On Trees, Chains and Fast Transactions in the Blockchain

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Abstract

A fundamental open problem in the area of blockchain protocols is whether the Bitcoin blockchain protocol is the optimal solution (in terms of efficiency, security) for building a secure transaction ledger. A recently proposed and widely deployed alternative is the GHOST protocol which, notably, is at the core of Ethereum as well as other recent proposals for improved Bitcoin-like systems. The GHOST variant is touted as offering superior performance compared to Bitcoin (block production in ethereum has been sped up by a factor of more than 40) without a security loss. Motivated by this, in this work, we study from both a provable security and attack susceptibility point of view the problem of transaction processing time for both GHOST and Bitcoin.

We introduce a new formal framework for the analysis of blockchain protocols that relies on trees (rather than chains) and we showcase the power of the framework by providing a unified description of the GHOST and Bitcoin protocols, the former of which we extract and formally describe in our framework. We then prove that GHOST implements a "robust transaction ledger" (i.e., possesses liveness and persistence) and hence it is a provably secure alternative to Bitcoin.

We then focus on the liveness property of both Bitcoin and GHOST, i.e., the worst-case transaction confirmation time that can be expected when playing against an adversary. We present a general attack methodology against liveness and we instantiate it with two attacks for Bitcoin and GHOST. We prove that our attack for Bitcoin is essentially optimal. Furthermore, we perform simulation results and we demonstrate that for a wide range of confirmation parameter choices and hashing power bounds for the adversary, GHOST, when under our attack, performs about the same or worse than Bitcoin in terms of transaction confirmation time. Our results highlight the importance of provable security analysis in the context of blockchain protocols.

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Contents

1 Introduction

The popularity of Bitcoin [17] has lead to a surge in the interest about its core protocol that maintains a distributed data structure called the "blockchain". In [10], the core of the Bitcoin protocol was abstracted under the moniker "Bitcoin Backbone" and it was demonstrated to be a powerful tool for solving consensus, [21, 14], in a synchronous, anonymous and Byzantine setting where (unreliable) broadcast is the communication operation available to the participants, (a problem first considered in [2, 18]). In [10], it was shown that the core protocol provably guarantees two properties: (i) *persistence:* if a transaction is reported as stable by one node, then it will be also reported by any other honest node of the system in the same ledger position, (ii) *liveness:* all honestly generated transactions that are broadcasted are eventually reported as stable by some honest node. This provides a formal framework for proving the security of systems like Bitcoin, since their security can be reduced to the persistence and liveness of the underlying transaction ledger. Furthermore, it provides a way to argue formally about transaction confirmation time since the liveness property is equipped with a delay parameter that specifies the maximum transaction delay that can be caused by an adversary.

Naturally, implementing a robust transaction ledger may be achieved in various other ways, and it is a fundamental open question of the area whether the Bitcoin protocol itself is an optimal implementation of a robust transaction ledger. Indeed, many researchers have challenged various aspects of the Bitcoin system and they have proposed modifications in its core operation. Some of the modified systems maintain the protocol structure but modify the hard-coded parameters (like the block generation rate) or the basic primitives, e.g., the way proof of work is performed (a number of alternative proof of work implementations have been proposed using functions like scrypt [22], lyra2 [24] and others). However, more radical modifications are possible that alter the actual operation of the protocol.

One of the most notable such variants is the `GHOST` protocol, which was suggested by Sompolinsky and Zohar in [25]. After the initial suggestion many cryptocurrencies using variants of the `GHOST` rule were proposed and implemented. The most popular among them, Ethereum [7] has received substantial research attention [13, 12, 4, 23, 11, 19] and also media attention.[1] Ethereum is essentially a Bitcoin-like system where transaction processing is Turing-complete and thus it can be used to implement any public functionality in a distributed way. Bitcoin-NG [8] is another popular Bitcoin-like system relying on `GHOST` that separates blocks in two categories, namely key blocks and microblocks, reflecting the fact that transaction serialization and leader election are two different parts of the system. Bitcoin-NG, due to its structured blockchain, can potentially offer higher throughput compared to Bitcoin.

Unfortunately, the security analysis of [25] is not as general as [10] (e.g., their attacker does not take advantage of providing conflicting information to different honest parties), while the analysis of [10] does not carry to the setting of `GHOST`. This is because the `GHOST` rule is a natural, albeit radical, reformulation of how each miner determines the main chain. In `GHOST`, miners adopt blocks in the structure of a *tree*. Note that in both Bitcoin and `GHOST` one can consider parties collecting all mined blocks in a tree data structure. However, while in Bitcoin the miners would choose the most difficult chain as the main chain, in `GHOST`, they will determine the chain by greedily following the "heaviest observed subtree." This means that for the same subtree, a Bitcoin miner and a `GHOST` miner may choose a completely different main chain. Furthermore, it means that the difficulty of

[1] E.g., see news stories in `http://www.nytimes.com/2016/03/28/business/dealbook/ethereum-a-virtual-currency-enables-transactions-that-rival-bitcoins.html`, `http://www.coindesk.com/ethereum-launches-ether-coin-millions-already-sold/` and `http://www.wired.com/2014/09/ethereum-backers-raise-15-million/`

the main chain of honest parties does not necessarily increase monotonically (it may decrease at times) and thus a fundamental argument (namely that blockchains monotonically increase) that made the analysis of [10] possible, does not hold anymore.

Our Results. We propose a new analysis framework for blockchain protocols focusing on trees of blocks as opposed to chains as in [10]. Our framework enables us to argue about random variables on the trees of blocks that are formed by the participants. In our framework, we can express concepts like a node being **d-dominant**, which means that the block corresponding to that node would be preferred by a margin of d compared to other sibling nodes according to a specified weight metric. This actually enables us to unify the description of Bitcoin and `GHOST`.

Using our framework we then provide a first formal security proof of the `GHOST` rule for blockchain protocols. Specifically, we prove that `GHOST` is a robust transaction ledger that satisfies liveness and persistence. We achieve this result, by a new methodology, that reduces the properties of the robust transaction ledger to a single lemma, that we call the *fresh block lemma* and is informally stated as follows.

> *Fresh Block Lemma.* (Informally) At any point of the execution and for any past sequence of s consecutive rounds, there is an honest block mined in these rounds, that is contained in the chain of any honest player from this point on.

As we demonstrate, the fresh block lemma is a powerful tool in the presence of an adversary: we show easily that the properties of the robust transaction ledger reduce to it in a black-box fashion. This provides an alternative proof methodology for establishing the properties of a robust transaction ledger compared to [10], who reduced the properties of the robust transaction ledger to two other properties called common prefix and chain quality, and may be of independent interest as it could be applicable to other blockchain variants.

Observing the fact that the delay parameter we prove for `GHOST` liveness is inferior to that shown for Bitcoin we turn our focus on the liveness property, and more specifically the delay parameter that specifies the worst-case confirmation time that can be caused by an adversary. We present a general attack methodology for attacking transaction confirmation time. Our attack method has three stages: (i) the attack preparation stage, (ii) the transaction denial stage and (iii) the blockchain retarder stage. In the attack preparation stage, our attacker prepares the attack and waits for the transaction that she dislikes to appear in the network (e.g., the attacker may mine a private chain or may interfere with block adoption of the honest nodes). When the disliked transaction appears, the attacker moves to the transaction delay phase where she tries to prevent honest nodes from adopting it. At any moment, the attacker may switch to the third phase where she gives up on preventing the honest nodes from adopting the transaction and tries to slow down the blockchain growth so that the confirmation time might be extended. Using this template, we present two attacks for Bitcoin and `GHOST` respectively.

We prove that our attack for Bitcoin is essentially optimal and it essentially matches the delay parameter for the liveness of the Bitcoin backbone as proven in [10]. It follows that the liveness property for Bitcoin is tight and our attack can be used as a yardstick to show whether a protocol can improve blockchain liveness compared to Bitcoin. Our attack for `GHOST`, is more involved, and exploits the way that honest nodes pick the main chain in a way that is intrinsic to the `GHOST` rule providing a powerful blockchain retarder phase.

We proceed to perform experiments in order to compare the two attacks. Our main finding is the following.

> The `GHOST` protocol, under our attack, is either outperformed by Bitcoin (when subjected to the optimal attack), or performs about the same. This is true for a wide range of

hashing power levels of the adversary and parameter settings that specify the number of blocks that one should wait in order to confirm a transaction.

The gap between the two protocols in favor of Bitcoin becomes particularly significant when the number of blocks required for confirmation is very high (at the level that is required by various exchanges[2] see Figure 6). Given that the main claims for the GHOST protocol is its alleged superior capability to allow faster transactions compared to Bitcoin, cf. [25], it is important to reflect that this appears to be untrue when the protocol is subjected to our attack. We note that, in order to compare "apples to apples," we compare the two protocols, GHOST and Bitcoin, using the same equally accelerated block production rate. Comparing the two at an equal rate is justified from our provable security analysis for the persistence property which does not enable us to show a security advantage of GHOST over Bitcoin for accelerated rates; put differently, Bitcoin does not appear to lose security at a higher rate than GHOST when accelerated.[3]

We remark that the current implementation of Ethereum utilizes a variant of GHOST, termed "uncles-only GHOST". We show that our transaction confirmation time attack against GHOST easily extends against uncles-only GHOST (albeit with a slightly milder effect). There are other ways to modify GHOST that can be considered (e.g., [15]) and these may be also cast and analyzed both from a provable perspective in our framework as well as from an attack potential perspective using our attack template. Our work highlights the benefits of provable security analysis in the domain of designing blockchain protocols and provides ways to differentiate such protocols from the perspective of the fundamental properties that a blockchain protocol should satisfy.

On the generality of the adversarial model. The adversarial model we adopt in this work is the one proposed by Garay et al. [10]. This model is quite general in the sense that, it can captures many attack models that were proposed in the literature. For example, it captures the double spending attacker of [17], the block withholding attacker of [9] (which can be simulated because the adversary can change the order that messages arrive for each honest player) and the eclipse attacker of [6] where the communication of a portion of the honest nodes in the network is completely controlled (eclipsed) by the adversary (this can be simulated by simply considering the eclipsed nodes to be controlled by the adversary and having the adversary honestly execute their program while dropping their incoming messages).

Limitations and directions for future research. Our analysis is in the standard Byzantine model where parties fall into two categories, those that are honest (and follow the protocol) and those that are dishonest that may deviate in an arbitrary (and coordinated) fashion as dictated by the adversary. It is an interesting direction for future work to consider the rational setting where all parties wish to optimize a certain utility function. Designing suitable incentive mechanisms, for instance see [16] for a suggestion related to the GHOST protocol, or examining the requirements for setup assumptions, cf. [1], are related important considerations. Our analysis is in the static setting, i.e., we do not take into account the fact that parties change dynamically and that the protocol calibrates the difficulty of the POW instances to account for that; we note that this may open the possibility for additional attacks, say [3], and hence it is an important point for consideration and future work. While we discover an optimal attack against the liveness property for bitcoin, the provable security bound for the delay in the liveness property of GHOST is not matched by an

[2]Kraken and Poloniex are currently the biggest Ethereum exchanges. Kraken initially had used 6000 blocks for confirmation time, while Poloniex 375 blocks.

[2]Currently the Ethereum Frontier reports an average of about 14 seconds, cf. `https://etherchain.org`; the 12 seconds rate was discussed by Buterin in [5]. In contrast, Bitcoin block generation rate is 10 minutes.

[3]We note that even though the analysis of [25] suggests that there is an advantage, their analysis is performed in a much more restricted attack model than ours.

attacker. Even though, we demonstrate that our GHOST attacker causes higher delays than Bitcoin for most choices of the parameters, it does not match the worst case provable bound, something that means that the bound might be lowered (or alternatively the attack may be improved). Finally, it is interesting to consider our results in more general models such as the semi-synchronous model of [20].

Organization. In section 2 we overview the model that we use for expressing the protocols and the theorems regarding the security properties. In section 3 we introduce our new tree-based framework. Then, in section 4 we present our security analysis of an abstraction of the GHOST protocol that demonstrates it is a robust transaction ledger in the static setting. In section 5 we present our liveness attacks against Bitcoin and GHOST variants, we prove the optimality of the attack against Bitcoin and we compare the two attacks by performing simulations for various parameter choices.

2 Preliminaries and the GHOST Backbone protocol

2.1 Model

For our model we adopt the abstraction proposed in [10]. Specifically, in their setting, called the q-bounded setting, synchronous communication is assumed and each party is allowed q queries to a random oracle. The network supports an anonymous message diffusion mechanism that is guaranteed to deliver messages of all honest parties in each round. The adversary is rushing and adaptive. Rushing here means that in any given round he gets to see all honest players' messages before deciding his own strategy. However, after seeing the messages he is not allowed to query the hashing oracle again in this round. In addition, he has complete control of the order that messages arrive to each player. The model is "flat" in terms of computational power in the sense that all honest parties are assumed to have the same computational power while the adversary has computational power proportional to the number of players that it controls.

The total number of parties is n and the adversary is assumed to control t of them (honest parties don't know any of these parameters). Obtaining a new block is achieved by finding a hash value that is smaller than a difficulty parameter D. The success probability that a single hashing query produces a solution is $p = \frac{D}{2^\kappa}$ where κ is the length of the hash. The total hashing power of the honest players is $\alpha = pq(n-t)$, the hashing power of the adversary is $\beta = pqt$ and the total hashing power is $f = \alpha + \beta$. A number of definitions that will be used extensively are listed below.

Definition 1. A round is called:

- successful if at least one honest player computes a solution in this round.
- uniquely successful if exactly one honest player computes a solution in this round.

Definition 2. In an execution blocks are called:

- honest, if mined by an honest party.
- adversarial, if mined by the adversary.

Definition 3. (chain extension) We will say that a chain $\mathcal{C}'$ extends another chain $\mathcal{C}$ if a prefix of $\mathcal{C}'$ is a suffix of $\mathcal{C}$.

In [10], a lower bound to the probabilities of two events, that a round is successful or that is uniquely successful (defined bellow), was established and denoted by $\gamma_{\mathsf{u}} = \alpha - \alpha^2$. While this bound is sufficient for the setting of small f, here we will need to use a better lower bound to the probability of those events, denoted by γ, and with value approximately $\alpha e^{-\alpha}$ (see Appendix B). Observe that $\gamma > \gamma_{\mathsf{u}}$.

2.2 The GHOST Backbone Protocol

In order to study the properties of the core Bitcoin protocol, the term *Backbone Protocol* was introduced in [10]. On this level of abstraction we are only interested on properties of the blockchain, independently from the data stored inside the blocks. In the same work, the Bitcoin backbone protocol is described in a quite abstract and detailed way. The main idea is that honest players, at every round, receive new chains from the network and pick the longest valid one to mine. Then, if they obtain a new block (by finding a small hash), they broadcast their chain at the end of the round. For more details we refer to [10, Subsection 3.1].

The same level of abstraction can also be used to express the GHOST protocol. The GHOST backbone protocol, as presented in [25], is based on the principle that blocks that do not end up in the main chain, should also matter in the chain selection process. In order to achieve this, players store a tree of all mined blocks they have received, and then using the greedy heaviest observed subtree (GHOST) rule, they pick which chain to mine.

Algorithm 1 The chain selection algorithm. The input is a block tree T. The $|\cdot|$ operator corresponds to the number of nodes of a tree.

```
function GHOST(T)
    B ← GenesisBlock
    if children_T(B) = ∅ then
        return C = (GenesisBlock, ..., B)
    else
        B ← argmax_{c ∈ children_T(B)} |subtree_T(c)|
        return GHOST(subtree_T(B))
    end if
end function
```

At every round, players update their tree by adding valid blocks sent by other players. The same principle as Bitcoin applies; for a block to be added to the tree, it suffices to be a valid child of some other tree block. The adversary can add blocks anywhere he wants in the tree, as long as they are valid. Again, as on Bitcoin, players try to extend the chains they choose by one or more blocks. Finally, in the main function, a tree of blocks is stored and updated at every round. If a player updates his tree, he broadcasts it to all other players.

The protocol is also parameterized by three external functions $V(\cdot)$, $I(\cdot)$, $R(\cdot)$ which are called: the input validation predicate, the input contribution function, and the chain reading function, respectively. $V(\cdot)$ dictates the structure of the information stored in each block, $I(\cdot)$ determines the data that players put in the block they mine, $R(\cdot)$ specifies how the data in the blocks should be interpreted depending on the application.

2.3 Security properties

Two crucial security properties of the Bitcoin backbone protocol were considered in previous works: the *common prefix* and the *chain quality* property. The common prefix property ensures that two honest players have the same view of the blockchain, if they prune a small number of blocks from the tail of their respective chains. On the other hand, the chain quality property ensures that honest players chains' do not contain long sequences of adversarial blocks. These two properties were shown to hold for the Bitcoin backbone protocol.

Algorithm 2 The GHOST backbone protocol, parameterized by the *input contribution function* $I(\cdot)$ and the *reading function* $R(\cdot)$. $\mathbf{x}_{\mathcal{C}}$ is the vector of inputs of all block in chain $\mathcal{C}$.

$T \leftarrow GenesisBlock$ ▷ T is a tree.
$state \leftarrow \varepsilon$
$round \leftarrow 0$
while TRUE **do**
  $T_{new} \leftarrow \mathsf{update}(T, \text{blocks found in } \textsc{Receive}())$
  $\tilde{\mathcal{C}} \leftarrow \mathtt{GHOST}(T_{new})$
  $\langle state, x\rangle \leftarrow I(state, \tilde{\mathcal{C}}, round, \textsc{Input}(), \textsc{Receive}())$
  $\mathcal{C}_{\mathsf{new}} \leftarrow \mathsf{pow}(x, \tilde{\mathcal{C}})$
  if $\tilde{\mathcal{C}} \neq \mathcal{C}_{\mathsf{new}}$ or $T \neq T_{\mathsf{new}}$ **then**
    $T \leftarrow \mathsf{update}(\mathrm{T}_{\mathsf{new}}, \mathrm{head}(\mathcal{C}_{\mathsf{new}}))$
    $\textsc{Broadcast}(\mathrm{head}(\mathcal{C}_{\mathsf{new}}))$
  end if
  $round \leftarrow round + 1$
  if $\textsc{Input}()$ contains $\textsc{Read}$ **then**
    write $R(\mathbf{x}_{\mathcal{C}})$ to $\textsc{Output}()$
  end if
end while

Also in the same work, the *robust public transaction ledger* primitive was presented. This primitive captures the notion of a book, in which transactions are recorded, and it is used to implement Byzantine Agreement in the honest majority setting. The primitive satisfies two properties: *persistence*, and *liveness*. Persistence ensures that, if a transaction is seen in a block deep enough in the chain, it will stay there. And liveness ensures that if a transaction is given as input to all honest players, it will eventually be inserted in a block, deep enough in the chain, of an honest player. The Bitcoin backbone was shown to be sufficient to construct this kind of ledger. More details about the security properties and the primitive are given in Appendix A.

3 A unified description of the Bitcoin and GHOST backbones

Next, we introduce our new analysis framework for backbone protocols that is focusing on trees of blocks and we show how the description of the Bitcoin and GHOST can be unified. In this model, every player stores all blocks "he hears" on a tree, starting from a pre-shared block called the *Genesis* (or v_{root}) block. This is the model where GHOST was initially described. Bitcoin, and other possible backbone variants, can also be seen in this model and thus a unified language can be built. We first define block trees (or just trees) that capture the knowledge of honest players (regarding the block tree on different moments at every round).

Definition 4. We denote by T_r^P (resp. $T_r^{\exists}$) the tree that is formed from the blocks that player P (resp. at least one honest player) has received until the beginning of round r. Similarly, T_r^+ is the tree that contains $T_r^{\exists}$ and also includes all blocks mined by honest players at round r. For any tree T and block $b \in T$, we denote by $T(b)$ the subtree of T rooted on b.

Notice that, due to the fact that broadcasts of honest players always succeed, blocks in T_r^+ are always in T_{r+1}^P. Thus for every honest player P it holds that:

$$T_r^P \subseteq T_r^{\exists} \subseteq T_r^{+} \subseteq T_{r+1}^P$$

Intuitively, heavier trees represent more proof of work. But there is more than one way to define the weight of a tree. For example, in Bitcoin the heaviest tree is the longest one. But for GHOST a heavy tree is one with many nodes. To capture this abstraction we condition our definitions on a norm w that assigns weights on trees. This norm will be responsible for deciding which tree has more proof of work, and thus which tree is favored by the chain selection rule. We choose to omit w from the notation since it will always be clear from the context which norm we use.

Definition 5. Let w be a norm defined on trees. For any tree T let $siblings(v)$ denote the set of nodes in T that share the same parent with v. Then node v is **d-dominant** in T (denoted by $\mathrm{dom}_T(v,d)$) iff

$$w(T(v)) \geq d \wedge \forall v' \in siblings(v) : w(T(v)) \geq w(T(v')) + d$$

The chain selection rule in the Bitcoin protocol can be described using the notion of the d-dominant node. Let $w(T)$ be the height of some tree T. Each player P, starting from the root of his T_r^P tree, greedily decides on which block to add on the chain by choosing one of its 0-dominant children and continuing recursively[4] (ties are broken based on time-stamp, or based on which block was received first). Interestingly, the GHOST selection rule can also be described in exactly the same way by setting w to be the number of nodes of the tree. Thus we have a unified way for describing the chain selection rule in both protocols. Building upon this formalism we can describe the paths that fully informed honest players may choose to mine at round r (denoted by HonestPaths(r)) in a quite robust way, thus showcasing the power of our notation.

$$\text{HonestPaths}(r) = \{p = v_{\text{root}} v_1 \dots v_k | p \text{ is a root-leaf path in } T_r^{\exists} \wedge \forall i \in \{1,..,k\}\ \mathrm{dom}_{T_r^{\exists}}(v_i, 0)\}$$

We conclude this section by presenting two crucial properties that both the Bitcoin and GHOST backbones satisfy. The first property states that by broadcasting k blocks the adversary can decrease the dominance of some block at most by k. Intuitively, it tells us if the adversary's ability to mine new blocks is limited, then his influence over the block tree is also limited. On the other hand, the second property states that uniquely successful rounds increase the dominance only of nodes in the path from the root to the new block.

Proposition 6. For the Bitcoin and GHOST backbones protocols it holds that:

- If the adversary publishes $k \leq d$ blocks at round $r-1$ then for every block $v \in T_{r-1}^{+}$ it holds that $\mathrm{dom}_{T_{r-1}^{+}}(v,d)$ implies $\mathrm{dom}_{T_r^{\exists}}(v, d-k)$.

- If r is a uniquely successful round and the newly mined block extends a path in HonestPaths(r), then for any block v in $T_r^{\exists}$ it holds that: $\mathrm{dom}_{T_r^{\exists}}(v,d)$ implies $\mathrm{dom}_{T_r^{+}}(v, d+1)$ if and only if v is in the path from v_{root} to the new block.

4 Security Analysis and Applications of the GHOST Backbone

In this section, we prove that the GHOST backbone protocol is sufficient to construct a robust transaction ledger. From now on we assume that $w(T)$ is the total number of nodes of tree T.

[4] This is exactly algorithm 1 with a minor modification. At line 6 the subtree T that is chosen maximizes $w(T)$.

4.1 The Fresh Block Lemma

In [10], it was shown that the Bitcoin backbone satisfies two main properties: common prefix and chain quality. However, a fundamental property needed for their proof, is that the chain of honest players grows at least at the rate of successful rounds. This does not hold for GHOST. The reason is that, if a chain received by an honest player is heavier than the one he currently has, he will select it, *even if it is shorter*. To reflect these facts, we develop an argument that is a lot more involved and leads to a power lemma that can be shown for a backbone protocol, that we call the "fresh block lemma." First, we introduce a new notion, that of a path that all of its nodes are dominant up to a certain value. The more dominant a path is, the harder it gets for the adversary to stop honest players from choosing it.

Definition 7. ($\mathrm{p_{dom}}(r,d)$) For $d>0$, $\mathrm{p_{dom}}(r,d)$ is the longest path $p=v_{\mathrm{root}}v_1\ldots v_k$ in T_r^+ s.t.

$$p\neq v_{\mathrm{root}}\wedge\forall i\in\{1,\ldots,k\}:\ \mathrm{dom}_{T_r^+}(v_i,d)$$

If no such path exists $\mathrm{p_{dom}}(r,d)=\perp$.

Note that the dominant path $\mathrm{p_{dom}}(r,d)$, if it is not $\perp$, will be unique (this stems from the requirement that $d>0$).

In the next lemma, we show that the effort that uniquely successful rounds impose on the adversary is cumulative. For any sequence of m (not necessarily consecutive) uniquely successful rounds starting at some round r', no matter the strategy of the adversary, at round r there will be at least one honest block in $\mathrm{p_{dom}}(r,m-k)$ where k is the number of adversarial blocks that have been released during rounds $[r'-1,r-1]$ (and as a result, in such case, it will be $\mathrm{p_{dom}}(r,m-k)\neq\perp$). This establishes the robustness of $\mathrm{p_{dom}}$ in the sense that only adversarial blocks can change it and they do so in a linear dependency to the degree of its dominance at worst.

Lemma 8. *Let $r_1,..,r_m$ be uniquely successful rounds from round r' until round r. If the adversary broadcasts $k<m$ blocks from round $r'-1$ until round $r-1$, then there exists an honest block b, mined in one of the rounds $r_1,..,r_m$ such that b is in $\mathrm{p_{dom}}(r,m-k)$.*

Proof sketch. The proof is based on two observations. Firstly, if the adversary does not broadcast a block in the round before a uniquely successful round s, then the newly mined honest block will be in $\mathrm{p_{dom}}(s,1)$. Secondly, if the adversary broadcasts $k<d$ blocks in the round before a uniquely successful round s, all blocks in $\mathrm{p_{dom}}(s-1,d)$ at round $s-1$ will also be in $\mathrm{p_{dom}}(s,d+1-k)$. It follows that for each uniquely successful round, unless the adversary publishes a block, there will be an honest block introduced in the dominant path and such block will be maintained in the dominant path unless the adversary broadcasts as many blocks as the number of uniquely successful rounds that follow. As a result, in the period from round r' until round r, our assumption that the adversary broadcasts k blocks that are strictly less than m, the number of uniquely successful rounds, implies that at least one block will be maintained in $\mathrm{p_{dom}}(r,m-k)$. □

The fresh block lemma is stated next. Informally, it states that at any point in time, in any past sequence of s consecutive rounds, at least one honest block was mined on these rounds that is permanently inserted in the chain that every honest player chooses to adopt, with overwhelming probability on s.

Lemma 9. (Fresh Block Lemma) *Assume $\gamma\geq(1+\delta)\beta$, for some real $\delta\in(0,1)$. Then, for all $s\in\mathbb{N}$ and $r\geq s$ it holds that there exists a block mined by an honest player on and after[5] round*

[5]Throughout this work, we only consider executions that run for a polynomial number of rounds in the security parameter κ.

$r-s$, that is contained in the chain which any honest player adopts on and after round r with probability $1-e^{-\Omega(\delta^2 s)}$.

Proof sketch. The main idea here is that after s rounds, due to lemma 8, a block mined by an honest player in a uniquely successful round will be in the chain which any honest player adopts on the following rounds. This is because the adversary does not have the required resources to compensate for all uniquely successful rounds that are going to occur in this round interval. However, as the time passes the adversary may use blocks mined in the past to compensate for the uniquely successful rounds in a given interval. We prove that in order to use blocks mined from some point in the past, he must compensate all uniquely successful rounds from that point on, which again is impossible if this point is more than s rounds old. □

4.2 A robust public transaction ledger based on GHOST

In [10] it is shown how to instantiate the functions V, R, I so that the resulting protocol, denoted by Π_{PL}, built on top of the Bitcoin backbone, implements a robust transaction ledger (see Appendix A, Definition 18). In this section we show how we can achieve the same goal, using exactly the same instantiation of V, R, I, but on top of the GHOST backbone. We call the resulting protocol, $\Pi_{\mathsf{PL}}^{\mathtt{GHOST}}$.

Having established that every s rounds a fresh and honest block is inserted in the chain of all players, we are in a position to prove the main properties of a robust transaction ledger Liveness stems from the fact that after s^2 rounds, s fresh honest blocks mined on this interval will be in the chain of any honest player. On the other hand, Persistence is implied by the fact that all honest players share a freshly mined block. This block will stay in their chains for the subsequent rounds, therefore the history until this block has become persistent. But this block cannot be very deep in the main chain, because the number of blocks until the head of the chain is bounded by the number of blocks generated from the time the fresh block was mined.

Lemma 10 (Liveness)**.** *Assume $\gamma \geq (1+\delta)\beta$, for some $\delta \in (0,1)$. Further, assume oracle* Txgen *is unambiguous. Then for all $k \in \mathbb{N}$ protocol $\Pi_{\mathsf{PL}}^{\mathtt{GHOST}}$ satisfies Liveness with wait time $u = k(k+1)$ rounds and depth parameter k with probability at least $1-e^{-\Omega(\delta^2 k)}$.*

Proof. We prove that assuming all honest players receive as input the transaction tx for at least u rounds, there exists an honest party at round r with chain $\mathcal{C}$ such that tx is included in $\mathcal{C}^{\lceil k}$. From Lemma 9 it follows that for all round intervals of the form $[r-(i+1)k, r-ik]$ and for $i \in \{0, ..., k\}$, there exists at least one block in chain $\mathcal{C}$ that was computed on this interval by an honest player with probability at least $1-e^{-\Omega(\delta^2 k)}$. By the union bound it follows that a total of $k+1$ blocks where one of them contains tx are included in $\mathcal{C}$ with probability at least $1-e^{-\Omega(\delta^2 k)}$. Thus there exists an honest party at round r with chain $\mathcal{C}$ such that tx is included in $\mathcal{C}^{\lceil k}$. □

Lemma 11 (Persistence)**.** *Suppose $\gamma \geq (1+\delta)\beta$, for some real $\delta \in (0,1)$. Then for all $k \in \mathbb{N}$ protocol $\Pi_{\mathsf{PL}}^{\mathtt{GHOST}}$ satisfies Persistence with probability $1-e^{-\Omega(\delta^2 k)}$, where $k/((1+\delta)f)$ is the depth parameter.*

Proof. Let $\mathcal{C}$ be the chain that an honest player adopts at round r. It is sufficient to show that the head of $\mathcal{C}^{\lceil k}$ has been computed before round $r-k/((1+\delta)f)$, because then from Lemma 9 there exists an honest block computed at least at this round that is on the chain that players adopt from round r and afterwards.

Suppose, for the sake of contradiction, that the head of $\mathcal{C}^{\lceil k}$ is computed after round $r-k/((1+\delta)f)$. The length of $\mathcal{C}$ cannot be greater than the number of solutions Y obtained from the oracle in this amount of rounds. By the Chernoff bound,

$$\Pr[Y \geq (1+\delta)f(k/((1+\delta)f))] \leq e^{-\delta^2 fs/3}.$$

It follows that, with probability $1 - e^{-\delta^2 fs/3}$, $Y < k$ which is a contradiction and thus the lemma follows. □

Corollary 12. *The protocol* $\Pi_{\mathsf{PL}}^{\mathtt{GHOST}}$ *is a robust transaction ledger.*

5 Liveness Attacks

In this section we explore through simulation[6] a novel attack on the transaction confirmation time of GHOST, against the optimal attack on Bitcoin, providing some interesting insights on the optimality of the two protocols. The attacks we are going to consider follow a simple template depicted in Figure 1. First, at the attack preparation phase, the attacker tries to build the maximum possible advantage against honest players, until the time the target transaction tx is broadcast. Next, in the transaction denial phase, he tries to delay a new honest block containing tx to enter honest players' chains. When a given condition is met, he proceeds to the blockchain retarder phase, where he tries to decrease the rate at which the block containing tx gets deeper in the chain. Remember, that the verifier waits until tx is buried k-blocks deep in order to verify. Therefore, by slowing down the chain growth speed, confirmation time is extended furtherer more.

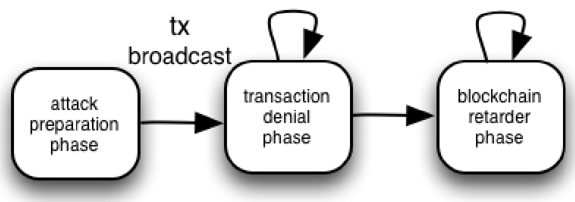


Figure 1: The template of our attacks on confirmation time.

5.1 On Bitcoin Liveness

An attack based on this template against the confirmation time of Bitcoin is quite straightforward. In the attack preparation phase, honest players want to ensure that at the point the target transaction is released they would have the maximum advantage. Advantage here is interpreted as the number of blocks the adversary's secret chain is ahead compared to the honest players' chains. The selfish mining attack, where the attacker (1) tries to mine a secret chain ahead of the honest players, so when they surpass him, he adopts their chain and (2) only broadcasts blocks from his chain when the honest parties have mined a block in the same height, has exactly this property. Observe that the adversary is rushing and thus always wins ties. Also, the chains of honest parties grow at least at the rate of successful rounds. Thus, the rest of our attack should focus on extending as much as possible the transaction denial phase, while not helping honest parties extend their chains faster. Hence, the worst the adversary can do, is to prevent tx from getting in honest players' chains after it has been broadcast in the network. In order to achieve this, he continues to mine the secret chain from the previous phase. The only difference from before, is that he does not leave his chain when

[6]All simulations are carried under the assumptions of our model.

honest players get ahead, but instead he persists on mining his own chain with the hope that he will surpass the honest miners. This way the chains of honest players grow as slow as possible, and the height at which the first honest block is added on the main chain is maximized.

Proposition 13. The attack presented above against Liveness is optimal with respect to confirmation time.

Proof sketch. The optimality of this attack stems from two facts: (1) the advantage of honest players when tx is released is maximum and (2) during the transaction denial phase, honest players chains' grow at least as slow as against any other attack. For the sake of contradiction suppose that the first fact does not hold. Then there exists some attacker A' such that at the time tx is released, has a secret chain that gives him greater advantage. But in order for any attacker to have advantage d, there must be a round prior to the release of tx, where the number of blocks that the adversary has mined since that round is greater or equal to the number of successful rounds plus d. This is a contradiction, since our attacker would also have at least the same advantage.

Suppose now that there exists a better attacker A'' against confirmation time than the one described above. From the second fact, it follows that he must have added the target transaction at an increased height in the chains of honest players compared to our attack. Otherwise, at the time where the transaction is confirmed for our attacker, a chain with sufficient length would have been broadcast such that the target transaction would have been confirmed by A'' too. But in order to increase the height at which the transaction enters the chain, the adversary must have mined more blocks than the number of successful rounds. This is a contradiction, since our attacker would have taken advantage of this fact and would have also added the transaction at least at the height that A'' had. Hence, our proposition follows. □

5.2 On GHOST Liveness

GHOST was designed to prevent selfish-mining type of attacks. Hence, the attack we described for Bitcoin is going to be much less efficient here. Instead, a weak point of GHOST is that chain length is not strictly increasing as time goes by. The key idea of our scheme is that the attacker tries to reduce the speed that the chains of honest players grow (from now on chain growth speed) and thus is named the GHOST-retarder attack. By succeeding, he can effectively decrease the transaction confirmation time for any observer waiting for a transaction to be k blocks deep in his chain.

The GHOST-retarder attack exploits the fact that in GHOST thin and long trees may have the same or less weight than short and wide trees. So in the blockchain retarder phase of our attack, the goal of the adversary is to mine, in secret, a subtree of height two that is heavier than the naturally longer subtree that the honest players are mining by themselves. If the adversary's subtree gets heavier, he can publish it and following the GHOST rule force the honest players to switch to a shorter chain. By doing this repeatedly, every time starting from a recently mined block, and by restarting if honest miners get too far ahead, a concrete reduction of the chain growth speed is achieved as shown in Figure 3, that increases as the adversaries power increases. Thus, contrary to the attack on Bitcoin, we shift our focus to the blockchain retarder phase of the attack template presented earlier. A more detailed description of the blockchain retarder phase of our attack is given in Appendix C.

In the attack preparation phase of our attack, the attacker behaves just like in Bitcoin; he tries to mine a secret chain that weights more than the tree the honest parties build. Notice, that this process is less efficient compared to the same attack on Bitcoin, because the adversary has to compensate for all blocks the honest parties mine, and not only for the number of successful rounds. Next, the adversary using the advantage obtained from the previous phase, tries to stop the transaction from being adopted by honest players (as in the Bitcoin attack). The only difference is that when the honest parties mine a heavier subtree, he immediately proceeds to the blockchain retarder phase that was described earlier.

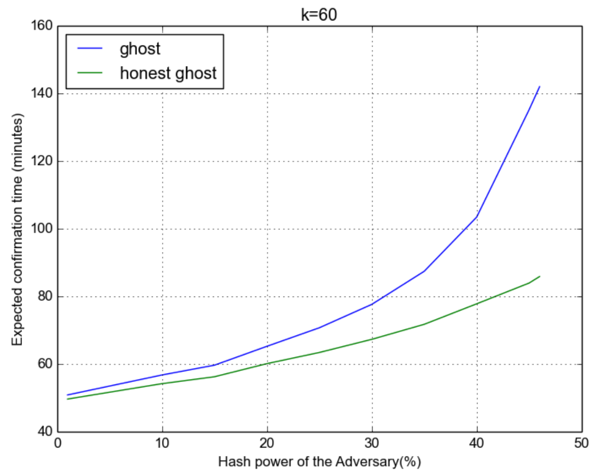


Figure 2: The expected confirmation time of GHOST against the retarder attack for adversaries of increasing power. The power of the honest players is $f = 0.3$.

5.3 Analysis of the chain growth speed reduction

To supplement our earlier claims, in this subsection we argue that the speed at which the chains of honest players grows for the GHOST backbones is slower than that of Bitcoin.

Let the random variable $NB(r, 1-p)$ denote the number of successes in a sequence of i.i.d. Bernoulli trials, with probability of success $1-p$, until r failures occur. The random variable will follow the well known negative binomial distribution. It holds that:

$$Pr[NB(r, 1-p) \leq k] = 1 - I_{1-p}(k+1, r)$$

where $I_{1-p}(k+1, r)$ is the regularized incomplete beta function.

Suppose that we launch the GHOST-retarder phase with parameter r (see Appendix C); the adversary tries to mine a short and wide tree with r nodes before the honest players manage to mine a tree with the same number of nodes. Let E_1 be the event where the number of rounds that the adversary needs to have r successes is less than s and E_2 be the event where the number of rounds that the honest parties needs to have r successes is more than s. Then the probability that the adversary will win the race after at least s rounds is greater than the intersection of E_1 and E_2.

$$Pr[E_1 \wedge E_2] \geq Pr[NB(r, 1-p) \leq \frac{\beta s}{p} \wedge NB(r, 1-p) > \frac{\alpha s}{p}] =$$
$$(1 - I_{1-p}(\frac{\beta s}{p} + 1, r))I_{1-p}(\frac{\alpha s}{p} + 1, r)$$

The last equality follows from the fact that the two events are independent. For $f = 0.3, \alpha = 0.17, \beta = 0.13, p = 10^{-4}, s = 37$ and $r = 6$ we get that $Pr[E_1 \wedge E_2] \geq 0.14$. Thus the average number of rounds at which a new block is added to the chain of an honest player in the GHOST backbone is at least:

$$E[rounds] \geq (1 - 0.14)/\gamma' + 0.14 \cdot 37/2 \geq 8.1$$

where γ' is the probability of a successful round. But $\gamma' = 1 - (1-p)^{q(n-t)} \approx 1 - e^{-\alpha} = 0.156$, for sufficiently small p. On the other hand, the expected number of rounds it takes for a new block to be added to the chain of an honest player for the Bitcoin backbone is $1/\gamma' \approx 6.4$ rounds. Hence, as our experimental results also showed, there exists a clear difference at the rate at which the chains of honest players grow between the two protocols.

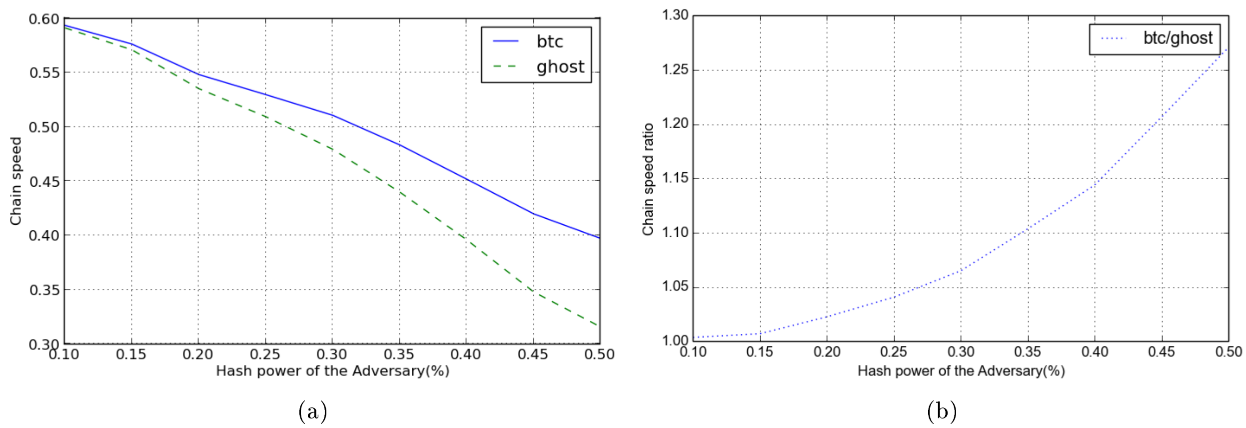


Figure 3: Chain speed from experimental analysis for $f = 1$. Note that as the hashing power of the adversary increases both Bitcoin and GHOST speed decrease. However, Bitcoin is clearly favorable to GHOST (a) and in fact the ratio of Bitcoin to GHOST chain speed increases (b).

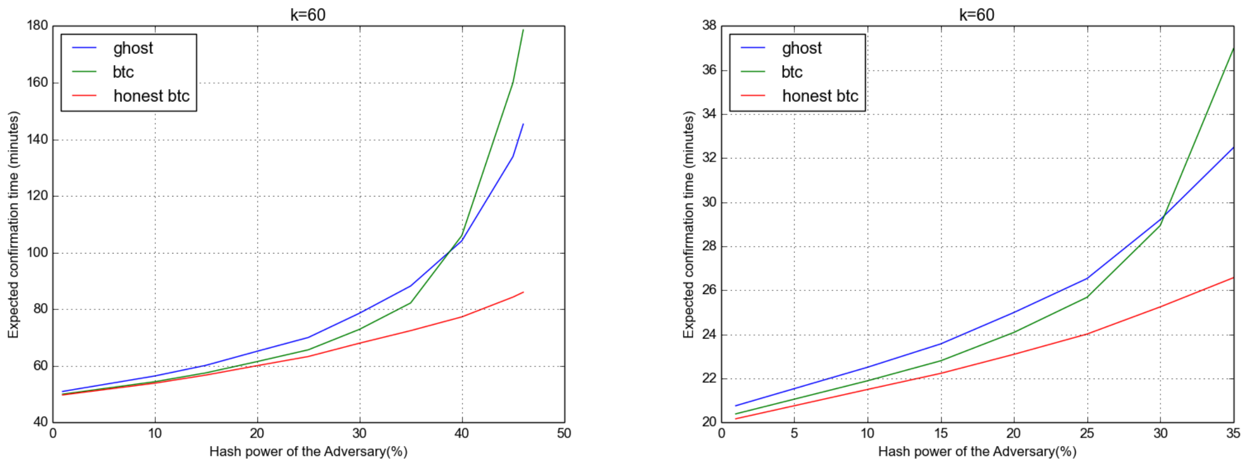
5.4 Comparison

The analysis of the previous section shows that the chain growth speed of GHOST is significantly smaller than that of Bitcoin. Hence, as k grows bigger we also expect the confirmation time of Bitcoin to be smaller than that of GHOST (see Figure 5 and 6), since new blocks will take more time to be added. Our simulation also shows (see Figure 4) that our attack performs worse for β approaching γ compared to Bitcoin. So the optimal scenario for our attack, is an adversary who does not have enough power to break security, but using the attack can slow down confirmation times significantly for the entire network. Since it is not clear whether the GHOST-retarder attack is optimal, it remains an open question whether a more efficient attack on confirmation time can be devised β approachesγ.

5.5 Uncle-only GHOST

A prominent GHOST variant is uncle-only GHOST. It was introduced along with Ethereum as a variant between GHOST and Bitcoin. The way uncle-only GHOST works is that each block can refer to a number of uncles (siblings of his ancestor blocks), and for each uncle referred, the chain gains one more unit of weight. Obviously in the same chain, the same uncle can be referred only once. Moreover, in order to reduce the computational overhead of counting uncles deep in the tree, only uncles that are at most 7 levels above the referring block are counted.

Interestingly, our GHOST-retarder attack still applies to this variant with a small modification. The adversary again tries to mine a short and wide tree. When he decides to attack he has to mine a block under the short tree, in order to capitalize on the blocks mined previously (see Figure 7 for an example). The analysis we did on subsection 5.3 still applies with a small added factor; the



(a)

(b)

Figure 4: The expected confirmation time of GHOST and Bitcoin for (a) $f = 0.3$, (b) $f = 1$ against the two attacks described in section 5, as well as the expected confirmation of Bitcoin when the attacker stays silent. Notice that when the hashing power of the adversary approaches γ, Bitcoin's confirmation becomes worse than that of GHOST.

uncle-base GHOST would need on expectation at least :

$$E[rounds] \geq (1 - 0.14)/\gamma' + 0.14 \cdot 37/3 \geq 7.24 \text{ rounds}$$

while Bitcoin would need 6.1 rounds and plain GHOST would need 8.1 rounds, in order for the chains of honest players to grow by a block.

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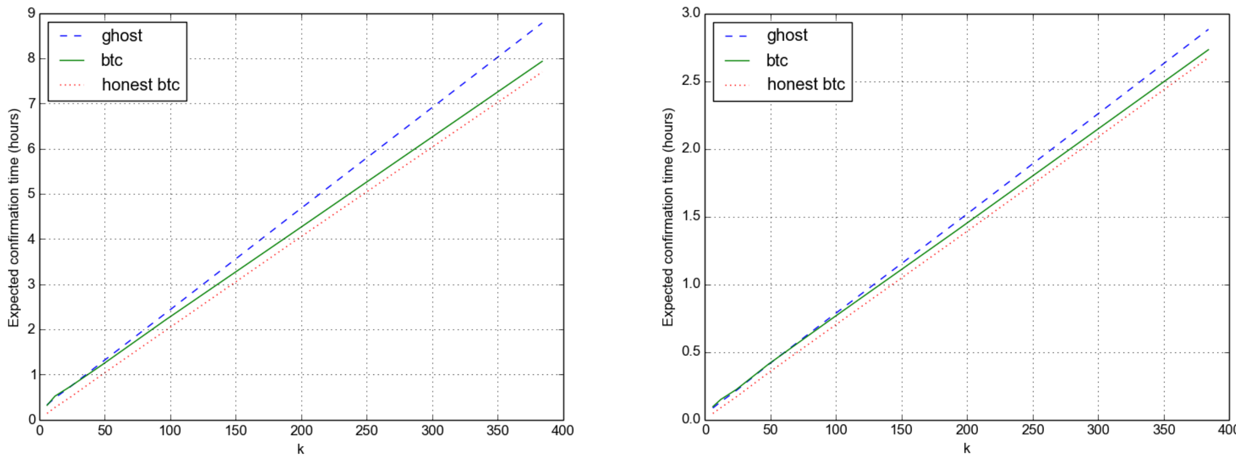

(a) (b)

Figure 5: The expected confirmation time plotted for different values of k and parameters (a)$f = 0.3, \beta/f = 36\%$ and (b)$f = 1, \beta/f = 30\%$. Observe that the delay difference when Bitcoin is not attacked and when Bitcoin is attacked remains the same, while the difference with GHOST grows.

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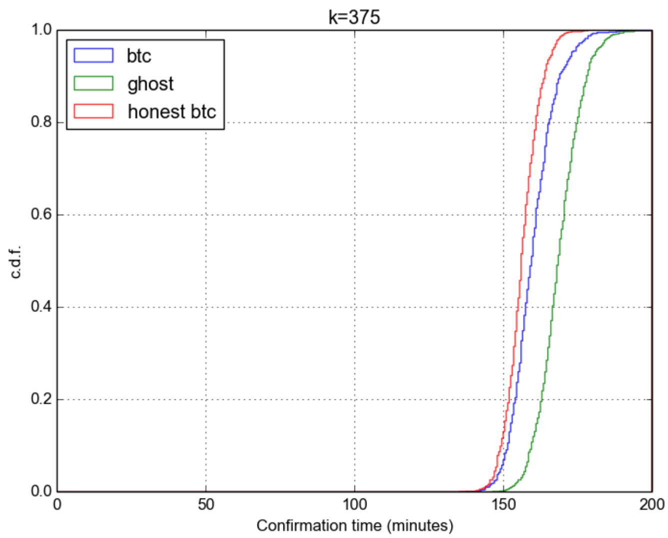

Figure 6: The charts depicts the cumulative distribution function of the confirmation time of the Bitcoin and GHOST based ledgers against the two attacks described in section 5, as well as the expected confirmation of Bitcoin when the attacker stays silent. The parameters used in the experiments are $f = 1$, and adversary hashing power $\beta/f = 30\%$.

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A Security Properties

A.1 Security Properties of the Backbone protocols

In [10, Definitions 2&3] two crucial security properties of the Bitcoin backbone protocol were considered, the common prefix and the chain quality property. The common prefix property ensures

Figure 7: The GHOST-retarder attack against (a) the original and (b) uncle-only GHOST. Honest (resp. adversarial) blocks are shown with blue (resp. red). Referred uncles are shown with the dotted lines. The score of a chain in uncle-only GHOST, is the score of the last block, and the chain ending in the heaviest block is chosen.

that two honest players have the same view of the blockchain if they prune a small number of blocks from the tail. On the other hand the chain quality property ensures that honest players chains' do not contain long sequences of adversarial blocks. These properties are defined as predicates over the random variable formed by the concatenation of all parties views' denoted by $\text{VIEW}^{H(\cdot)}_{\Pi,\mathcal{A},\mathcal{Z}}(\kappa, q, z)$.

Definition 14 (Common Prefix Property)**.** The common prefix property Q_{cp} with parameter $k \in \mathbb{N}$ states that for any pair of honest players P_1, P_2 maintaining the chains $\mathcal{C}_1, \mathcal{C}_2$ in $\text{VIEW}^{H(\cdot)}_{\Pi,\mathcal{A},\mathcal{Z}}(\kappa, q, z)$, it holds that

$$\mathcal{C}_1^{\lceil k} \preceq \mathcal{C}_2 \text{ and } \mathcal{C}_2^{\lceil k} \preceq \mathcal{C}_1.$$

Definition 15 (Chain Quality Property)**.** The chain quality property Q_{cq} with parameters $\mu \in \mathbb{R}$ and $\ell \in \mathbb{N}$ states that for any honest party P with chain $\mathcal{C}$ in $\text{VIEW}^{H(\cdot)}_{\Pi,\mathcal{A},\mathcal{Z}}(\kappa, q, z)$, it holds that for any ℓ consecutive blocks of $\mathcal{C}$ the ratio of adversarial blocks is at most μ.

These two properties were shown to hold for the Bitcoin backbone protocol. Formally, in [10, Theorems 9&10] the following were proved:

Theorem 16. *Assume $f < 1$ and $\gamma_{\mathsf{u}} \geq (1+\delta)\lambda\beta$, for some real $\delta \in (0,1)$ and $\lambda \geq 1$ such that $\lambda^2 - f\lambda - 1 \geq 0$. Let $\mathcal{S}$ be the set of the chains of the honest parties at a given round of the backbone protocol. Then the probability that $\mathcal{S}$ does not satisfy the common-prefix property with parameter k is at most $e^{-\Omega(\delta^3 k)}$.*

Theorem 17. *Assume $f < 1$ and $\gamma_{\mathsf{u}} \geq (1+\delta)\lambda\beta$ for some $\delta \in (0,1)$. Suppose $\mathcal{C}$ belongs to an honest party and consider any ℓ consecutive blocks of $\mathcal{C}$. The probability that the adversary has contributed more than $(1 - \frac{\delta}{3})\frac{1}{\lambda}\ell$ of these blocks is less than $e^{-\Omega(\delta^2 \ell)}$.*

A.2 Robust public transaction ledgers

In [10] the robust public transaction ledger primitive was presented. It tries to capture the notion of a book where transactions are recorded, and it is used to implement Byzantine Agreement in the honest majority setting.

A *public transaction ledger* is defined with respect to a set of valid ledgers $\mathcal{L}$ and a set of valid transactions $\mathcal{T}$, each one possessing an efficient membership test. A ledger $\mathbf{x} \in \mathcal{L}$ is a vector of sequences of transactions tx $\in \mathcal{T}$. Each transaction tx may be associated with one or more *accounts*, denoted $a_1, a_2, \ldots$ Ledgers correspond to chains in the backbone protocols. An oracle Txgen is allowed in the protocol execution that generates valid transactions (this represents transactions that are issued by honest parties). For more details we refer to [10].

Definition 18. A protocol Π implements a *robust public transaction ledger* in the q-bounded synchronous setting if it satisfies the following two properties:

- *Persistence:* Parameterized by $k \in \mathbb{N}$ (the "depth" parameter), if in a certain round an honest player reports a ledger that contains a transaction tx in a block more than k blocks away from the end of the ledger, then tx will always be reported in the same position in the ledger by any honest player from this round on.
- *Liveness:* Parameterized by $u, k \in \mathbb{N}$ (the "wait time" and "depth" parameters, resp.), provided that a transaction either (i) issued by Txgen, or (ii) is neutral, is given as input to all honest players continuously for u consecutive rounds, then there exists an honest party who will report this transaction at a block more than k blocks from the end of the ledger.

These two properties were shown to hold for the ledger protocol Π_{PL} build on top of the Bitcoin backbone protocol. Formally, in [10, Lemma 15&16] the following were proved:

Lemma 19 (Persistence)**.** *Suppose $f < 1$ and $\gamma_{\mathsf{u}} \geq (1+\delta)\lambda\beta$, for some real $\delta \in (0,1)$ and $\lambda \geq 1$ such that $\lambda^2 - f\lambda - 1 \geq 0$. Protocol Π_{PL} satisfies Persistence with probability $1 - e^{-\Omega(\delta^3 k)}$, where k is the depth parameter.*

Lemma 20 (Liveness)**.** *Assume $f < 1$ and $\gamma_{\mathsf{u}} \geq (1+\delta)\lambda\beta$, for some $\delta \in (0,1)$, $\lambda \in [1,\infty)$ and let $k \in \mathbb{N}$. Further, assume oracle Txgen is unambiguous. Then protocol Π_{PL} satisfies Liveness with wait time $u = 2k/(1-\delta)\gamma_{\mathsf{u}}$ and depth parameter k with probability at least $1 - e^{-\Omega(\delta^2 k)}$.*

B Probability of uniquely successful rounds

In this section we demonstrate a new lower bound on the probability of uniquely successful rounds. This bound allows us to argue about the security of `GHOST` even when f is larger than 1.

Lemma 21. *For $p < 0.1$ and $a \in (p, 2k)$: $e^{-a-kp} \leq (1-p)^{\frac{a}{p}-k} \leq e^{-a+kp}$*

Proof. The second inequality is well studied and holds for $p > 0$. For the first inequality by solving for a we get $a \leq k\frac{ln(1-p)}{1+\frac{ln(1-p)}{p}}$ which holds for $p < 0.1$ and $a \in (p, 2k)$. □

Let γ be a lower bound on the probability of a uniquely successful round (a round where only one block is found). From the event where $(n-t)$ players throw q coins each and exactly one coin toss comes head, the probability of a uniquely successful rounds is at least:

$$(n-t)qp(1-p)^{q(n-t)-1} \geq \alpha e^{-\alpha-kp}$$

We set $\gamma = ae^{-a-kp}$, for the minimum k that satisfies the relation $\alpha \in (p, 2k)$. This is a substantially better bound that γ_u and is also a lower bound for the event that at a round is successful. The relation of the two bounds is depicted in Figure 8.

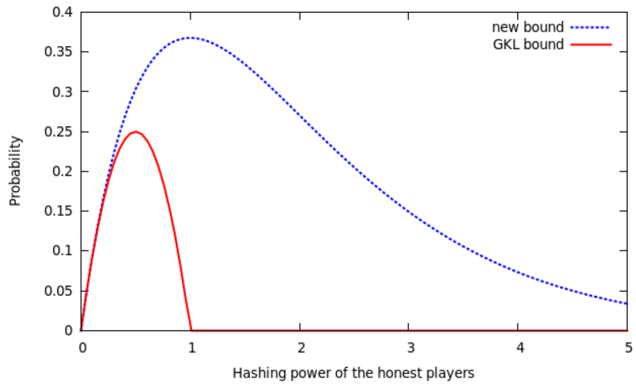

Figure 8: Comparison of the lower bounds of uniquely successful rounds γ and γ_u, used respectively in this work and [10]. Notice that the new lower bound allows as to argue about security when f is greater than 1.

C GHOST-retarder phase

Algorithm 3 The algorithm of the adversary on the chain growth attack with parameter r (r must be greater or equal to 3).

$\langle t_H, t_A \rangle \leftarrow \langle 0, 0 \rangle$ ▷ The weight of the competing trees.
Update the block tree
$\mathcal{C} \leftarrow argmin_{\mathcal{C} \in \text{HonestPaths}} |C|$
Mine head($\mathcal{C}$)
if |blocks mined| $= 0$ **then**
  go to 1
else
  $b \leftarrow$ newly mined block
  Mine b
end if
while $t_H < r$ **do**
  Update the block tree
  $\langle t_H, t_A \rangle \leftarrow \langle t_H + \text{new honest blocks}, t_A + \text{new adversarial blocks} \rangle$
  if $(t_A > t_H)$ and (length of honest subtree $\geq r$) **then**
    Broadcast $subtree(b)$
    $\langle t_H, t_A \rangle \leftarrow \langle 0, 0 \rangle$
    go to 1
  end if
  Mine b
end while

D GHOST Backbone protocol

In this section we present for completeness the remaining procedures of the GHOST backbone protocol. The function `pow` is the same as the one defined in [10]. The function `update` gets a block tree and a set of blocks and returns the updated tree containing all new blocks.

Algorithm 4 The *proof of work* function, parameterized by q, D and hash functions $H(\cdot), G(\cdot)$. The input is $(x, \mathcal{C})$.

function $\mathsf{pow}(x, \mathcal{C})$
    if $\mathcal{C} = \varepsilon$ **then** ▷ Determine proof of work instance
        $s \leftarrow 0$
    else
        $\langle s', x', ctr' \rangle \leftarrow \text{head}(\mathcal{C})$
        $s \leftarrow H(ctr', G(s', x'))$
    end if
    $ctr \leftarrow 1$
    $B \leftarrow \varepsilon$
    $h \leftarrow G(s, x)$
    while $(ctr \leq q)$ **do**
        if $(H(ctr, h) < D)$ **then**
            $B \leftarrow \langle s, x, ctr \rangle$
            break
        end if
        $ctr \leftarrow ctr + 1$
    end while
    $\mathcal{C} \leftarrow \mathcal{C}B$ ▷ Extend chain
    return $\mathcal{C}$
end function

Algorithm 5 The tree update function, parameterized by q, D and hash functions $H(\cdot), G(\cdot)$. The inputs are a block tree T and an array of blocks.

function $\mathsf{update}(T, B)$
    foreach $\langle s, x, ctr \rangle$ **in** T
    foreach $\langle s', x', ctr' \rangle$ **in** B
    if $((s' = H(ctr, G(s, x))) \wedge (H(ctr', G(x', ctr')) < D))$ **then**
        $children_T(\langle s, x, ctr \rangle) = children_T(\langle s, x, ctr \rangle) \cup \langle s', x', ctr' \rangle$ ▷ Add to the tree.
    end if
    return T
end function

E Proofs

E.1 Proposition 6

Proof. The lemma stems from the fact that adding only one block in the tree reduces or increases the dominance of some block by at most 1. For the first bullet, adding k blocks one by one, implies that the dominance of any node will reduce or increase its dominance by at most k. For the second bullet, notice that dominance increases only for blocks that get heavier. The only blocks that get heavier in this case are the ones in the path from the root to the newly mined block. Since these blocks are in HonestPaths(r), they are at least 0-dominant and so their dominance will further increase. Furthermore, the newly mined block is 1-dominant since he does not have any siblings. □

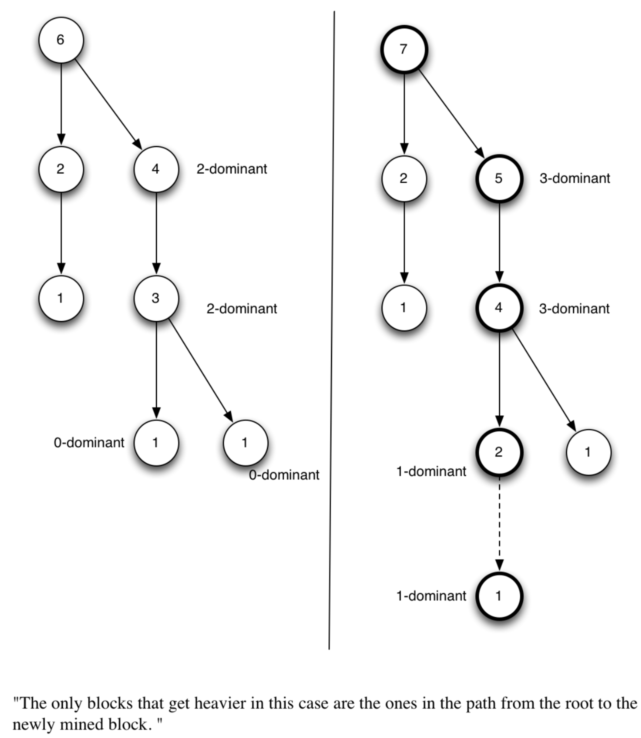

Figure 9: An example of Proposition 6.

E.2 Lemma 8

Proof. We are first going to prove two preliminary claims that show the effect of a uniquely successful round to $\mathrm{p_{dom}}$. The first claim shows that if a uniquely successful round s is not compensated accordingly by the adversary, a newly mined block will be forced into $\mathrm{p_{dom}}(s, 1)$.

Claim 1. *Let round s be a uniquely successful round and b be the honest block mined at round s. If the adversary does not broadcast any block at round $s-1$ then $b \in \mathrm{p_{dom}}(s, 1)$.*

Proof of Claim. First, notice that since the adversary does not broadcast any block it holds that for any honest player P, $T_s^{\exists} = T_r^P$. Therefore, all nodes in the path from v_{root} to b are at least

0-dominant in $T_s^\exists$. Since only this block is added in T_s^+ it holds that all nodes up to the newly mined block are 1-dominant. Thus it follows that $b \in \mathrm{p_{dom}}(s, 1)$. $\dashv$

The second claim shows the effect of a uniquely successful round s to an existing $\mathrm{p_{dom}}(s-1, d)$ path. Notice that if the adversary broadcasts less than d blocks the same nodes continue to be at least 1-dominant in the following round.

Claim 2. *Let round s be a uniquely successful round, b be the honest block mined at round s and $\mathrm{p_{dom}}(s-1, d) \neq \bot$. If the adversary broadcasts (i) $k < d$ blocks at round $s-1$ then $\mathrm{p_{dom}}(s-1, d) \subseteq \mathrm{p_{dom}}(s, d+1-k)$, (ii) $k = d$ blocks at round $s-1$ then either $b \in \mathrm{p_{dom}}(s, 1)$ or $\mathrm{p_{dom}}(s-1, d) \subseteq \mathrm{p_{dom}}(s, 1)$ and b is a descendant of the last node in $\mathrm{p_{dom}}(s-1, d)$.*

Proof of Claim. There are two cases. In the first case suppose the adversary broadcasts $k < d$ blocks. Then with these blocks the adversary can lower the dominance in $T_s^\exists$ of nodes in $\mathrm{p_{dom}}(s-1, d)$ by k. Thus $\mathrm{p_{dom}}(s-1, d)$ will be a prefix of all the chains in HonestPaths(s). But because s is a uniquely successful round, the dominance in T_s^+ of all nodes in $\mathrm{p_{dom}}(s-1, d)$ will increase by one. Therefore $\mathrm{p_{dom}}(s-1, d) \subseteq \mathrm{p_{dom}}(s, d+1-k)$ and b will be a descendant of the last node in $\mathrm{p_{dom}}(s-1, d)$.

In the second case suppose the adversary broadcasts $k = d$ blocks. If he does not broadcast all of these blocks to reduce the dominance in $T_s^\exists$ of the nodes in $\mathrm{p_{dom}}(s-1, d)$, then $\mathrm{p_{dom}}(s-1, d)$ will be a prefix of all the chains in HonestPaths(s) and as in the previous case, $\mathrm{p_{dom}}(s-1, d) \subseteq \mathrm{p_{dom}}(s, d+1-k)$ and b will be a descendant of the last node in $\mathrm{p_{dom}}(s-1, d)$.

Otherwise the adversary will reduce the dominance in $T_s^\exists$ of at least one node in $\mathrm{p_{dom}}(s-1, d)$ to zero. If b is a descendant of the last node in $\mathrm{p_{dom}}(s-1, d)$, then all nodes in $\mathrm{p_{dom}}(s-1, d)$ will be 1-dominant in T_s^+ and $\mathrm{p_{dom}}(s-1, d) \subseteq \mathrm{p_{dom}}(s, 1) = \mathrm{p_{dom}}(s, d+1-d)$. If b is not a descendant of the last node in $\mathrm{p_{dom}}(s-1, d)$, then for the player P that mined this block it holds that $T_s^P = T_s^\exists$, because he would have not mined a chain that does not contain $\mathrm{p_{dom}}(s-1, d)$ at round s otherwise. Therefore, P at round s was mining a chain that belonged to HonestPaths(s, v_{root}) and thus all nodes in the chain are at least 0-dominant in $T_s^\exists$. But because s is a uniquely successful round the dominance of all nodes in the chain that b belongs to will increase by one and thus $b \in \mathrm{p_{dom}}(s, 1)$. $\dashv$

Let b_i denote the honest block mined at round r_i. Let us assume that $r = r_m$. We are going to prove the lemma using induction on the number of uniquely successful rounds m.

For the base case suppose $m = 1$. The adversary does not broadcast any block until round $r_1 - 1$ and from the first claim $b_1 \in \mathrm{p_{dom}}(r_1, 1)$. Thus the base case is proved. Suppose the lemma holds for $m-1$ uniquely successful rounds and let k_1 be the number of blocks that the adversary broadcasts in the round interval $[r'-1, r_{m-1}-1]$. We have two cases.

(First case) $k_1 = m-1$ and the adversary broadcasts no blocks in the rest of the rounds. From the first claim it follows that $b_m \in \mathrm{p_{dom}}(r_m, 1)$.

(Second case) $k_1 < m-1$ and from the induction hypothesis there exist blocks $b'_1, ..., b'_{m-1-k_1}$ mined by honest players at the uniquely successful rounds $r_1, .., r_{m-1}$ where $b'_i \in \mathrm{p_{dom}}(r_{m-1}, i)$. Let k_2 be the number of blocks that the adversary broadcasts until round $r_m - 2$ and k_3 the number of blocks he broadcasts at round $r_m - 1$. If $k_2 = m-1$ then again from the first claim it follows that $b_m \in \mathrm{p_{dom}}(r_m, 1)$. If $k_2 < m-1$ then if $k_3 + k_2 = m-1$ then from the second claim either $b_m \in \mathrm{p_{dom}}(r_m, 1)$ or $b'_{m-1-k_1} \in \mathrm{p_{dom}}(r_m, 1)$. If $k_3 + k_2 < m-1$ then again from the second claim at round r_m, $b'_i \in \mathrm{p_{dom}}(r_m - 1, i)$ for i in $\{k_2 + k_3 + 1, .., m-1-k_1\}$ and either $b'_{k_2+k_3}$ is in $\mathrm{p_{dom}}(r_m, 1)$ or b_m is in $\mathrm{p_{dom}}(r_m, 1)$. This completes the induction proof.

We proved that if $k_4 < m$ is the number of blocks the adversary broadcasts until round $r_m - 1$, then there exists honest blocks $b'_1, .., b'_{m-k_4}$ s.t. b'_i is in $\mathrm{p_{dom}}(r_m, i)$. Now in the case $r > r_m$, let

$k_5 < m - k_4$ be the number of blocks the adversary broadcasts in the remaining rounds. The lemma follows easily from the second claim.

Remark 1. Let $r_1, .., r_m$ be uniquely successful rounds up to round r and the honest block mined at round r_1 is in $\mathrm{p_{dom}}(r_1, 1)$. If the adversary broadcasts $k < m$ blocks from round r_1 until round $r-1$, then there exists an honest block b mined in one of the rounds $r_1, .., r_m$ such that b in $\mathrm{p_{dom}}(r, m-k)$. (to see why the remark holds notice that that blocks that the adversary broadcasts before round r_1 affect only the dominant path at round r_1, and not at the following rounds)

□

E.3 Lemma 9

Proof. Let random variable Z_{s_1,s_2} (resp. $Z^{pub}_{s_1,s_2}$) denote the number of blocks the adversary computes (resp. broadcasts) from round s_1 until round s_2, and random variable X_{s_1,s_2} denote the number of rounds that are uniquely successful in the same interval.

We are first going to prove two preliminary claims. We show that as long as from some round r and afterwards the adversary broadcasts less blocks than the total number of uniquely successful rounds, the chain that any honest player adopts after round r extends $\mathrm{p_{dom}}(r, X_{1,r} - Z_{1,r})$. More generally we can prove the following claim.

Claim 3. *Consider any execution such that for all $s_2 \geq s_1$ it holds that $Z_{1,s_2} < X_{1,s_2}$. Then, the chain that any honest player adopts after round s_1 extends $\mathrm{p_{dom}}(s_1, X_{1,s_1} - Z_{1,s_1})$.*

Proof of Claim. Since $X_{1,s_1} > Z_{1,s_1}$ from Lemma 8 if follows that $p = \mathrm{p_{dom}}(s_1, X_{1,s_1} - Z_{1,s_1-1}) \neq \perp$. As long as the number of blocks that the adversary broadcasts at round s_2 are less than the dominance of the nodes in p in $T^+_{s_2-1}$, all honest players at round s_2 will adopt chains containing p. Thus uniquely successful rounds will increase the dominance of these nodes. But since from the assumptions made, $Z_{1,s_2} < X_{1,s_2}$, in all rounds after round s_1, the nodes in p are at least 1-dominant in every $T^P_{s_2}$ where P is an honest player; the claim follows. ⊣

Next we will show that if successive u.s. rounds occur such that the blocks mined are on different branches, then the adversary must broadcast an adequate number of blocks, as specified below.

Claim 4. *Consider any execution where $s_1 < s_2 < ... < s_m$ are u.s. rounds and s_k is the first u.s. round such that the honest block mined in this round is not a descendant of the honest block mined in round s_{k-1}, for $k \in \{2, .., m\}$. Then either $Z^{pub}_{s_1-1,s_m-1} > X_{s_1,s_m-1}$ or $Z^{pub}_{s_1-1,s_m-1} = X_{s_1,s_m-1}$ and the honest block mined at round s_m will be in $\mathrm{p_{dom}}(s_m, 1)$.*

Proof of Claim. Let $b_1, .., b_m$ denote the honest blocks mined at rounds $s_1, .., s_m$ respectively. We are going to prove the claim for $m = 2$. Suppose, for the sake of contradiction, that $Z^{pub}_{s_1-1,s_2-1} < X_{s_1,s_2-1}$. By the definition of s_2, the honest blocks mined on all u.s. rounds until round $s_2 - 1$ are descendants of b_1. From Lemma 8 at least one honest block b computed in one of the u.s. rounds in $[s_1, s_2 - 1]$ will be in $\mathrm{p_{dom}}(s_2 - 1, X_{s_1,s_2-1} - Z^{pub}_{s_1-1,s_2-2})$. Since from our hypothesis the adversary will broadcast less than $Z^{pub}_{s_2-1,s_2-1} < X_{s_1,s_2-1} - Z^{pub}_{s_1-1,s_2-2}$ blocks at round s_2-1, it is impossible for b_2 not to be a descendant of b and thus of b_1 which is a contradiction. Hence, $Z^{pub}_{s_1-1,s_2-1} \geq X_{s_1,s_2-1}$. If $Z^{pub}_{s_1-1,s_2-1} > X_{s_1,s_2-1}$ the base case follows. Otherwise, $Z^{pub}_{s_1-1,s_2-1} = X_{s_1,s_2-1}$ and we have two cases. In the first case, $X_{s_1,s_2-1} = Z^{pub}_{s_1-1,s_2-2}$ and at round round $s_2 - 1$ the adversary does not broadcast any block. From Claim 1 of Lemma 8, b_2 will be in $\mathrm{p_{dom}}(s_2, 1)$. In the second case, it holds that the adversary broadcasts exactly $X_{s_1,s_2-1} - Z^{pub}_{s_1-1,s_2-2}$ blocks at round $s_2 - 1$. From

Claim 2 of Lemma 8, since b_2 cannot be a descendant of the last node of $\mathrm{p_{dom}}(s_2-1,1)$, b_2 will be in $\mathrm{p_{dom}}(s_2,1)$. Hence, the base case follows.

Suppose the lemma holds until round s_m. By the inductive hypothesis we have two cases. In the first case $Z^{pub}_{s_1-1,s_m-1} > X_{s_1,s_m-1}$ which implies $Z^{pub}_{s_1-1,s_m-1} \geq X_{s_1,s_m}$. If no u.s. round happens during rounds $s_m+1,\ldots,s_{m+1}-1$ then from Claim 1 in the proof of Lemma 8 the claim follows. Otherwise, a u.s. round s' happens during these rounds, where the honest block mined is a descendant of b_m. Then we can make the same argument as for the base case starting from round s' and get that either $Z^{pub}_{s'-1,s_{m+1}-1} > X_{s',s_{m+1}-1}$ or $Z^{pub}_{s'-1,s_{m+1}-1} = X_{s',s_{m+1}-1}$ and the honest block mined at round s_{m+1} will be in $\mathrm{p_{dom}}(s_{m+1},1)$. Since $Z^{pub}_{s'-1,s_{m+1}-1} < Z^{pub}_{s_m-1,s_{m+1}-1}$ and $X_{s',s_{m+1}-1} = X_{s_m+1,s_{m+1}-1}$, by the inequality of the inductive hypothesis the claim follows.

In the second case $Z^{pub}_{s_1-1,s_m-1} = X_{s_1,s_m-1}$ and the honest block b_m mined at round s_m will be in $\mathrm{p_{dom}}(s_m,1)$. From Remark 1 of the proof of claim Lemma 8, for an application of this Lemma from rounds s_m until $s_{m+1}-1$ we can count the adversarial blocks starting from round s_m. Thus from the same argument as for the base case starting from round s_m we get that either $Z^{pub}_{s_m,s_{m+1}-1} > X_{s_m,s_{m+1}-1}$ or $Z^{pub}_{s_m,s_{m+1}-1} = X_{s_m,s_{m+1}-1}$ and the honest block mined at round s_m will be in $\mathrm{p_{dom}}(s_m,1)$. By the equality of the inductive hypothesis the claim follows. ⊣

Next, we observe that Lemma 8 as well as Claim 3 and 4 can be applied on a subtree of the block tree, if all honest blocks mined after the round the root of the subtree was mined are on this subtree.

Observation 5. Let b be an honest block computed at round s_1 that is in the chains adopted by all honest players after round s_2. Also, all the blocks mined at u.s. rounds after round s_1 are descendants of b. Then the following hold:

1. Regarding applications of Lemma 8 and Claim 4 on the subtree of the block tree rooted on b after round s_1, we can ignore all blocks that the adversary has mined up to round s_1.

2. Regarding applications of Claim 3 after round s_2, we can ignore all blocks that the adversary has mined up to round s_1.

To see why the observation holds consider the following. Since the adversary receives block b for the first time at round s_1+1, all blocks that the adversary mines before round s_1+1 cannot be descendants of b. Regarding the first point, blocks that are not descendants of b do not affect the validity of Lemma 8 and Claim 4 on the subtree of the block tree rooted on b; this is because blocks that are not descendants of b, do not affect the dominance of the nodes of the subtree rooted at b. Regarding the second point, consider the dominant path at round $s_3 > s_2$ in the subtree that is rooted on b. Then, this path can be extended up to the root node, since, by our assumption, b is in the chains adopted by all honest players after round s_2.

We are now ready prove the lemma. First, we are going to define a set of bad events which we will show that hold with probability exponentially small in s. Let $BAD(s_1,s_2)$ be the event that $X_{s_1,s_2} \leq Z_{s_1,s_2}$. In [10, Lemma 5], by an application of the Chernoff bounds it was proved that assuming that $\gamma \geq (1+\delta)\beta$ for some $\delta \in (0,1)$, then with probability at least $(1-e^{-\frac{\beta}{75}\delta^2 s'})(1-e^{-\frac{\gamma}{32}\delta^2 s'}) \geq 1-e^{-(\min(\frac{\beta}{75},\frac{\gamma}{32})\delta^2 s'-\ln(2))}$ for any $r'>0, s'\geq s$:

$$X_{r',r'+s'-1} > (1+\frac{\delta}{2})Z_{r',r'+s'-1} \tag{1}$$

Thus, there exists an appropriate constant $\epsilon = \delta^2\min(\frac{\beta}{75},\frac{\gamma}{32})$, independent of r, such that it holds that for any $r'>0, s'\geq s$, $BAD(r',r'+s'-1)$ occurs with probability at most $e^{-\epsilon\delta^2 s'+\ln 2}$.

Let $BAD(s_1)$ denote the event $\bigvee_{r' \geq s} BAD(s_1+1, s_1+r')$. From an application of the union bound, we get that for the function $g(s) = \epsilon\delta^2 s - \ln 2 + \ln(1-e^{-\epsilon\delta^2})$, the probability that $BAD(s_1)$ happens is:

$$Pr[\bigvee_{r' \geq s} BAD(s_1+1, s_1+r')] \leq \sum_{r' \geq s} e^{-\epsilon\delta^2 r' + \ln 2}$$
$$\leq e^{\ln 2} \sum_{r' \geq s} e^{-\epsilon\delta^2 r'} \leq e^{\ln 2} \frac{e^{-\epsilon\delta^2 s}}{1-e^{-\epsilon\delta^2}} \leq e^{-g(s)}$$

We will use the convention that block b_i is mined at round r_i. Let b_1 be the most recent honest block that is in the chains that all honest players have adopted on and after round r, such that the blocks mined at all u.s. rounds after round r_1 are descendants of b_1. This block is well defined, since in the worst case it is the genesis block. If r_1 is greater or equal to $r - s$, then the lemma follows for block b_1 with probability 1.

Suppose round r_1 is before round $r - s$ and that $BAD(r_1)$ does not happen. The negation of $BAD(r_1)$ implies that $X_{r_1+1,r-1+c} > Z_{r_1+1,r-1+c}$, for $c \geq 0$. By Lemma 8 and Claim 3 there exists at least one honest block b_2, mined in a u.s. round and contained in the chains of all honest players on and after round r. W.l.o.g. let b_2 be the most recently mined such block. By the definition of b_1, b_2 is a descendant of b_1. If r_2 is greater or equal to $r - s$ then the lemma follows, since b_2 is an honest block mined on and after round $r - s$ that satisfies the conditions of the lemma.

Suppose round r_2 is before round $r - s$. Let r_3 be the earliest u.s. round, such that b_3 and the blocks mined at all u.s. rounds afterwards are descendants of b_2. Since b_2 will be in the chains of all honest players after round r, round r_3 is well defined. Also let $s_1 < \ldots < s_m < \ldots$ be the sequence of u.s. rounds after round r_1 that satisfy the conditions of Claim 4. That is, s_k is the first u.s. round such that the honest block mined in this round is not a descendant of the honest block mined in round s_{k-1}, for $k \in \{2, .., m\}$. The first u.s. round after round r_1 corresponds to s_1.

We will argue that r_3 is equal to some $s_i > s_1$ in the aforementioned sequence. Suppose, for the sake of contradiction that it does not. This implies that the honest block mined at round r_3 (denoted by b_3) is a descendant of the honest block mined at some round s_i of the sequence. W.l.o.g. suppose that s_i is the largest such round that is before round r_3. There are three cases. In the first case, $r_2 < s_i < r_3$. By the definition of s_i and r_3, the block mined at round s_i is an ancestor of b_3 and also a descendant of b_2. Hence, s_i satisfies the definition of r_3 which is a contradiction (there is an earlier round than r_3 with the same property). In the second case, $s_i = r_4$, where b_4 is a descendant of b_1 and either $b_2 = b_4$ or b_4 is an ancestor of b_2. Then b_4 is a block that satisfies the definition of b_1, and is more recent, which is a contradiction. In the third case, $r_1 < s_i < r_2$ and the block mined at round s_i is not an ancestor of b_2. By the definition of s_i, the honest block mined at round s_i is an ancestor of b_3, that has been mined before round r_2. But this is contradictory, since no honest block can be an ancestor of b_3, mined before round r_2, but not be an ancestor of b_2.

Since we proved that r_3 is equal to some s_i we can apply Claim 4 from round $r_1 + 1$ until round r_3. Again, from Observation 5, regarding applications of Claim 4 after round r_1 we can ignore blocks that were mined before round $r_1 + 1$. Then either $Z_{r_1+1,r_3-1} \geq Z^{pub}_{r_1+1,r_3-1} > X_{r_1+1,r_3-1}$ or $Z_{r_1+1,r_3-1} \geq Z^{pub}_{r_1+1,r_3-1} = X_{r_1+1,r_3-1}$ and the honest block mined at round r_3 will be in $\mathrm{p_{dom}}(r_3, 1)$.

Suppose, for the sake of contradiction, that round r_3 is after round $r_2 + s$. Then $(r_3 - 1) - (r_1 + 1) \geq s$ and $Z_{r_1+1,r_3-1} \geq X_{r_1+1,r_3-1}$. This is a contradiction, since in this case $\neg BAD(r_1)$ implies $Z_{r_1+1,r_3-1} < X_{r_1+1,r_3-1}$. Therefore, $r_3 \leq r_2 + s < r$. In addition, notice that $\neg BAD(r_1)$ also implies

$$X_{r_1+1,r_2+s} > Z_{r_1+1,r_2+s} \tag{2}$$

We are going to apply Lemma 8 and Observation 5 from round r_3 until round $r_2 + s$ in the subtree rooted at b_2. According to the analysis we made previously there are two cases. In the first case, $Z^{pub}_{r_1+1,r_3-1} > X_{r_1+1,r_3-1}$ or equivalently $Z^{pub}_{r_1+1,r_3-1} \geq X_{r_1+1,r_3}$. Suppose, for the sake of contradiction, that $r_3 = r_2 + s$. Then $Z_{r_1+1,r_2+s-1} \geq X_{r_1+1,r_2+s}$. But this is a contradiction, since $\neg BAD(r_1)$ implies Inequality 2. Therefore, $r_3 < r_2 + s$. From Inequality 2:

$$
\begin{aligned}
X_{r_3+1,r_2+s} &\geq X_{r_1+1,r_2+s} - X_{r_1+1,r_3} \\
&> Z_{r_1+1,r_k+s} - Z^{pub}_{r_1+1,r_3-1} \geq Z^{pub}_{r_3,r_2+s}
\end{aligned}
$$

The last inequality, stems from two facts: that we can ignore blocks that were mined before round $r_1 + 1$ regarding applications of Lemma 8 and also that the blocks that the adversary broadcasts at distinct rounds are different (adversaries that broadcast the same block multiple times can be ignored without loss of generality).

In the second case, $Z^{pub}_{r_1+1,r_3-1} = X_{r_1+1,r_3-1}$ and the honest block mined at round r_3 will be in $\mathrm{p}_{\mathrm{dom}}(r_3, 1)$. Again from Inequality 2:

$$
\begin{aligned}
X_{r_3,r_2+s} &= X_{r_1+1,r_2+s} - X_{r_1+1,r_3-1} \\
&> Z_{r_1+1,r_k+s} - Z^{pub}_{r_1+1,r_3-1} \geq Z^{pub}_{r_3,r_2+s}
\end{aligned}
$$

The same analysis holds for all rounds after $r_2 + s$. By an application of Claim 3, an honest block b, computed in one of the u.s. rounds after round r_2 and before round r, will be in the chains that honest players adopt on and after round r. Since b_2 is the most recently mined block, before round $r - s$, included in the chain of all honest players, b must have been mined on and after round $r - s$ (since $r_3 > r_2$). Let A be the event that there exists a block mined by an honest player on and after round $r - s$, that is contained in the chain which any honest player adopts after round r. We have proved that $(\neg BAD(r_1))$ implies A. Then:

$$
\begin{aligned}
Pr[A] =& Pr[A \wedge BAD(r_1)] + Pr[A \wedge \neg BAD(r_1)] \\
\geq& Pr[A \wedge \neg BAD(r_1)] \\
=& Pr[A | \neg BAD(r_1)] Pr[\neg BAD(r_1)] \\
=& Pr[\neg BAD(r_1)] \\
\geq& 1 - e^{-g(s)}
\end{aligned}
$$

Hence, the lemma holds with probability at least $1 - e^{-g(s)}$. □