}}$ respectively.

Since $\|\mathbf{x}\|\le\rho\sqrt{\theta}$, and $\beta\ge\sqrt{2\ln(2n(1+1/\epsilon))/\pi}/q\ge\sqrt{2}\eta_\epsilon(\mathbb{Z}^n)/q$ from lemma 3, it follows that the statistical distance between $-\mathbf{N}^T\mathbf{x}+\hat{\mathbf{e}}$ and $D^m_{\alpha'}$ is at most $4m\epsilon$ by lemma 2. This gives $Adv[\mathcal{B}_0]\le 4m\epsilon$. The last $Adv[\mathcal{B}_2]$ is bounded by δ from the Leftover hash lemma. To sum up, $Adv[\mathcal{A}]\le 2Adv[\mathcal{C}_1]+Adv[\mathcal{C}_2]+4m\epsilon+\delta$ with trivial reduction to $\mathsf{extLWE}^m_{k,n,q,\beta,\{0^n\}}$ from $\mathsf{extLWE}^m_{k,n,q,\beta,X}$. □

4.2 Attacks for spLWE

There exist many attacks for LWE including dual attack, primal attacks. ([3], [17]). Here, we do not consider the combinatorial BKW algorithm, its variants or attacks based on the Arora and Ge algorithm that are not suitable in our case.([1], [5], [19], [28], [25]). Since the analysis of traditional dual attack is based on the (discrete) Gaussian error (and secret in the LWE normal form), these traditional attacks are not directly applicable for spLWE. Therefore we modified those attacks for analysis of spLWE including random guess about 0 component in a sparse secret vector $\mathbf{s}$.

Dual (distinguish) Attack Assume that we are given $(\mathbf{A}, \boldsymbol{b})\in\mathbb{Z}_q^{m\times n}\times\mathbb{Z}_q^m$, and want to distinguish whether they are uniform random samples or spLWE samples. For a constant $c\in\mathbb{R}$ with $c\le q$, consider a lattice $L_c(\mathbf{A})$ defined by

$$L_c(\mathbf{A}) = \left\{(\boldsymbol{x}, \boldsymbol{y}/c)\in\mathbb{Z}^m\times(\mathbb{Z}/c)^n : \boldsymbol{x}^T\mathbf{A} = \boldsymbol{y} \mod q\right\}.$$

If the samples $(\mathbf{A}, \boldsymbol{b})$ are come from spLWE, for $(\boldsymbol{x}, \boldsymbol{y})\in L_c(\mathbf{A})$, we have

$$\begin{aligned}\langle\boldsymbol{x}, \boldsymbol{b}\rangle &= \langle\boldsymbol{x}, \mathbf{A}\boldsymbol{s}+\boldsymbol{e}\rangle\\ &= \langle\boldsymbol{x}, \mathbf{A}\boldsymbol{s}\rangle+\langle\boldsymbol{x}, \boldsymbol{e}\rangle\\ &= c\langle\boldsymbol{y}, \boldsymbol{s}\rangle+\langle\boldsymbol{x}, \boldsymbol{e}\rangle \mod q\end{aligned}$$

We see that, for a sufficiently small vector $(\boldsymbol{v}, \boldsymbol{w})\in L_c(\mathbf{A})$, the value $\langle\boldsymbol{v}, \boldsymbol{b}\rangle \mod q$ becomes small when the samples are spLWE ones, and $\langle\boldsymbol{v}, \boldsymbol{b}\rangle \mod q$ is uniformly distributed when $(\mathbf{A}, \boldsymbol{b})$ are come from the uniform distribution. Hence one can decide whether the samples come from spLWE distribution or uniform distribution from the size of $\langle\boldsymbol{v}, \boldsymbol{b}\rangle \mod q$ with some success probability. We now determine how small a vector $(\boldsymbol{v}, \boldsymbol{w})$ must be found as follows. First, we estimate the length of $(\boldsymbol{v}, \boldsymbol{w})\in L_c(\mathbf{A})$. One can easily check that

$$\left(\begin{array}{c|c} I_m & 0 \\ \hline \frac{1}{c}\mathbf{A}^T & \frac{q}{c}I_n \end{array}\right)$$

is a basis matrix of $L_c(\mathbf{A})$. Hence we know $\dim(L_c(\mathbf{A})) = m+n$ and $\det(L_c(\mathbf{A})) = (q/c)^n$.

Therefore a lattice reduction algorithm with a root Hermite factor δ_0 gives $(\boldsymbol{v}, \boldsymbol{w}) \in L_c(\mathbf{A})$, such that

$$||(\boldsymbol{v}, \boldsymbol{w})|| = \delta_0^{m+n}(q/c)^{\frac{n}{m+n}}, \tag{1}$$

and the length is minimized when $m = \sqrt{n(\log q - \log c)/\log \delta_0} - n$.

Next, we consider the distribution of $c\langle \boldsymbol{w}, \boldsymbol{s}\rangle + \langle \boldsymbol{v}, \boldsymbol{e}\rangle \mod q$. Here, we assume that the coefficients of sparse vector $\boldsymbol{s}$ are sampled independently by $(b_1 d_1, b_2 d_2, \ldots, b_n d_n)$ where $d_i \leftarrow \mathrm{Ber}(n, \theta/n)$, $b_i \leftarrow \{\pm 1, \pm 2, \pm 4, \ldots, \pm\rho\}$, and $\rho = 2^l$ for some $l \in \mathbb{Z}_{\geq 0}$. Since $c\langle \boldsymbol{w}, \boldsymbol{s}\rangle$ is the sum of many independent random variables, asymptotically it follows a Gaussian distribution with mean 0, and variance $(c||\boldsymbol{w}||)^2 \cdot \frac{2\theta(4^{l+1}-1)}{3n(2l+2)}$. From that $\langle \boldsymbol{v}, \boldsymbol{e}\rangle$ follows a Gaussian distribution with mean 0, variance $(\sigma||\boldsymbol{v}||)^2$, and lemma 4, we have distinguishing advantage

$$\exp(-\pi(s'/q)^2) \text{ where } s' = \sqrt{2\pi}\sqrt{\sigma^2||\boldsymbol{v}||^2 + c^2\frac{2\theta(4^{l+1}-1)}{3n(2l+2)}||\boldsymbol{w}||^2}. \tag{2}$$

From above equations 1, 2 with distinguishing advantage ϵ, the attacker need to find small δ_0 such that

$$\delta_0 = (\frac{c}{q})^n(\frac{q\ln(1/\epsilon)}{M\pi})^{1/m+n} \text{ where } M = \sqrt{2\pi}\sqrt{\sigma^2\frac{m}{m+n} + c^2\frac{2\theta(4^{l+1}-1)}{3n(2l+2)}\frac{n}{m+n}}$$

5 Parameter Selection and Implementation Result

5.1 Parameter Selection

To deduce some parameters, we assume that the best known classical and quantum SVP (sieving) algorithm in dimension k runs in time $2^{0.292k+o(k)}$ and $2^{0.265k+o(k)}$ respectively[9, 29]. The BKZ 2.0 algorithm gives the root Hermite factor $\delta_0 \approx (\frac{k}{2\pi e}(\pi k)^{1/k})^{1/2(k-1)}$ for block size k [14], and the iteration number of exact SVP solver is $\frac{n^3}{k^2}\log n$ [26].

We consider direct CVP attack by sieving [30], the modified dual (distinguish), and embedding attack. We also consider a trade-off, ignoring components on secret vectors, specified to spLWE. More details are provided in appendix 6.2. To sum up, the parameters must satisfy the following for the quantum security:

- $n\log q \cdot (2l+1)^\theta \cdot \binom{n}{\theta} > 2^{2\lambda}$ from bruteforce attack (grover algorithm), where $\binom{n}{\theta} = \frac{n!}{(\theta!)(n-\theta!)}$ (For classical security, 2λ becomes λ)
- Let $T(n, q, \theta, s, l)$ be a BKZ 2.0 running time to get root Hermite factor δ_0 satisfying following equation:

$$\delta_0 = \max_{1<c<q, 1\leq m\leq n}\left\{(\frac{c}{q})^n(\frac{q\ln(1/\epsilon)}{M\pi})^{1/m+n}\right\}$$

where

$$M = \sqrt{2\pi}\cdot\sqrt{\sigma^2\frac{m}{m+n} + c^2\frac{2\theta(4^{l+1}-1)}{3n(2l+2)}\frac{n}{m+n}}.$$

Since our secret key is a sparse vector, one can guess some components to be zero. To take λ-bit secure parameters, it should satisfy the following:

$$\min_k \left\{ \frac{1}{p_{n,k,\theta}} \cdot T(n-k, q, \theta, s, l) \right\} > 2^\lambda \text{ where } p_{n,k,\theta} = \binom{n-\theta}{k} / \binom{n}{k}$$

.

- To prevent direct CVP attack, n, θ should satisfy following equation:

$$\min_k \left\{ \frac{1}{p_{n,k,\theta}} \cdot 2^{0.265(n-k)} \right\} > 2^\lambda \text{ for classical security, 0.265 becomes 0.292.}$$

- For the correctness, $q \geq 8s\rho\sqrt{\frac{2\theta}{\pi}\ln(2/\gamma)}$ by the Lemma 7. We set the $\gamma = 0.01$ in our parameter setting.

Proposed parameters. From numerical optimization, we propose various parameters for security parameter λ from 72 to 128. The result is in the table below.

λ	n	q	s	θ	$\log(t_{dual})$	$\log(t_{SVP})$
72	225	228	5	22	82	73
96	290	261	5	29	102	97
128	390	302	5	39	130	130

Table 1: Parameters with sparse secret key for $s_i \in \{0, \pm 1\}$ (classical).

λ	n	q	s	θ	$\log(t_{dual})$	$\log(t_{SVP})$
72	300	233	5	30	85	76
96	375	265	5	37	105	100
128	500	305	5	50	133	134

Table 2: Parameters with sparse secret key for $s_i \in \{0, \pm 1\}$ (quantum).

5.2 Implementation Result

We use C++ on a Linux-based system, with GCC compiler and apply the Eigen library (www.eigen.tuxfamily.org) which makes vector and matrix operations fast. To sample $\boldsymbol{u}$ efficiently in our encryption algorithm, we assume that there are only one non-zero element in each n/θ-size block. To follow the previous reduction and security proof, we need sampling of discrete Gaussian distribution when we generate error vectors in key generation and encryption algorithm. We use *box-muller transformation* to generate discretized Gaussian distribution. In below case, message space length is 32-byte and secret key is ternary vector. We used PC with CPU 2.6GHz Intel Core i5 without parallelization.

					IND-CPA				IND-CCA		
λ	n	q	s	θ	Setup(ms)	Enc(μs)	Dec(μs)	Cptx(byte)	Enc(μs)	Dec(μs)	Cptx(byte)
72	225	270	5	22	3.4	43	15	259	57	69	279
96	290	320	5	29	5.1	52	17	333	60	70	365
128	390	400	5	39	8.4	81	21	453	83	91	501

Table 3: Implementation result in classical hardness with 256 bit message

					IND-CPA				IND-CCA		
λ	n	q	s	θ	Setup(ms)	Enc(μs)	Dec(μs)	Cptx(byte)	Enc(μs)	Dec(μs)	Cptx(byte)
72	300	320	5	30	5.4	55	16	344	61	69	400
96	375	370	5	37	7.9	70	26	431	89	96	511
128	500	470	5	50	13.2	135	26	586	135	142	698

Table 4: Implementation result in quantum hardness with 256 bit message

We also compare with software implementation in [24], which implements LWE based PKE [38] and Ring version PKE [32]. Their implementation is executed on an Intel Core 2 Duo CPU running at 3.00 GHz PC. Parameters in each rows are secure in same security parameters.

Our scheme			[24]	LWE		RLWE	
(n, q, s, θ)	Enc	Dec	(n, q, s)	Enc	Dec	Enc	Dec
(150,240,5,15)	0.02	0.012	(128,2053,6.77)	3.01	1.24	0.76	0.28
(290,320,5,29)	0.04	0.015	(256,4093,8.87)	11.01	2.37	1.52	0.57
(390,400,5,39)	0.05	0.017	(384,4093,8.35)	23.41	3.41	2.51	0.98
(550,470,5,49)	0.08	0.02	(512,4093,8.0)	46.05	4.52	3.06	1.18

Table 5: Our scheme vs. LWE vs. RLWE: Time in milliseconds for encryption, and decryption for a 32-byte plaintext.

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6 Appendix

6.1 Attacks for search spLWE

Dual (search) Attack In this section, we assume the Geometric Series Assumption (GSA) on q-ary lattices, introduced by Schnorr [40], and this will be used to estimate the length of last vector of BKZ 2.0 reduced basis. Let $\mathbf{B} = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ be a basis for an n-dimensional lattice Λ which is reduced by the BKZ 2.0 with root Hermite factor δ_0, then the GSA says:

$$\|\boldsymbol{b}_i^*\| = \beta^{i-1} \cdot \|\boldsymbol{b}_1^*\| \text{ for some constant } 0 < \beta \leq 1,$$

where $\{\boldsymbol{b}_1^*, \cdots, \boldsymbol{b}_n^*\}$ is the Gram-schmidt orthogonalization of $\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$. From $\|\boldsymbol{b}_1\| = \delta_0^n \cdot \det(\mathbf{B})^{1/n}$, we have:

$$\det(\mathbf{B}) = \prod_{i=1}^{n} \|\boldsymbol{b}_i^*\| = \prod_{i=1}^{n} \beta^{i-1} \cdot \|\boldsymbol{b}_1^*\| = \beta^{\frac{(n-1)n}{2}} \cdot \delta_0^{n^2} \cdot \det(\mathbf{B}).$$

From the above equation, it follows that $\beta = \delta_0^{-2n^2/(n-1)n}$. Since BKZ reduced basis satisfies $\boldsymbol{b}_i = \boldsymbol{b}_i^* + \sum_{j=0}^{i-1} \mu_{ij} \cdot \boldsymbol{b}_j^*$ with $|\mu_{ij}| \leq 1/2$, one can show that,

$$\|\boldsymbol{b}_i\| \leq ||\boldsymbol{b}_1|| \cdot \sqrt{\frac{1-\beta^{2i-2}}{4-4\beta^2} + \beta^{2i-2}}.$$

We now describe the dual attack against a small number of LWE instances $(\mathbf{A}, \mathbf{A}\boldsymbol{s} + \boldsymbol{e}) = (\mathbf{A}, \boldsymbol{b}) \in \mathbb{Z}^{m \times n} \times \mathbb{Z}^m$. For some constant $c \in \mathbb{N}$ with $c \leq q$, we consider a scaled lattice $\Lambda_c(\mathbf{A})$.

$$\Lambda_c(\mathbf{A}) = \{(\boldsymbol{x}, \boldsymbol{y}/c) \in \mathbb{Z}^m \times (\mathbb{Z}^n/c) : \boldsymbol{x}\mathbf{A} = \boldsymbol{y} \bmod q\}.$$

A dimension and determinant of the lattice $\Lambda_c(\mathbf{A})$ is $n + m$ and $(q/c)^n$, respectively. With the above assumptions, we can obtain vectors $\{(\boldsymbol{v}_i, \boldsymbol{w}_i)\}_{1 \leq i \leq n}$ in $\Lambda_c(\mathbf{A})$ such that,

$$\|(\boldsymbol{v}_i, \boldsymbol{w}_i)\| \leq \delta_0^{m+n} \cdot (q/c)^{\frac{n}{m+n}} \cdot \sqrt{\frac{1-\beta^{2i-2}}{4-4\beta^2} + \beta^{2i-2}} \approx \delta_0^{m+n} (q/c)^{\frac{n}{m+n}} \cdot \sqrt{\frac{1}{4-4\beta^2}}.$$

Clearly, the element $(\boldsymbol{v}_i, \boldsymbol{w}_i)$ in $\Lambda_c(\mathbf{A})$ satisfies

$$\boldsymbol{v}_i \cdot \boldsymbol{b} = \boldsymbol{v}_i \cdot (A \cdot \boldsymbol{s} + \boldsymbol{e}) = \langle c \cdot \boldsymbol{w}_i, \boldsymbol{s} \rangle + \langle \boldsymbol{v}_i, \boldsymbol{e} \rangle = \langle (\boldsymbol{v}_i, \boldsymbol{w}_i), (\boldsymbol{e}, c \cdot \boldsymbol{s}) \rangle \bmod q.$$

If, for $1 \leq i \leq n$, $(\boldsymbol{v}_i, \boldsymbol{w}_i)$ is short enough to satisfy $\|(\boldsymbol{v}_i, \boldsymbol{w}_i)\| \cdot \|(\boldsymbol{e}, c \cdot \boldsymbol{s})\| < q/2$, the above equation are hold over $\mathbb{Z}$. Then we can recover $\boldsymbol{e}$ and $\boldsymbol{s}$ by solving a system of linear equations. Since, $\|(\boldsymbol{e}, c\boldsymbol{s})\| \approx \sqrt{n \cdot \sigma^2 + c^2 \cdot \|\boldsymbol{s}\|^2}$, condition for attack is following:

$$\delta_0^{n+m} \cdot (q/c)^{\frac{n}{m+n}} \cdot \sqrt{\frac{n \cdot \sigma^2 + c^2 \cdot \|\boldsymbol{s}\|^2}{4-4\beta^2}} < \frac{q}{2}$$

for constant $0 < c \leq q$. To find an optimized constant c we assume $m = n$. In this case, the size is optimized with $c = \sqrt{n \cdot \sigma^2 / ||s||^2}$. Therefore final condition to success attack is following:

$$2\delta_0^{4n} \cdot \sigma \cdot \|\boldsymbol{s}\| \cdot \sqrt{n} < q(1-\beta^2).$$

Modified Embedding Attack One can reduce the LWE problem to unique-SVP problem via Kannan's embedding technique. First, we consider a column lattice

$$\Lambda_q(\mathbf{A}') = \{\boldsymbol{y} \in \mathbb{Z}^{m+1} : \boldsymbol{y} = \mathbf{A}'\boldsymbol{x} \bmod q\} \text{ for } \mathbf{A}' = \begin{pmatrix} 1 & 0 \\ -\boldsymbol{b} & \mathbf{A} \end{pmatrix}.$$

The vector $(1, \boldsymbol{e})^T$ is in lattice $\Lambda_q(\mathbf{A}')$, and its size is approximately $\sigma\sqrt{m}$. If this value is sufficiently smaller than $\lambda_2(\Lambda_q(\mathbf{A}'))$ ($\approx \sqrt{\frac{m}{2\pi e}} q^{(m-n)/m}$). In [2], they showed a shortest vector of $m+1$ dimension lattice Λ_{m+1} can be found with high probability if

$$\frac{\lambda_2(\Lambda_{m+1})}{\lambda_1(\Lambda_{m+1})} = \frac{\lambda_2(\Lambda_q(\mathbf{A}))}{||(1, \boldsymbol{e})||} \geq \tau \cdot \delta_0^m,$$

where $\tau \approx 0.4$. In this case, the BKZ algorithms are used as a SVP solver. For spLWE case, we can obtain more larger gap than the ordinary attack. We also consider a scaled lattice $\Lambda_c(\mathbf{B})$ generated by following matrix:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\mathbf{I}_n & 0 \\ -\boldsymbol{b} & \mathbf{A} & q\mathbf{I}_m \end{pmatrix}$$

for a constant $0 < c < 1$. The vector $(1, c\boldsymbol{s}, \boldsymbol{e})^T$ is in this lattice, and its size is approximately $\sqrt{m \cdot \sigma^2 + c^2\|\boldsymbol{s}\|^2}$. Let define a matrix $\mathbf{B}'$ as following,

$$\mathbf{B}' = \begin{pmatrix} c\mathbf{I}_n & 0 \\ \mathbf{A} & q\mathbf{I}_m \end{pmatrix}.$$

Now we have $\lambda_1(\Lambda_c(\boldsymbol{B})) = \sqrt{m \cdot \sigma^2 + c^2 \cdot ||\boldsymbol{s}||^2}$ and $\lambda_1(\Lambda_c(\mathbf{B}')) = \sqrt{\frac{n+m}{2\pi e}} \cdot \det(\Lambda_c(\mathbf{B}'))^{1/(n+m)} = \sqrt{\frac{n+m}{2\pi e}} \cdot (q^m c^n)^{1/n+m}$. Therefore it is needed to find the root Hermite factor δ_0 such that:

$$\sqrt{\frac{n+m}{2\pi e}} \cdot (q^m c^n)^{1/n+m} \geq 0.4 \cdot \delta_0^{n+m} \cdot \sqrt{m \cdot \sigma^2 + c^2\|\boldsymbol{s}\|^2} \tag{3}$$

$$\Leftrightarrow \sqrt{\frac{n+m}{2\pi e \cdot (m \cdot \sigma^2 + c^2\|\boldsymbol{s}\|^2)}} \cdot (q^m c^n)^{1/n+m} \geq 0.4 \cdot \delta_0^{n+m} \tag{4}$$

The left part of the above inequality is maximized when $c = \sqrt{n\sigma^2}/\|\boldsymbol{s}\|$, so we have:

$$\sqrt{\frac{1}{2\pi e \cdot \sigma^2}} \left(q^m \cdot \left(\frac{\sigma\sqrt{n}}{\|\boldsymbol{s}\|} \right)^n \right)^{1/(n+m)} \geq 0.4 \cdot \delta_0^{n+m}$$

6.2 Improving Lattice Attacks for spLWE

Since the attacks described in section 4.2, 6.1, and 6.2, time complexity of them are heavily rely on the lattice reduction algorithm used. Namely, if one can reduce the dimension of lattice in attacks, one can obtain a high advantage to solve the LWE problem. In this section, we introduce two dimension reducing techniques to improve lattice attacks for spLWE instances. The first one is ignoring components of a sparse secret and the other one is the dimension modulus switching technique in [11]. For convenience, we denote $T(m)$ as the expected time of solving m-dimensional LWE.

Ignoring Components on Secret Vectors Most entries of secret vector $\boldsymbol{s}$ are zero. Therefore, ignoring some components, one can reduce the dimension of LWE. More precisely, we delete k entries of secret vector $\boldsymbol{s}$ and its corresponding column of $\mathbf{A}$. For convenience, we denote it as $\boldsymbol{s}'$ and $\mathbf{A}'$, respectively. If the deleted components of $\boldsymbol{s}$ are zero, the following equation is also hold:

$$\mathbf{A} \cdot \boldsymbol{s} + \boldsymbol{e} = \mathbf{A}' \cdot \boldsymbol{s}' + \boldsymbol{e} \bmod q.$$

The probability P_k that the selected k entries are zero is $\binom{n-\theta}{k}/\binom{n}{k}$. It implies that one can reduce the n-dimensional LWE to $(n-k)$-dimensional LWE with probability P_k. Namely, solving $1/P_k$ instances in $(n-k)$-dimensional LWE, one can expect to solve the n dimension LWE. Hence to guarantee λ bits security it gives:

$$T(n-k)/P_k \geq 2^{\lambda}. \tag{5}$$

Modulus Dimension Switching In [11], for LWE instances, the authors describe a modulus dimension switching technique. Using the corollary 3.4 in [11], for n, q, θ, w that divides n and $\epsilon \in (0, 1/2)$, one can reduce a $\mathsf{LWE}_{n,q,\leq\alpha}$ instance to $\mathsf{LWE}_{n/w,q^w,\leq\beta}$ instances, where β is a constant satisfying $\beta^2 \geq \alpha^2 + (4/\pi)\ln(2n(1+1/\epsilon))\cdot\theta/q^2 \approx \alpha^2$. Especially, in this reduction, a secret vector $\boldsymbol{s} = (s_1, s_2, \cdots, s_n)$ of $\mathsf{LWE}_{n,q,\leq\alpha}$ is changed to $\boldsymbol{s}'' = (s_1 + qs_2 + \cdots + q^{w-1}s_w, \cdots, s_{n-w+1} + \cdots + q^{w-1}s_n)$ of $\mathsf{LWE}_{n/w,q^w,\leq\beta}$. It means that if one can recover the $\boldsymbol{s}''$ solving $\mathsf{LWE}_{n/w,q^w,\leq\beta}$ instances, one can also reveal the vector $\boldsymbol{s}$. Let t be the number of set $W = \{s_{wi} | s_{wi} \neq 0, 1 \leq i \leq n/w\}$ and P'_w be the probability of $t = 0$. P'_w is equal to $\dfrac{\binom{n-\theta}{n/w}}{\binom{n}{n/w}}$. When t is not zero, the expected size of $\|\boldsymbol{s}''\|$ is $\sqrt{tq^w}$. Thus, in that case, its size is not sufficiently short and applying the solving LWEalgorithms in section 4.2, 6.1, and 6.2 to new n/w-dimensional LWE instances are not appropriate to gain the advantage. Hence, we only consider the case $t = 0$. As a similar reason to ignoring components on secret vectors, to get λ-bit security, we can obtain the following conditions:

$$T(n/w)/P'_w \geq 2^{\lambda}. \tag{6}$$

Combining the ignoring k components and modulus dimension switching techniques for n/w-dimension, one can reach the final conditions to obtain the λ-bit security:

$$T((n-k)/w)/(P_k P'_w) \geq 2^{\lambda}. \tag{7}$$