PROBABILITY THAT THE K-GCD OF PRODUCTS OF POSITIVE INTEGERS IS B-SMOOTH

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ABSTRACT. In 1849, Dirichlet [5] proved that the probability that two positive integers are relatively prime is $1/\zeta(2)$. Later, it was generalized into the case that positive integers has no nontrivial kth power common divisor. In this paper, we further generalize this result: the probability that the gcd of m products of n positive integers is B-smooth is $\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]$ for $m \geq 2$. We show that it is lower bounded by $\frac{1}{\zeta(s)}$ for some $s > 1$ if $B > n^{\frac{m}{m-1}}$, which completes the heuristic proof in the cryptanalysis of cryptographic multilinear maps by Cheon et al. [2]. We extend this result to the case of k-gcd: the probability is $\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\left(1+\frac{{}_nH_1}{p}+\cdots+\frac{{}_nH_{k-1}}{p^{k-1}}\right)\right\}^m\right]$, where ${}_nH_i = \binom{n+i-1}{i}$.

1. Introduction

In 1849, Dirichlet [5] proved that the probability that two positive integers are relatively prime is $1/\zeta(2)$. To be precise,

$$\lim_{N\to\infty} \frac{\left|\left\{(x_1, x_2) \in \{1, 2, .., N\}^2 : \gcd(x_1, x_2) = 1\right\}\right|}{N^2} = \frac{1}{\zeta(2)}.$$

Lehmer [7] and more recently Nymann [10] extended this result that the probability that the r positive integers are relatively prime is $1/\zeta(r)$.

Meanwhile, in 1885, Gegenbauer [6] proved that the probability that a positive integer is not divisible by rth power for an integer $r \geq 2$ is $1/\zeta(r)$. In 1976, Benkoski [1] combined Gegenbauer and Lehmer's results and obtain that the probability that r positive integers are relatively k-prime is $1/\zeta(rk)$. For positive integers $x_1, ..., x_r$ and k, we denote by $\gcd_k(x_1, .., x_r)$ or k-gcd of $x_1, ..., x_r$ the largest kth power that divides $x_1, ..., x_r$. If $\gcd_k(x_1, .., x_r) = 1$, we call $x_1, .., x_r$ are relatively k-prime.

Later, study on the probability of gcd was extended by changing domain from $\mathbb{Z}$ to other Principal Ideal Domains. One extension is the result of Collins and Johnson [3] in 1989 that the probability that two Gaussian integers are relatively prime is $1/\zeta_{\mathbb{Q}(i)}(2)$. In 2004, Morrison and Dong [8] extended Benkoski's result to the ring $\mathbb{F}_q[x]$ for a finite field $\mathbb{F}_q$. More recently, in 2010, Sittinger [11] extended Benkoski's result to the algebraic integers over the algebraic number field K: the probability that k algebraic integers are relative r-prime is $1/\zeta_{O_K}(rk)$ while O_K is the ring of algebraic integers in K, and $\zeta_O(rk)$ denotes the Dedekind zeta function over O_K.

Key words and phrases. gcd of products of positive integers, B-smooth, k-gcd.

In this paper, we move our question to the probability that the gcd of products of positive integers is B-smooth. We investigate the probability that the gcd of *products* of positive integers is B-smooth. Given positive integers $m \geq 2$ and n, assume that r_{ij}'s are positive integers chosen randomly and independently in $[1, N]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Our theorem states that the probability that $\gcd(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ is B-smooth converges to $\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]$ as $N \to \infty$. This is proved by using Lebesgue Dominated Convergence Theorem and the inclusion and exclusion principle.

We show that the value of $\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]$ is lower bounded by $\prod_{B<p\leq\hat{n}}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right] \cdot \prod_{\hat{n}<p\leq\hat{r}}\left\{1-\left(\frac{n}{p}\right)^m\right\} \cdot \frac{1}{\zeta(s)}$ for $\hat{n} = \max\{n, B\}$, $r = \lfloor n^{\frac{m}{m-1}} + 1\rfloor$, $\hat{r} = \max\{\hat{n}, r\}$ and $s = m(1 - \log_{\hat{r}} n) > 1$. Note that the first product term is equal to 1 if $B = \hat{n}$, and the second product term is equal to 1 if $\hat{n} = \hat{r}$. Thus our theorem proves the heuristic argument in the lemma in [2, page 10] to tell that this probability is lower bounded by $1/\zeta(s)$ in case of $B = 2n$ and $\frac{m}{\log_2 2n} > 1$. The lemma is used to guarantee the success probability of the cryptanalysis of cryptographic multilinear maps proposed by Coron et al. [4].

Finally, we extend the theorem to the case of k-gcd. When r_{ij}'s are chosen randomly and independently from $\{1, \cdots, N\}$, we show that the probability that $\gcd_k(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ is B-smooth converges to

$$\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\left(1+\frac{{}_nH_1}{p}+\cdots+\frac{{}_nH_{k-1}}{p^{k-1}}\right)\right\}^m\right]$$

as $N \to \infty$, where ${}_nH_i = \binom{n+i-1}{i}$. This result is another generalized form of Benkoski's.

Notations. For an integer x, if x has no prime divisor larger than B, we say that x is B-smooth. For a finite set X, the number of elements of X is denoted by $|X|$. All of the error terms in this paper are only about the positive integer N, *i.e.* O is actually O_N. For positive integers $x_1, ..., x_r$, and k, we denote by $\gcd_k(x_1, ..., x_r)$ or the k-gcd of $x_1, ..., x_r$ the largest kth power that divides $x_1, ..., x_r$. Note that the usual gcd is 1-gcd. From now on, alphabet p always denotes a prime number, and $\bigsqcup$ is a disjoint union.

2. Probability that the gcd of products of positive integers is B-smooth

2.1. **The gcd of products of positive integers.** In this section, we fix the positive integers $m \geq 2$ and n. For a positive integer N, r_{ij}'s are integers uniformly and independently chosen in $[1, N]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The aim of this section is to compute the probability that $\gcd(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ is B-smooth when $N \to \infty$. Denote by $p_1, p_2, p_3....$ the prime numbers larger than B in increasing order, and define $T(\ell, N)$ be the number of ordered pairs (r_{ij}) such that $\gcd(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ is coprime to $p_1, ..., p_\ell$ for $1 \leq r_{ij} \leq N$. Note that $\lim_{\ell\to\infty} T(\ell, N)/N^{mn}$ is the probability that $\gcd(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ is

B-smooth where r_{ij} are chosen randomly and independently in $\{1,2,...,N\}$. By following two steps, we obtain the value of $\lim_{N\to\infty}\lim_{\ell\to\infty} T(\ell,N)/N^{mn}$.

Theorem 2.1. *Let $p_1, p_2, ...$ be the prime numbers larger than B in increasing order. Then,*

$$\lim_{N\to\infty}\frac{T(\ell,N)}{N^{mn}} = \prod_{i=1}^{\ell}\left[1-\left\{1-\left(1-\frac{1}{p_i}\right)^n\right\}^m\right]. \tag{2.1}$$

Proof. Let $X_\ell = \{p_1, p_2, ..., p_\ell\}$ and $1 \le r_{ij} \le N$ for a positive integer N. By the inclusion and exclusion principle,

$$\begin{aligned}
&\left|\left\{(r_{ij}) : \gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj}) \text{ is coprime to } p_1, ..., p_\ell\right\}\right| \\
&= \sum_{P\subset X_\ell}(-1)^{|P|}\left|\left\{(r_{ij}) : \prod_{p\in P} p \mid \gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj})\right\}\right| \\
&= \sum_{P\subset X_\ell}(-1)^{|P|}\left|\left\{(r_{1j}) : \prod_{p\in P} p \mid \prod_{j=1}^{n} r_{1j}\right\}\right|^m.
\end{aligned}$$

where $\prod_{p\in P} p = 1$ for $P = \phi$. Applying the inclusion and exclusion principle again, we obtain

$$\begin{aligned}
\left|\left\{(r_{1j}) : \prod_{p\in P} p \mid \prod_{j=1}^{n} r_{1j}\right\}\right| &= \sum_{Q\subset P}(-1)^{|Q|}\left|\left\{(r_{1j}) : p \nmid \prod_{j=1}^{n} r_{1j}, \forall p\in Q\right\}\right| \\
&= \sum_{Q\subset P}(-1)^{|Q|}\left(\sum_{R\subset Q}(-1)^{|R|}\left\lfloor\frac{N}{\prod_{p\in R} p}\right\rfloor\right)^n.
\end{aligned}$$

Consequently, we have

$$T(\ell,N) = \sum_{P\subset X_\ell}(-1)^{|P|}\left\{\sum_{Q\subset P}(-1)^{|Q|}\left(\sum_{R\subset Q}(-1)^{|R|}\left\lfloor\frac{N}{\prod_{p\in R} p}\right\rfloor\right)^n\right\}^m.$$

Finally, using $\lfloor N/\prod_{p\in R} p\rfloor/N = 1/\prod_{p\in R} p + O(1/N)$, we have

$$\begin{aligned}
\frac{T(\ell,N)}{N^{mn}} &= \sum_{P\subset X_\ell}(-1)^{|P|}\left\{\sum_{Q\subset P}(-1)^{|Q|}\left(\sum_{R\subset Q}(-1)^{|R|}\frac{1}{\prod_{p\in R} p}\right)^n\right\}^m + O\left(\frac{1}{N}\right) \\
&= \prod_{i=1}^{\ell}\left[1-\left\{1-\left(1-\frac{1}{p_i}\right)^n\right\}^m\right] + O\left(\frac{1}{N}\right),
\end{aligned}$$

which gives the theorem as $N\to\infty$. □

Theorem 2.1 gives the probability that the gcd of products of positive integers is not divisible by the first ℓ primes greater than B. To obtain the probability that this gcd is B-smooth, we need to take $\ell\to\infty$ before taking $N\to\infty$ in Theorem 2.1. To swap the orders of limits, we use the Lebesgue Dominated Convergence Theorem for counting measure on set of natural numbers, which states:

Let $\{f_n : \mathbb{N} \to \mathbb{R}\}$ be a sequence of functions. Suppose that $\lim_{n\to\infty} f_n$ exists pointwisely and there exists a function $g : \mathbb{N} \to \mathbb{R}$ s.t $|f_n| \le g$, and $\sum_{x=1}^{\infty} g(x) < \infty$. Then we have

$$\lim_{n\to\infty} \sum_{x=1}^{\infty} f_n(x) = \sum_{x=1}^{\infty} \lim_{n\to\infty} f_n(x).$$

Theorem 2.2. *When r_{ij}'s are chosen randomly and independently from $\{1, 2, ..., N\}$, the probability that $\gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj})$ is B-smooth converges to*

$$\prod_{p>B} \left[1 - \left\{1 - \left(1 - \frac{1}{p}\right)^n\right\}^m\right]$$

as $N \to \infty$.

Proof. Define $g_N(\ell) = (T(\ell-1, N) - T(\ell, N))/N^{mn}$ and $T(0, N) = N^{mn}$. Note that $g_N(\ell)$ is the probability that $\gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj})$ is coprime to $p_1, ..., p_{\ell-1}$ and divisible by p_ℓ for randomly and independently chosen r_{ij}'s from $\{1, ..., N\}$, and so is non-negative.

We claim that

$$\lim_{N\to\infty} \sum_{\ell=1}^{\infty} g_N(\ell) = \sum_{\ell=1}^{\infty} \lim_{N\to\infty} g_N(\ell). \tag{2.2}$$

Since $\sum_{1\le s\le \ell} g_N(s) = 1 - T(\ell, N)/N^{mn}$, this claim gives the proof of the theorem.

To prove the claim, we show that $g_N(\ell)$ is bounded by the function $g(\ell) = \frac{n^m}{p_\ell^m}$ and $\sum_{\ell=1}^{\infty} g(\ell) \le n^m \zeta(m) < \infty$. As the final step, we have

$$\begin{aligned}
g_N(\ell) &= \Pr\left[\gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj}) \text{ coprime to } p_1, ..., p_{\ell-1} \text{ and divisible by } p_\ell\right] \\
&\le \Pr\left[p_\ell \mid \gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj})\right] \\
&= \frac{\left|\left\{(r_{1j}) : p_\ell \mid \prod_{j=1}^{n} r_{1j}\right\}\right|^m}{N^{mn}} = \frac{\left(N^n - \left|\left\{(r_{1j}) : p_\ell \nmid \prod_{j=1}^{n} r_{1j}\right\}\right|\right)^m}{N^{mn}} \\
&= \frac{(N^n - |\{r_{11} : p_\ell \nmid r_{11}\}|^n)^m}{N^{mn}} = \left\{1 - \left(1 - \frac{1}{N}\left\lfloor\frac{N}{p_\ell}\right\rfloor\right)^n\right\}^m \\
&\le \left\{1 - \left(1 - \frac{1}{p_\ell}\right)^n\right\}^m \le \frac{n^m}{p_\ell^m},
\end{aligned}$$

where the last inequality is from Bernoulli's inequality. □

Corollary 2.3. *Let $\hat{n} = \max\{n, B\}$, $r = \lfloor n^{\frac{m}{m-1}} + 1\rfloor$ and $\hat{r} = \max\{\hat{n}, r\}$. Then the probability that $\gcd(\prod_{j=1}^{n} r_{1j}, ..., \prod_{j=1}^{n} r_{mj})$ is B-smooth is upper bounded by*

$$\frac{1}{\zeta(m)} \cdot \prod_{p>B} \left(1 - \frac{1}{p^m}\right)^{-1},$$

and lower bounded by

$$\prod_{B<p\le \hat{n}}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\cdot \prod_{\hat{n}<p\le \hat{r}}\left\{1-\left(\frac{n}{p}\right)^m\right\}\cdot \frac{1}{\zeta(s)},$$

for $s=m(1-\log_{\hat{r}} n)>1$*. The first product term is equal to 1 if* $B=\hat{n}$*, and the second product term is equal to 1 if* $\hat{n}=\hat{r}$*.*

Proof. Since $\prod_{p>B}[1-\{1-(1-1/p)^n\}^m]$ decreases as n increases, we can obtain an inequality

$$\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\le \prod_{p>B}\left(1-\frac{1}{p^m}\right)=\frac{1}{\zeta(m)}\prod_{p\le B}\left(1-\frac{1}{p^m}\right)^{-1}.$$

Using Bernoulli's inequality, we can also obtain

$$\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\ge \prod_{B<p\le\hat{n}}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\cdot\prod_{p>\hat{n}}\left\{1-\left(\frac{n}{p}\right)^m\right\}.$$

We can easily check that $n^m/p^m\le 1/p^s$ for prime p larger than $\hat{r}$. Therefore, we obtain

$$\begin{aligned}&\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\\ &\ge \prod_{B<p\le\hat{n}}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\cdot\prod_{\hat{n}<p\le\hat{r}}\left\{1-\left(\frac{n}{p}\right)^m\right\}\cdot\prod_{p>\hat{r}}\left(1-\frac{1}{p^s}\right)\\ &\ge \prod_{B<p\le\hat{n}}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\cdot\prod_{\hat{n}<p\le\hat{r}}\left\{1-\left(\frac{n}{p}\right)^m\right\}\cdot\frac{1}{\zeta(s)}.\end{aligned}$$

Finally, $s=m(1-\log_{\hat{r}} n)>1$ since $\hat{r}\ge r>n^{\frac{m}{m-1}}$, and the proof is completed. □

Remark 2.4. Suppose $B=2n$, and $\frac{m}{\log_2 2n}$ is a positive integer larger than 1. Then we can check that $B>n^{\frac{m}{m-1}}$, so $\hat{r}=B\ge r\ge n$. Applying Corollary 2.3, we have

$$\prod_{p>B}\left[1-\left\{1-\left(1-\frac{1}{p}\right)^n\right\}^m\right]\ge\frac{1}{\zeta(s)},$$

for $s=m(1-\log_{\hat{r}} n)=m(1-\log_{2n} n)=\frac{m}{\log_2 2n}$. This is exactly same lower bound suggested in the lemma of [2, page 10].

2.2. **Generalization to k-gcd.** Now, we extend Theorem 2.1 and 2.2 to the case of k-gcd. For a positive integer N, r_{ij}'s are chosen randomly and independently in $\{1,2,...,N\}$. We compute the probability that $\gcd_k(\prod_{j=1}^n r_{1j},...,\prod_{j=1}^n r_{mj})$ is B-smooth when $N\to\infty$. Define $T_k(\ell,N)$ be the number of ordered pairs (r_{ij}) such that $\gcd_k(\prod_{j=1}^n r_{1j},...,\prod_{j=1}^n r_{mj})$ is coprime to $p_1,...,p_\ell$ for $1\le r_{ij}\le N$. Note that $\lim_{\ell\to\infty}T_k(\ell,N)/N^{mn}$ is the probability that $\gcd_k(\prod_{j=1}^n r_{1j},...,\prod_{j=1}^n r_{mj})$ is B-smooth where r_{ij}'s are chosen randomly and independently in $\{1,2,...,N\}$. Similarly to Theorem 2.1 and 2.2, we obtain the value of $\lim_{N\to\infty}\lim_{\ell\to\infty}T_k(\ell,N)/N^{mn}$ by following two steps.

Theorem 2.5. *Let $p_1, p_2, ...$ be the prime numbers larger than B in increasing order. Then,*

$$\lim_{N\to\infty} \frac{T_k(\ell, N)}{N^{mn}} = \prod_{i=1}^{\ell} \left[1 - \left\{ 1 - \left(1 - \frac{1}{p_i}\right)^n \left(1 + \frac{{}_nH_1}{p_i} + \cdots \frac{{}_nH_{k-1}}{p_i^{k-1}}\right)\right\}^m \right].$$

Proof. Similarly to Theorem 2.1, we apply the inclusion and exclusion principle. Note that $\prod_{p\in P} p \mid \gcd_k(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ if and only if $\prod_{p\in P} p^k \mid \prod_j r_{ij}$ for any i. For $X_\ell = \{p_1, ..., p_\ell\}$ and $1 \le r_{ij} \le N$, we can get

$$\begin{aligned} &\left| \left\{ (r_{ij}) : \gcd{}_k(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj}) \text{ is coprime to } p_1, ..., p_\ell \right\} \right| \\ &= \sum_{P\subset X_\ell} (-1)^{|P|} \left(\sum_{Q\subset P} (-1)^{|Q|} \left| \left\{ (r_{1j}) : p^k \nmid \prod_{j=1}^n r_{1j}, \forall p \in Q \right\} \right| \right)^m . \end{aligned}$$

Since the summation is finite, after dividing by N^{mn} and $N \to \infty$,

$$\lim_{N\to\infty} \frac{T_k(\ell, N)}{N^{mn}} = \sum_{P\subset X_\ell} (-1)^{|P|} \left(\sum_{Q\subset P} (-1)^{|Q|} \Pr\left[p^k \nmid \prod_{j=1}^n r_{1j}, \forall p \in Q \right] \right)^m .$$

Let $p^a \parallel x$ denotes that $p^a \mid x$ and $p^{a+1} \nmid x$, and $a_{p,j}$'s be the non-negative integers for $p \in Q$ and $1 \le j \le n$. Note that the number of n-tuples of non-negative integers $(a_{p,1}, ..., a_{p,n})$ satisfying $a_{p,1} + \cdots + a_{p,n} = i$ is ${}_nH_i = \binom{n+i-1}{i}$. Then we have

$$\begin{aligned} \Pr\left[p^k \nmid \prod_{j=1}^n r_{1j} \text{ for all } p \in Q \right] &= \sum_{a_{p,1}+\cdots a_{p,n}<k} \Pr\left[p^{a_{p,j}} \parallel r_{1j} \text{ for all } p, j \right] \\ &= \sum_{a_{p,1}+\cdots a_{p,n}<k} \prod_{p\in Q, j} \Pr\left[p^{a_{p,j}} \parallel r_{1j} \right] \\ &= \prod_{p\in Q} \left(\sum_{a_{p,1}+\cdots a_{p,n}<k} \prod_{j=1}^n \Pr\left[p^{a_{p,j}} \parallel r_{1j} \right] \right) \\ &= \prod_{p\in Q} \left(\sum_{a_{p,1}+\cdots a_{p,n}<k} \prod_{j=1}^n \frac{p-1}{p^{a_{p,j}+1}} \right) \\ &= \prod_{p\in Q} \left\{ \left(1 - \frac{1}{p}\right)^n \sum_{a_{p,1}+\cdots a_{p,n}<k} \frac{1}{p^{a_{p,1}+\cdots+a_{p,n}}} \right\} \\ &= \prod_{p\in Q} \left(1 - \frac{1}{p}\right)^n \left(1 + \frac{{}_nH_1}{p} + \cdots + \frac{{}_nH_{k-1}}{p^{k-1}}\right), \end{aligned}$$

which gives the theorem when substituting in above equation. □

Theorem 2.6. *When r_{ij}'s are chosen randomly and independently from $\{1, 2, ..., N\}$, the probability that $\gcd_k(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj})$ is B-smooth converges to*

$$\prod_{p>B} \left[1 - \left\{ 1 - \left(1 - \frac{1}{p}\right)^n \left(1 + \frac{{}_nH_1}{p} + \cdots \frac{{}_nH_{k-1}}{p^{k-1}}\right) \right\}^m \right]$$

as $N \to \infty$.

Proof. The statement is proved by exactly the same way with Theorem 2.2. Since

$$\begin{aligned} \frac{T_k(\ell-1, N) - T_k(\ell, N)}{N^{mn}} &\leq \Pr\left[p_\ell \mid \gcd{}_k(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj}) \right] \\ &= \Pr\left[p_\ell^k \mid \gcd(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj}) \right] \\ &\leq \Pr\left[p_\ell \mid \gcd(\prod_{j=1}^n r_{1j}, ..., \prod_{j=1}^n r_{mj}) \right] \\ &\leq \frac{n^m}{p_\ell^m}, \end{aligned}$$

we can apply Lebesgue Dominated Convergence Theorem in the same way to Theorem 2.2 to obtain the theorem. □

Theorem 2.6 is a generalized form of Benkoski's theorem [1] and Theorem 2.2. As we mentioned in Introduction, Benkoski's theorem is that the probability that r positive integers are relatively k-prime is $1/\zeta(rk)$. When $k = 1$, $1 + \frac{{}_nH_1}{p} + \cdots + \frac{{}_nH_{k-1}}{p^{k-1}} = 1$, so the result is same with Theorem 2.2. Also when $B = n = 1$, the same condition with Benkoski's theorem, $\left(1 - \frac{1}{p}\right)^n \left(1 + \frac{{}_nH_1}{p} + \cdots + \frac{{}_nH_{k-1}}{p^{k-1}}\right) = 1 - \frac{1}{p^k}$. Therefore,

$$\prod_{p>B} \left[1 - \left\{ 1 - \left(1 - \frac{1}{p}\right)^n \left(1 + \frac{{}_nH_1}{p} + \cdots + \frac{{}_nH_{k-1}}{p^{k-1}}\right) \right\}^m \right] = \prod_{p} \left(1 - \frac{1}{p^{mk}}\right) = \frac{1}{\zeta(mk)}.$$

This is exactly the same result of Benkoski.

The value of $\prod_{p>B} \left[1 - \left\{ 1 - \left(1 - \frac{1}{p}\right)^n \left(1 + \frac{{}_nH_1}{p} + \cdots + \frac{{}_nH_{k-1}}{p^{k-1}}\right) \right\}^m \right]$ can be lower bounded by the case of $k = 1$. Therefore, we can conclude

$$\begin{aligned} &\prod_{p>B} \left[1 - \left\{ 1 - \left(1 - \frac{1}{p}\right)^n \left(1 + \frac{{}_nH_1}{p} + \cdots + \frac{{}_nH_{k-1}}{p^{k-1}}\right) \right\}^m \right] \\ &\geq \prod_{B<p\leq\hat{n}} \left[1 - \left\{ 1 - \left(1 - \frac{1}{p}\right)^n \right\}^m \right] \cdot \prod_{\hat{n}<p\leq\hat{r}} \left\{ 1 - \left(\frac{n}{p}\right)^m \right\} \cdot \frac{1}{\zeta(s)}, \end{aligned}$$

for $\hat{n} = \max\{n, B\}$, $r = \lfloor n^{\frac{m}{m-1}} + 1 \rfloor$, $\hat{r} = \max\{\hat{n}, r\}$, and $s = m(1 - \log_{\hat{r}} n)$.

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