

# Determining the Nonexistent Terms of Non-linear Multivariate Polynomials: How to Break Grain-128 More Efficiently

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**Abstract.** In this paper, we propose a reduction technique that can be used to determine the density of IV terms of a complex multivariable boolean polynomial. Using this technique, we revisit the dynamic cube attack on Grain-128. Based on choosing one more nullified state bit and one more dynamic bit, we are able to obtain the IV terms of degree 43 with various of complicated reduction techniques for polynomials, so that the nonexistent IV terms can be determined. As a result, we improve the time complexity of the best previous attack on Grain-128 by a factor of  $2^{16}$ . Moreover, our attack applies to all keys.

**Keywords:** Stream ciphers, Grain-128, Polynomial reduction, Dynamic cube attack

## 1 Introduction

Most cryptanalytic problems of symmetric ciphers can be reduced to the problem of solving large non-linear multivariate polynomial systems. This problem is NP-complete [6]. Solving the system using linearization or relinearization methods directly will result in space and time complexities that are beyond the power of current computers. As a result, algebraic attacks such as cube attack [3], cube tester [5,1] and dynamic cube attack [4] were proposed in order to reduce the complexities.

Stream cipher Grain-128 [7] is a refined version of Grain scheme, one of the finalists of eSTREAM Project. The output bit is a high degree boolean function in initial vector (IV) bits and key bits. Since the proposal of Grain-128, a number of cryptanalytic results have been presented in the literatures. Fischer et. al.

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applied the statistical analysis to recover the key of reduced Grain-128 up to 180 out of 256 iterations with a complexity slightly better than brute force [5]. They pointed out that Grain-128 may be immune to this attack due to the very high degree of the polynomial of the output bit. Knellwolf et al. proposed the conditional differential cryptanalysis on NLFSR-based stream ciphers including Grain scheme ciphers [8]. Conditional differential cryptanalysis exploited the message modification technique introduced in [10,9], which controlled the diffusion by controlling the plaintexts. It was applied to recover 3 bits' key of the Grain-128 reduced to 213 rounds with a probability up to 0.59 and distinguish Grain-128 reduced to 215 rounds from random primitives [8]. Another message controlling method is to nullify some state bits which may be more important than the others for reducing the degree or enhancing the sparsity. After nullifying some state bits, the representation of the output bit with IV bits can be simplified; and the output bit becomes nonrandom. This technique is called dynamic cube attack which combines the conditional differential and cube tester. With dynamic cube attack, Dinur and Shamir [4] proposed two key-recovery attacks on reduced-round Grain-128 for arbitrary keys and an attack on the full Grain-128 that holds for  $2^{-10}$  of the key space. Furthermore, Dinur et al. [2] improved the dynamic cube attack on full Grain-128, and tested the main component with a dedicated hardware. Their experimental results showed that for about 7.5% of the keys, the proposed attack beat the exhaustive search by a factor of  $2^{38}$ .

In this paper, we further improve the attack in [2] by obtaining the IV terms. The time complexity of our attack is about  $2^{74}$  cipher executions, which is applicable to all keys of Grain128.

**Our Contributions.** The contributions of this paper are three fold. Firstly, we exploit the nullification technique introduced in [2] and improve the nullification by a better choice of nullified state bits and dynamic IV bits. Secondly, we propose IV controlling techniques to reduce the IV terms of high degrees. Thirdly, we give several reduction techniques for boolean functions in order to reduce the number of terms we need to process. The primary techniques are those to remove the repeated terms and covered terms. Furthermore, we present the data partition techniques so that we do the removal in parallel with super computers.

With the aforementioned techniques, we mathematically compute the IV terms of degree 43. The major difference between our attack and the previous dynamic cube attacks is that we focus on obtaining the IV terms mathematically instead of choosing the suitable (sub)cubes with testing technique. In this way, our method not only improves the previous attacks but also can naturally be applied to all keys with the same complexity.

Due to the difference, we also simplified the attack procedure of the dynamic cube attack. Briefly, our attack can be decomposed to the following phases:

1. Determine the dynamic variables, state/IV bits to be nullified, as well as the key bits to be guessed. Calculate the IV terms of degree 43 afterwards. The cube can be chosen freely from those disappeared IV terms. This is the preprocessing phase of the attack. However, most of the dedicated work in this paper contributes to this phase.

In this phase, we exploit various of techniques to obtain the exact IV terms. We exploit two algorithms to remove the repeated and covered terms, which help reduce the polynomial dramatically. We also use IV representation to calculate the IV terms of degree 43.

2. Guess the key bits, and get the output bits with IVs chosen according to the principle in the previous phase. The summation of the output bits over the cube can be used as a distinguisher since for the correct key the summation will always be 0, while for wrong keys 0 and 1 are supposed to occur with the same probability. This is the only on-line phase of our attack.

The detail of our attack will be demonstrated in the rest of the paper. In Section 2, the outline of Grain-128 and basic concepts of cube attacks, cube tester and dynamic cube attack will be introduced. In Section 3, we present a couple of methods to control the IV terms and reduction techniques in order to reduce the number of terms to be processed. The preprocessing phase is introduced in Section 3 as well. Section 4 will give the on-line phase of the attack. Finally, Section 5 summarizes the paper.

## 2 Preliminaries

In this section, we will first briefly introduce Grain-128. Then the techniques related to our work will be introduced, including cube attack/tester, dynamic cube attack and nullification technique.

### 2.1 Outline of Grain-128

The state of Grain-128 is represented by a 128-bit LFSR and a 128-bit NLFSR. The feedback function of the LFSR and NLFSR are defined as

$$\begin{aligned} s_{i+128} &= s_i + s_{i+7} + s_{i+38} + s_{i+70} + s_{i+81} + s_{i+96}, \\ b_{i+128} &= s_i + b_i + b_{i+26} + b_{i+56} + b_{i+91} + b_{i+96} + b_{i+3}b_{i+67} + b_{i+11}b_{i+13} + \\ &\quad b_{i+17}b_{i+18} + b_{i+27}b_{i+59} + b_{i+40}b_{i+48} + b_{i+61}b_{i+65} + b_{i+68}b_{i+84}. \end{aligned}$$

The output function is

$$\begin{aligned} z_i &= b_{i+2} + b_{i+15} + b_{i+36} + b_{i+45} + b_{i+64} + b_{i+73} + b_{i+89} + s_{i+93} + b_{i+12}s_{i+8} + \\ &\quad s_{i+13}s_{i+20} + b_{i+95}s_{i+42} + s_{i+60}s_{i+79} + b_{i+12}b_{i+95}s_{i+95}. \end{aligned}$$

During the initialization step, the 128-bit key is loaded into the NLFSR and 96 bits of IV are loaded into the LFSR, with the other IV bits setting to 1. The state runs 256 rounds with the output feeding back, and the first output bit is  $z_{257}$ . For the detail of Grain-128, we refer to [7].

The output bit  $z_{257}$  is a boolean function over IV bits and key bits. We define in this paper the degree of the polynomial in terms of the IV variables, since the key bits are constant though they are unknown in our attack.

Next, we will give some more definitions used throughout the paper. Assuming that there are  $n$  IV bits, i.e.,  $v_1, v_2, \dots, v_n$  and  $m$  key bits, i.e.,  $k_1, k_2, \dots, k_m$ , the output bit  $x$  can be illustrated as

$$x = \sum_{I,J} t_I g_I(k), \quad (1)$$

where  $t_I$  is the multiplication of all the IV bits whose indices are in  $I$ , i.e., it can be represented by  $t_I = \prod_{i \in I} v_i$ ,  $I$  and  $J$  are subsets of  $\{1, 2, \dots, n\}$  and  $\{1, 2, \dots, m\}$  respectively.  $g_I(k)$  is defined as the coefficient function, which is a function over key bits:

$$g_I(k) = \prod_{l_1 \in J_1} k_{l_1} + \prod_{l_2 \in J_2} k_{l_2} + \dots + \prod_{l_s \in J_p} k_{l_s}, \quad (2)$$

where  $J_1, J_2, \dots, J_p$  are subsets of  $\{1, 2, \dots, m\}$ . In the rest of the paper, we will call it coefficient for short, when there is no ambiguity. Each  $t_I \prod_{l_j \in J_j} k_{l_j}$  in (1) is a term of  $x$ , and  $t_I$  is called an IV term. Note that any term has a unique IV term. But an IV term may occur in a large number of terms because  $g_I(k)$  is probably very complicated.

## 2.2 Cube Attack and Cube Tester

Cube attack [3] exploits the IV terms whose coefficient is linear over key bits, and then retrieves the keys once enough independent linear functions are obtained, by solving linear equations. A methodology is proposed [3,1] to test if a coefficient is linear and which key bits are involved.

Even if the coefficient is nonlinear but only involves a small number of key bits (where the degree on key bits is low), the coefficient can also be obtained by the technique called cube tester [1]. This technique is constitute of two steps: First testing which key bits are involved in the coefficient, and then determining the specific expression (if it is linear, then the first step is enough).

## 2.3 Dynamic Cube Attack

In [4], Dinur and Shamir proposed the dynamic cube attack to recover the secret key by exploiting distinguishers obtained from cube testers, with application to Grain-128.

In dynamic cube attack, some dynamic bits in the IV that are determined by key bits are chosen in order to nullify some state bits that will greatly simplify the output function. Then one expects to acquire certain nonrandom property which can be exploited by cube tester. The nonrandom property can be used as a distinguisher for key recovery.

As mentioned in Section 1, the attack in [2] is an improvement of that in [4]. The dynamic cube attacks introduced and exploited in [4,2] are based on the

nullification technique. The major difference between the two work is the different choices of nullified bits<sup>6</sup>. As a consequence, we will detail the nullification technique in the next subsection.

## 2.4 Nullification Technique

For Grain-128, the first output bit  $z_{257}$  can be represented by state bits as

$$z_{257} = b_{269}b_{352}s_{352} + b_{352}s_{299} + s_{317}s_{336} + s_{270}s_{277} + b_{269}s_{265} + s_{350} + b_{346} + b_{330} + b_{321} + b_{302} + b_{293} + b_{272} + b_{259}.$$

The most significant term of its ANF (algebraic normal form) is  $b_{269}b_{352}s_{352}$ . In fact, the terms resulted from  $b_{269}b_{352}s_{352}$  are much more than those from the other terms. Similarly, the most significant terms of the ANF of  $b_{269}$ ,  $b_{352}$  and  $s_{352}$  are  $b_{153}b_{236}s_{236}$ ,  $b_{236}b_{319}s_{319}$  and  $b_{236}b_{319}s_{319}$  respectively. The common factor is  $b_{236}$ , so  $b_{269}$ ,  $b_{352}$ ,  $s_{352}$  and  $z_{257}$  can be simplified if nullifying  $b_{236}$ . But  $b_{236}$  is too complicated, i.e.,

$$b_{236} = b_{120}b_{203}s_{203} + b_{203}s_{150} + b_{176}b_{192} + s_{168}s_{187} + b_{169}b_{173} + b_{148}b_{156} + b_{135}b_{167} + b_{125}b_{126} + b_{119}b_{121} + b_{111}b_{175} + s_{121}s_{128} + b_{120}s_{116} + b_{204} + s_{201} + b_{199} + b_{197} + b_{181} + b_{172} + b_{164} + b_{153} + b_{144} + b_{134} + b_{123} + b_{110} + b_{108} + 1.$$

Nullifying  $b_{236}$  needs too many guessed key bits, so the scheme in [4] retrieved a subset of  $2^{-10}$  of all possible keys by fixing 10 key bits to be zero.

Another approach is to simplify  $b_{236}$  by nullifying  $b_{203}$ , which is adopted by [2].  $b_{203}$  is still too complicated to be nullified directly, i.e.,

$$b_{203} = b_{87}b_{170}s_{170} + b_{170}s_{117} + b_{143}b_{159} + s_{135}s_{154} + b_{136}b_{140} + b_{115}b_{123} + b_{102}b_{134} + b_{92}b_{93} + b_{86}b_{88} + b_{78}b_{142} + s_{88}s_{95} + b_{87}s_{83} + b_{171} + s_{168} + b_{166} + b_{164} + b_{148} + b_{139} + b_{131} + b_{120} + b_{111} + b_{101} + b_{90} + b_{77} + b_{75} + s_{75}.$$

In order to nullify  $b_{203}$ , one should first nullify  $b_{170}$ ,  $b_{159}$ ,  $b_{138}$ ,  $s_{135}$ ,  $b_{136}$ ,  $b_{134}$ ,  $b_{133}$ , and  $b_{131}$ . All of these bits but  $b_{170}$  can be nullified directly by choosing IV bits. We know that

$$b_{170} = b_{54}b_{137}s_{137} + b_{137}s_{84} + b_{110}b_{126} + s_{102}s_{121} + b_{103}b_{107} + b_{82}b_{90} + b_{69}b_{101} + b_{59}b_{60} + b_{53}b_{55} + b_{45}b_{109} + s_{55}s_{62} + b_{54}s_{50} + b_{138} + s_{135} + b_{133} + b_{131} + b_{115} + b_{106} + b_{98} + b_{87} + b_{78} + b_{68} + b_{57} + b_{44} + b_{42} + s_{42}.$$

In order to nullify  $b_{170}$ ,  $b_{137}$  can be nullified first by setting  $s_9$  to be a dynamic value. In fact,  $b_{170}$  can be nullified directly as well since it is not too complicated. However, nullification of  $b_{137}$  can not only nullify  $b_{170}$  but also simplify  $b_{253}$  which contributes a lot to the degree and IV terms. The term  $b_{143}b_{159}$  can be nullified

<sup>6</sup> The authors also experimentally verified the main component of the attack by a dedicated hardware in [2]

**Table 1.** Nullification Scheme in [2]

nullification	$b_{131}, b_{133}, b_{134}, s_{135}, b_{136}, b_{137}, b_{138}, b_{145}, b_{153}, b_{159}, b_{170}, b_{176}, b_{203}$
dynamic bits	$s_3, s_5, s_6, s_{77}, s_8, s_9, s_{10}, s_{17}, s_{25}, s_{31}, s_{42}, s_{83}, s_1$

by nullifying either  $b_{143}$  or  $b_{159}$ . The authors chose to nullify  $b_{159}$  because it can help reduce  $b_{275}$  whose ANF significant term is  $b_{159}b_{242}s_{242}$ .  $b_{145}$ ,  $b_{153}$  and  $b_{176}$  are also nullified in order to simplify  $s_{261}$ ,  $b_{269}$  and  $b_{292}$ . The nullified state bits and dynamic IV bits of [2] are shown in Table 1.

### 3 Obtaining the IV terms of Boolean Polynomials

In this paper, the main purpose is to obtain all the IV terms of degree 43. In order to achieve this goal, we first propose new nullification (Section 3.1) and IV choosing (Section 3.2) techniques. Then term reduction techniques (Section 3.3) are also presented to remove the terms that will not contribute to the IV terms of degree 43 in the polynomial. With these techniques, we are able to obtain the exact IV terms of degree 43 in the output bit in Section 3.4.

#### 3.1 Nullification of State Bits

Motivated by the nullification technique in [4,2], we choose to nullify some state bits. In order to simplify  $b_{269}b_{352}s_{352}$ , we also need to nullify  $b_{203}$ . In addition to the nullified state bits in Table 1,  $b_{143}$  is also nullified in our attack in order to simplify  $b_{259}$ , where

$$\begin{aligned}
b_{259} = & b_{143}b_{226}s_{226} + b_{226}s_{173} + b_{199}b_{215} + s_{191}s_{210} + b_{192}b_{196} + b_{171}b_{179} + b_{158}b_{190} \\
& + b_{148}b_{149} + b_{142}b_{144} + b_{134}b_{198} + s_{144}s_{151} + b_{143}s_{139} + b_{227} + s_{224} + b_{222} + \\
& b_{220} + b_{204} + b_{195} + b_{187} + b_{176} + b_{167} + b_{157} + b_{146} + b_{133} + b_{131} + s_{131}.
\end{aligned}$$

Note that the nullification of  $b_{143}$  only leads to 1 more guessed key bit. The state bits to be nullified are shown in Table 2.

Next we will explain why we choose to nullify these state bits. First, in order to nullify  $b_{203}$ , some other bits should be nullified. We substitute the terms and preserve the terms with high degree, and then calculate the frequency of occurrence of each state bit involved. Nullifying the high frequency state bits and setting them to 0, then the corresponding terms will disappear. As a result, the nullification of these state bits decreases the degree and terms of high degree dramatically. Then we substitute again and repeat the procedure above. Finally we can determine the dynamic IV bits and the corresponding key bits that have to be guessed during the on-line attack. The dynamic IV bits are shown in Table 2 as well. The more details of the nullification are shown in Table 7 in Appendix A.

**Table 2.** Our Nullification Scheme

nullification	$b_{131}, b_{133}, b_{134}, s_{135}, b_{136}, b_{137}, b_{138}, b_{143}, b_{145}, b_{153}, b_{159}, b_{170}, b_{176}, b_{203}$
dynamic bits	$s_3, s_5, s_6, s_{77}, s_8, s_9, s_{10}, s_{15}, s_{17}, s_{25}, s_{31}, s_{42}, s_{83}, s_1$

### 3.2 IV Choosing Techniques

After the nullifications shown in Table 2, there are 82 IV bits left, which probably leads to a very high degree of the output bit polynomial. Thus in this subsection we will show how to choose some IV bits to reduce the degree and make some IV terms disappear as well.

For Grain-128, the output is generated by iteration of IV and key bits, some IV or key bits tend to (dis)appear simultaneously in high degree terms. For example, for state bits in the first 32 initialization rounds, the only degree 2 terms are  $s_{i+13}s_{i+20}$  and  $s_{i+60}s_{i+79}$ . Choosing to nullify  $s_{i+13}$  may result in disappearance of  $s_{i+20}$ .

Furthermore, the degrees of some state bits are decreased to 1 by nullifying some IV bits of the terms. However, some state bits may have higher degrees due to the higher-degree dynamic bits. As a result, the high frequency IV bits in the high frequency state bits and terms are chosen to be nullified, in order to make as many dynamic bits disappear as possible.

Since some IV bits will disappear in high degree terms after setting some other IV bits to zero, IV terms with these IV bits such as  $s_{80}$  will be sparse in high degree terms. We call these common IV bits that lead to sparse terms *low-frequency bits*. We find out at least 7 of them, which are listed in Table 5 along with the corresponding state bits in the first 32 rounds.

In addition, the IV nullification is an iterative process since nullifying any IV bit may result in changing the degree of many state bits.

In summary, carefully choosing the nullified IV bits will result in the disappearance of many IV terms. The fact of the disappearance of IV terms can be used to derive distinguishers.

### 3.3 Reduction Techniques for Polynomial Terms

We presented the methods to reduce the boolean function of the output bit in Section 3.1 and Section 3.2. However, the specific IV terms that appear are under consideration, so that distinguishers can be deduced. Thus we propose in this subsection the manners to obtain the IV terms of boolean functions of Grain-128.

**Degree estimation of state bits** After the nullification and IV choosing scheme introduced in Section 3.1 and Section 3.2, we first estimate the degree of some state bits. Degrees of some state bits are shown in Table 3, which can be

**Table 3.** Degree of partial state bits

$i$	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146
$\deg(b_i)$	1	2	1	0	1	0	0	1	0	0	0	1	2	2	2	0	1	0	1
$\deg(s_i)$	1	2	2	1	1	1	1	0	1	1	1	1	2	1	2	0	1	1	1
$i$	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165
$\deg(b_i)$	1	1	1	0	2	1	0	2	2	1	1	1	0	2	2	3	3	2	2
$\deg(s_i)$	1	1	1	1	2	1	1	2	2	1	1	1	0	2	2	3	3	2	2
$i$	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184
$\deg(b_i)$	2	2	2	1	0	1	2	3	3	3	0	2	2	2	2	2	2	2	3
$\deg(s_i)$	1	2	2	1	1	1	2	3	3	3	2	2	2	2	2	2	2	2	3
$i$	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203
$\deg(b_i)$	2	2	2	3	2	2	2	2	3	3	5	4	3	3	3	3	3	3	0
$\deg(s_i)$	2	2	2	3	2	2	2	2	3	3	5	4	3	3	3	3	3	3	2
$i$	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220	221	222
$\deg(b_i)$	3	3	5	4	4	3	4	4	4	3	3	4	3	5	4	4	4	5	5
$\deg(s_i)$	3	3	5	4	4	3	4	4	4	3	3	4	3	5	4	4	4	5	4

**Table 4.**  $\deg(b_i) + \deg(s_i) - \deg(b_i s_i)$

$i$	129	140	142	151	154	155	160	161	162	163	164	165	167
$\deg(b_i) + \deg(s_i) - \deg(b_i s_i)$	1	1	1	1	1	1	1	1	1	2	1	1	1
$i$	168	172	173	174	175	180	181	182	183	184	188	193	194
$\deg(b_i) + \deg(s_i) - \deg(b_i s_i)$	1	1	1	2	2	1	1	1	1	1	1	2	1
$i$	195	196	197	198	199	206							
$\deg(b_i) + \deg(s_i) - \deg(b_i s_i)$	2	3	1	1	1	1							

obtained directly on PC by obtaining their boolean function in IV bits and key bits. In fact, the exact degree of state bits before round 150 can be obtained, while the others can be estimated by substitution.

In addition, we can obtain  $\deg(b_i) + \deg(s_i) - \deg(b_i s_i)$ , which can help reduce terms in advance and is shown in Table 4.

**Remove the Repeated Terms** We know that it is much easier to illustrate the output with the state bits than using the key and IV bits directly. Moreover, the state bits can be expressed by earlier ones that are simpler functions of the IV and key bits. As a consequence, we can iteratively express the output bits. During this procedure, we will try to reduce the complexity of the polynomial by removing the so called **repeated terms** (RT).

Two terms are repeated terms if they are the same by ignoring  $s_i$  ( $i \in [96, 127]$ ) (which are all 1s). It is obvious that the boolean addition of two RT is zero.

Before introducing our RT algorithm, we will introduce a simple but useful property for Grain-128.

**Table 5.** Low frequency bits

low-frequency bits	corresponding state bits
$s_0$	$b_{128}, s_{128}, b_{160}, s_{160}$
$s_2$	$b_{130}, s_{130}$
$s_{37}$	$b_{157}, s_{157}, s_{158}$
$s_{43}$	$s_{133}$
$s_{60}$	$b_{146}, s_{146}, s_{150}$
$s_{80}$	$b_{129}, s_{129}, s_{138}, b_{148}, s_{148}$
$s_{90}$	$s_{137}, s_{148}, s_{158}$

*Property 1.* Let  $b_{i+128} = b'_{i+128} + z_i$  and  $s_{i+128} = s'_{i+128} + z_i$ , then  $b_{i+128}s_{i+128} = b'_{i+128}s'_{i+128} + z_i(b'_{i+128} + s'_{i+128}) + z_i$ , where  $z_i$  is the feedback bit,  $b'_{i+128}$  and  $s'_{i+128}$  are the state bits before the feedback of  $z_i$ .

Property 1 holds because there is a collision, i.e.,  $z_i \cdot z_i = z_i$ , which can result in a number of repeated terms when substituting the terms of  $z_i$ . Actually we find that the repeated term are mostly caused from collisions. Property 1 is trivial, however, it helps a lot when we remove the RT with computer program but not manually before our RT removing algorithm, which is shown in Algorithm 1. Suppose that there are  $n$  terms, the complexity of Algorithm 1 is  $O(n^2)$ .

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**Algorithm 1** Repeated-term Removing Algorithm

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**Input:** The vector  $\mathbf{T}$  with  $n$  terms, i.e.,  $T_1, T_2, \dots, T_n$ .

**Output:** Updated  $\mathbf{T}$  with  $m$  terms, where  $m \leq n$ .

```

1: for  $i \leftarrow 1 : n - 1$  do
2:   for  $j \leftarrow i + 1 : n$  do
3:     if  $(T_i = T_j)$  then
4:        $T_i \leftarrow 0$ 
5:        $T_j \leftarrow 0$ 
6:     end if
7:   end for
8: end for
9:  $m \leftarrow 1$ 
10: for  $i \leftarrow 1 : n$  do
11:   if  $T_i \neq 0$  then
12:      $T_m \leftarrow T_i$ 
13:      $m \leftarrow m + 1$ 
14:   end if
15: end for

```

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In our attack, we hope to use Algorithm 1 as much as possible. However, the number of terms can grow dramatically in the process of substitution even though we have removed some of them. Applying Algorithm 1 to one million

terms needs about half a day with a single core, which is practical. However, there are billions of terms; the complexity is beyond the power of a single PC.

In this paper, we present a method to partition the terms into non-overlapping sets so as to apply Algorithm 1 in parallel. In order to guarantee that all the repeated terms can be removed, we propose to partition the terms according to the common factor. The reason is the fact that two repeated terms must have common factor(s). Then the terms can be partitioned into different sets according to the common factor, which is shown in Algorithm 2.

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**Algorithm 2** Term Partition Algorithm

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**Input:** The vector  $\mathbf{T}$  of  $n$  terms, i.e.,  $T_1, T_2, \dots, T_n$ .

**Output:**  $m$  sets  $\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_m$  with  $n_1, n_2, \dots, n_m$  terms.

- 1: **for**  $k \leftarrow 1 : m$  **do**
  - 2:   Calculate the occurrence frequency of each state bit in  $\mathbf{T}$ .
  - 3:   Select a state bit  $x$  with frequency  $n_k$  that is closest to  $\frac{n}{m-(k-1)}$ .
  - 4:   Include the terms with common factor  $x$  in set  $\mathfrak{s}_k$ .
  - 5:    $n \leftarrow n - n_k$
  - 6: **end for**
- 

The complexity of Algorithm 2 is

$$m \times \frac{n}{m} + (m-1) \times \frac{n}{m} + \dots + \frac{n}{m} = \frac{m+1}{2}n.$$

If  $n$  is large, then  $m$  may be also large because the number of terms in a set cannot be too large due to the computing ability of a single core. Then we refine the partition scheme as follows. Suppose that  $n$  terms are going to be partitioned into  $m = m_1 \times m_2$  sets,  $n$  terms can be partitioned into  $m_1$  sets first, then each set can be partitioned into  $m_2$  sets. The complexity of the first partition is  $\frac{m_1+1}{2}n$  and the complexity of the second partition is  $\frac{(m_2+1)n}{2m_1}$  on average in a single core.

After partitioning  $n$  terms into  $m$  sets. Algorithm 1 is applied to each set, which will run in parallel with  $m$  cores. The total computing complexity on  $m$  cores is about  $\frac{1}{m}$  of the original total complexity.

**Identifying the IV Terms of Degree 43** Removing the RT is not enough since the number of terms still increases dramatically, which pushes us to reduce the terms furthermore.

**IV representation.** Replacing the terms with their corresponding IV terms is called *IV representation*. Due to the neglect of the key information, using IV representation will result in repeated IV terms.

**Repeated IV terms.** When two IV terms are the same, then one of them can be removed directly, which is shown in Algorithm 3. In Line 4 of Algorithm 3, the repeated IV term is set to  $-1$  but not  $0$  so that the constant information will not be lost.

---

**Algorithm 3** Repeated-IV term Removing Algorithm

---

**Input:** The vector  $\mathbf{T}$  with  $n$  IV terms, i.e.,  $T_1, T_2, \dots, T_n$ .

**Output:** Updated  $\mathbf{T}$  with  $m$  IV terms, where  $m \leq n$ .

```
1: for  $i \leftarrow 1 : n - 1$  do
2:   for  $j \leftarrow i + 1 : n$  do
3:     if  $(T_i = T_j)$  then
4:        $T_i \leftarrow -1$ 
5:     end if
6:   end for
7: end for
8:  $m \leftarrow 1$ 
9: for  $i \leftarrow 1 : n$  do
10:  if  $T_i \neq -1$  then
11:     $T_m \leftarrow T_i$ 
12:     $m \leftarrow m + 1$ 
13:  end if
14: end for
```

---

**Covered IV terms.** In addition to repeated IV terms, we give the definition of covered IV terms: If  $I \subseteq J$ , then IV term  $s_1 = \prod_{i \in I} v_i$  is said to be covered by IV term  $s_2 = \prod_{j \in J} v_j$ , where  $I$  and  $J$  are subsets of  $\{0, 1, \dots, 95\}$  respectively. So repeated IV terms are special cases of covered IV terms.

Covered IV terms will show up frequently using IV representation, so we develop an algorithm to remove the covered IV terms, which is shown in Algorithm 4.

Here we give an example to illustrate the use of IV representation, repeated IV term removing and covered IV term removing. Assume that  $x_1 = v_0(k_1 + k_0k_2) + v_0v_1k_2$ ,  $x_2 = v_1k_0 + v_1v_2k_1$ , then the IV representations of  $x_1$  and  $x_2$  are  $v_0 + v_0v_1$  and  $v_1 + v_1v_2$  respectively. After removing the repeated IV terms of  $x_1x_2 = v_0v_1 + v_0v_1v_2 + v_0v_1 + v_0v_1v_2$  using Algorithm 3, the resultant IV terms are  $v_0v_1$  and  $v_0v_1v_2$ , so that the IV terms can be determined. If only the highest degree is under consideration, then Algorithm 4 can be used. After removing the covered IV terms,  $\hat{x}_1 = v_0v_1$ ,  $\hat{x}_2 = v_1v_2$ . Then  $\hat{x}_1\hat{x}_2 = v_0v_1v_1v_2 = v_0v_1v_2$ . So the highest degree of  $x_1x_2$  is 3, within the IV term  $v_0v_1v_2$ . Using the IV representation and Algorithm 4, only 1 IV multiplication is needed for this concrete example, while 4 IV multiplication and 6 key multiplication are needed in the trivial way.

Since the coefficient function on key bits is usually quite complicated, the proposed method reduces the computing complexity dramatically, which enables us to obtain the IV terms of degree 43.

The computational complexity of Algorithm 3 and Algorithm 4 is  $O(n^2)$  if there are  $n$  IV terms. This is the worst-case complexity; while for Grain-128 the complexity will be much lower due to the fact that there are large number of repeated IV terms and that an higher degree term may cover a larger number of lower degree ones. Normally, processing 10 million IV terms by Algorithm 3

---

**Algorithm 4** Covered-(IV)term Removing Algorithm

---

**Input:** The vector  $\mathbf{T}$  of  $n$  terms, i.e.,  $T_1, T_2, \dots, T_n$ .

**Output:** Updated  $\mathbf{T}$  with  $m$  terms, where  $m \leq n$ .

```
1: for  $i \leftarrow 1 : n - 1$  do
2:   if  $T_i \neq 0$  then
3:     for  $j \leftarrow i + 1 : n$  do
4:       if  $T_j \neq 0$  then
5:         if ( $T_i$  is covered by  $T_j$ ) then
6:            $T_i \leftarrow 0$ 
7:            $i \leftarrow i + 1$ 
8:         else if ( $T_j$  is covered by  $T_i$ ) then
9:            $T_j \leftarrow 0$ 
10:           $j \leftarrow j + 1$ 
11:         end if
12:       end if
13:     end for
14:   end if
15: end for
16:  $m \leftarrow 1$ 
17: for  $i \leftarrow 1 : n$  do
18:   if  $T_i \neq 0$  then
19:      $T_m \leftarrow T_i$ 
20:      $m \leftarrow m + 1$ 
21:   end if
22: end for
```

---

and 30 million IV terms by Algorithm 4 only needs about several minutes on a single core, which is quite efficient.

It is obvious that Algorithm 3 will not lose any information of IV terms. Now we have to make sure that the above method does not lose any information about the degree, i.e., the degree of the original polynomial will be bounded by the deduced degree.

*Property 2.* The degree of the multiplication of two state bits is strictly bounded by the estimated one deduced by *IV representation* and Algorithm 4.

*Proof.* This proof is composed of two steps. First, we need to prove that IV representation of the state bits with Algorithm 4 will guarantee the keeping of the IV terms with the highest degree. Then, we need to prove that applying Algorithm 4 on the multiplication of two IV-represented state bits will guarantee the keeping of the IV terms with the highest degree. It is obvious right for the latter one, we just need to prove the former one.

Consider state terms  $A = a_1 + a_2 + \dots + a_m$  and  $B = b_1 + b_2 + \dots + b_n$ , where  $a_i$  and  $b_j$  are IV terms. Assuming that  $a_{i_1}$  is covered by  $a_{i_2}$ , then  $a_{i_1}B$  will be covered by  $a_{i_2}B$ . Hence,  $a_{i_1}$  can be removed from  $A$ , which is done by Algorithm 4. The removal of IV terms in  $B$  is similar.  $\square$

This property can be easily extended to the multiplication of multiple state bits, which holds by iteratively using Property 2.

Actually, Algorithm 4 can be used to remove the so called **covered terms** (CT), where CT is defined as follows:

Let state terms

$$t_1 = \prod_{i_1 \in I_1} b_{i_1} \prod_{i_2 \in I_2} b_{i_2} \prod_{i_3 \in I_3} s_{i_3}$$

and

$$t_2 = \prod_{j_1 \in J_1} b_{j_1} \prod_{j_2 \in J_2} b_{j_2} \prod_{j_3 \in J_3} s_{j_3},$$

where  $I_1, J_1 \subseteq [0, 127]$ ,  $I_2, J_2 \subseteq [128, 352]$ ,  $I_3, J_3 \subseteq [0, 95] \cup [128, 352]$ . Term  $t_1$  is covered by  $t_2$  if  $I_2 \subseteq J_2$  and  $I_3 \subseteq J_3$ .

If the degree of a state term can be bounded by a bound using *IV representation*, then IV terms produced by the covered state terms will also be covered and hence will be removed using Algorithm 4. So covered state terms can be removed first. As a result, the state terms are partitioned into two sets, of which degrees of state terms in the first one are bounded by our bound while degrees of state terms in the other may be higher. Of course, most state terms will be in the first set. Algorithm 4 can be applied to the first set and most terms would be dropped off.

When there are billions of terms, removing the covered terms by executing Algorithm 4 on all terms with a single core is difficult as well, so we remove the covered terms in parallel. Similarly, here we propose a partition scheme that

guarantees all covered terms can be removed. A term covers another if only the degree of the first is no less than that of the second. So the terms can be partitioned into different sets according to their degrees. First we apply Algorithm 4 to the highest degree terms, and remove the second highest degree terms that are covered by the highest ones. Then we operate Algorithm 4 on second highest degree terms, and so on, until all covered terms are removed. Algorithm 4 is much faster than Algorithm 1 although they have the same computational complexity in worst case. Normally, operating Algorithm 4 on 10 million terms needs just several minutes in a single core when analyzing Grain-128. This is because a term may cover a large number of terms. Therefore, these terms will be set to 0 after a scan of all the terms. In fact, two terms are repeated terms only if they have the same degree. So the partition scheme for covered terms can also be used to partition terms into different sets with no repeated terms. However, removing repeated terms in this way will lead to a higher complexity since it cannot partition the terms uniformly. Thus we will not use it for repeated terms removing.

For state terms in the second set, we need to substitute the terms and remove the repeated terms using Algorithm 1 until the IV terms can be obtained using *IV representation*. Combined with Algorithm 3, all IV terms of degree 43 can be obtained.

### 3.4 Preprocessing Phase of the Proposed Attack

Now we are ready to describe the preprocessing phase of our attack, using the techniques proposed in the previous three subsections.

1. We deduce the key bits to be guessed (the number is 40), as well as the corresponding dynamic IV bits, to nullify the state bits shown in Table 2. The nullified IV bits and low-frequency bits shown in Table 6 are also chosen.
2. Iteratively express the output bit and discard the terms whose degrees are likely to be below certain threshold. Algorithm 1 is then used to remove the repeated terms. Note that there is no information lost in Algorithm 1.
3. Use Algorithm 4 to reduce the number of state terms, and to deduce the IV terms that may exist.
4. Obtain the IV terms that may exist using IV representation, combined with Algorithm 3 and Algorithm 4.

In Step 2, we will remove as many repeated terms as possible within our computing and storage ability. After Step 2, a large number of state terms are discarded. However, most of the terms are of degrees lower than our bound, which is actually proved by using Algorithm 4.

The preprocessing process is quite complicated, so we use a computer cluster with 740 nodes (8880 cores in total) to do most of the time consuming work. A dynamic number of cores are used (which are between 600 and 4000), depending on the specific program to be paralleled. Finally, we can determine all the IV terms that may appear, which are shown in Appendix B. The IV terms

**Table 6.** Nullified and low-frequency IV bits.

nullified bits	$s_{14}, s_{16}, s_{20}, s_{22}, s_{23}, s_{24}, s_{28}, s_{30}, s_{32}, s_{33}, s_{35}, s_{36}, s_{38}, s_{41}, s_{44}, s_{50}, s_{51}, s_{53}, s_{55}, s_{56}, s_{61}, s_{64}, s_{67}, s_{68}, s_{69}, s_{70}, s_{71}, s_{75}, s_{76}, s_{79}, s_{81}, s_{82}, s_{84}, s_{85}, s_{86}, s_{94}$
low-frequency bits	$s_0, s_2, s_{37}, s_{43}, s_{60}, s_{80}, s_{90}$

in Appendix B occur with probability  $1/2$  while the others do not occur with probability 1 in  $z_{257}$ .

## 4 On-line Phase of the Attack

In this section, we will introduce the on-line phase of our attack on Grain-128 and analyze its complexity. Actually, the on-line phase of our attack is much simpler than the preprocessing phase.

Since we determine that totally 2581 IV terms of degree 43 are possible to be existent, which means there are  $C_{46-7}^{43-7} - 2581 = 6558$  nonexistent IV terms. The density of IV terms is about 28%. We choose cubes from these nonexistent IV terms. It is known that summing over each of the cubes, the output will be always zero, which results in distinguishers for key recovery. Since for the correct key guess the summation is always 0, while for the wrong ones the summation could be 0 or 1 with random probability. On average,  $40 \log 40 \approx 213$  cubes are needed to get the correct key.

Thus, in this phase, we first guess the 40 key bits and sum over the output bits with the first chosen cube. If the result is 1, then we conclude that the guess is wrong. About half of the keys will be discarded in this way. For the remaining keys, we repeat the procedure with the second cube. And so on. Then after all the 213 cubes are used, it is supposed that only the correct key will be kept.

The time complexity is about  $2^{43} \left( 2^{40} + 1 + \frac{2^{40}-1}{2} + 1 + \frac{2^{40}-1}{2^2} + \dots + 1 + \frac{2^{40}-1}{2^{213}} \right) \approx 2^{84}$  bit operations, equivalent to  $2^{74}$  cipher executions. The data complexity is  $213 \cdot 2^{14} \cdot 2^{43} \approx 2^{65}$ .

After recovering the 40 key bits, there are various of methods to recover the remaining key bits. For example,  $b_{236}$  can be easily nullified with 23 key guesses. Then cubes of dimension 42 can be chosen as distinguishers. So the complexity to recover these bits is about  $23 \cdot 2^{23} \cdot 2^{42} \approx 2^{72}$  bit operations. Then the other key bits can be recovered by guessing with a complexity of  $2^{65}$ . As a result, the complexity of our attack is dominated by the recovery of the first 40 bits.

## 5 Conclusion

In this paper, we improved the attack on full-round Grain-128. Our attack is based on the knowledge that a lot of IV terms will disappear, after nullifying some state bits and IV bits. In addition, we find out the low-frequency IV bits and exploit them in the high degree terms. We also propose a method to cancel

the terms with lower degree, and exploit the IV representation to obtain the IV terms with much lower complexity. Then the nonexistent IV terms are used as distinguishers so that we improved the attack in [2] by a factor of  $2^{16}$ . Our attack is not based on any key information, so we can attack Grain-128 with any arbitrary selected keys. Although the nonexistent IV terms can be tested by cube tester technique on super computers, our method can also work for higher dimensions, in which case the computing complexities for cube tester are beyond our ability.

In this paper, we have various of strategies in choosing IV bits such as choosing the low-frequency IV bits, so that the IV terms of degree 43 are very sparse. We believe that attacker can enhance the sparsity with lower degree, which is much more complicated. So finding the lowest degree for sparse IV terms is an open problem for further research.

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## A The details of nullifications

The nullification detail is shown in Table 7. The first column is the state bits to be nullified and the second column is the corresponding equation. The third column is the subkey bits guessed for nullifications. For example, in order to

nullify  $b_{131}$ , we just need to set  $s_3$  to  $b_{15}b_{98} + \underline{b_{98}}s_{45} + b_{71}b_{87} + s_{63}s_{82} + b_{64}b_{68} + b_{43}b_{51} + b_{30}b_{62} + b_{20}b_{21} + b_{14}b_{16} + b_6b_{70} + s_{16}s_{23} + \underline{b_{15}}s_{11} + b_{99} + b_{94} + b_{92} + b_{76} + b_{67} + b_{59} + b_{48} + b_{39} + b_{29} + b_{18} + b_5 + b_3 + 1$ , where  $b_{98}$  and  $b_{15}$  underlined are the key bits guessed. Besides these two bits, one expression on key bits indicated by \*, that is  $b_{15}b_{98} + b_{71}b_{87} + b_{64}b_{68} + b_{43}b_{51} + b_{30}b_{62} + b_{20}b_{21} + b_{14}b_{16} + b_6b_{70} + b_{99} + b_{94} + b_{92} + b_{76} + b_{67} + b_{59} + b_{48} + b_{39} + b_{29} + b_{18} + b_5 + b_3$  should be guessed. Hence, three bits need to be guessed to nullify  $b_{131}$ . Totally, 40 bits should be guessed for nullifying the state bits in Table 2.

## B All IV terms of degree 43 in $z_{257}$

The resulted IV terms that may appear are shown in Table 8, 9, 10, 11, 12 and 13. Each hexadecimal number in this table indicates a multiplication of 43 IV bits. Let  $H = H_0H_1H_2H_3H_4H_5H_6H_7H_8H_9$ , where  $H_i$  is a hexadecimal number with the range of  $[0, 15]$ . As there are 39 bits, so  $H_9$  is within the range of  $[0, 7]$ . Define  $h_{ij}$  as the  $j$ -th lowest bit of  $H_i$ . Let  $S$  be the vector whose elements are 4, 7, 11, 12, 13, 18, 19, 21, 26, 27, 29, 34, 39, 40, 45, 46, 47, 48, 49, 52, 54, 57, 58, 59, 62, 63, 65, 66, 72, 73, 74, 78, 87, 88, 89, 91, 92, 93 and 95 sequently, then the cube defined by  $H$  is  $v_0v_2v_{37}v_{43}v_{60}v_{80}v_{90} \prod_{i \in [0,9]} v_{S_{i*4+j}}^{h_{ij}}$ . We use this expression due to the convenience for programming.

Table 7. Nullification equations

Nullified bits	Equations for nullification	Subkey bits guessed
$b_{203}$	$s_1 = b_{115}b_{123} + b_{92}b_{93} + b_{86}b_{88} + s_{88}s_{95} + b_{87}s_{83} + b_{111}b_{127} + b_{104}b_{108} + b_{83}b_{91} + b_{48}$ $+ b_{60}b_{61} + b_{54}b_{56} + b_{46}b_{110} + s_{56}s_{63} + b_{107} + b_{99} + b_{88} + b_{79} + b_{58} + b_{45} + b_{43} + s_{14}s_{21}$ $+ s_{43} + b_{52}s_{48} + b_{113} + b_{104} + b_{85} + b_{55} + b_{42} + s_{78} + s_{47} + s_{40} + b_{106}b_{122} + b_{99}b_{103} + b_5$ $+ b_{78}b_{86} + b_{65}b_{97} + b_{55}b_{56} + b_{49}b_{51} + b_{41}b_{105} + b_{50}s_{46} + b_{102} + b_{83} + b_{64} + b_{53} + b_{96}s_{43}$ $+ b_{40} + b_{104}b_{120} + b_{97}b_{101} + b_{76}b_{84} + b_{63}b_{95} + b_{53}b_{54} + b_{47}b_{49} + b_{39}b_{103} + b_{120} + s_3b_{13}b_{96}$ $+ b_{101} + b_{90} + b_{125} + b_{109} + b_{100} + b_{92} + b_{81} + b_{72} + b_{62} + b_{51} + b_{36} + b_{77} + b_{75} + s_{82}$ $+ s_{75} + b_{32}b_{115} + b_{115}s_{62} + b_{88}b_{104} + s_{80} + b_{81}b_{85} + b_{60}b_{68} + b_{47}b_{79} + b_{37}b_{38} + s_{39} + s_8$ $+ b_{31}b_{33} + b_{23}b_{87} + b_{111} + b_{109} + b_{93} + b_{84} + b_{76} + b_{65} + b_{56} + b_{46} + b_{70}b_{102} + b_{16} + b_3$ $+ b_{35} + b_{22} + b_{20} + b_{100}s_{47} + s_{18}s_{25} + b_{17}s_{13} + b_{78} + b_{50} + b_{41} + b_{13}s_9 + s_{94} + b_{90} + b_{65}$ $+ b_{20} + b_7 + s_{101} + s_{86} + s_{43} + s_{12} + s_5b_{15}b_{98} + b_{98}s_{45} + b_{15}s_{11} + b_{92} + b_{67} + b_{46} + b_{37}$ $+ b_{18} + b_{39} + s_{73} + s_{10} + 1$	$b_{87}, b_{52}, b_{115}, b_{96},$ $b_{13}, *$
$b_{176}$	$s_{83} = s_{48} + b_{48} + b_{74} + b_{104} + s_{11} + b_{11} + b_{37} + b_{67} + b_{14}b_{78} + b_{22}b_{24} + b_{28}b_{29} + b_{38}b_{70}$ $+ b_{51}b_{59} + b_{72}b_{76} + b_{79}b_{95} + b_{13} + b_{26} + b_{47} + b_{56} + b_{75} + b_{100} + s_{19}b_{23} + b_{23}b_{106} + b_{16}$ $+ b_{42} + b_{72} + b_{19}b_{83} + b_{27}b_{29} + b_{33}b_{34} + b_{43}b_{75} + b_{56}b_{64} + b_{77}b_{81} + b_{84}b_{100} + b_{18} + b_{31}$ $+ b_{52} + b_{61} + b_{80} + b_{89} + b_{105} + b_{28}b_{111} + s_{58}b_{111} + b_{51}b_{115} + b_{59}b_{61} + b_{65}b_{66} + b_{75}b_{107}$ $+ b_{88}b_{96} + b_{109}b_{113} + s_4b_{116} + b_4b_{116} + b_{30}b_{116} + b_{60}b_{116} + b_{95}b_{116} + b_{100}b_{116} + b_7b_{71}b_{116}$ $+ b_{15}b_{17}b_{116} + b_{21}b_{22}b_{116} + b_{31}b_{63}b_{116} + b_{44}b_{52}b_{116} + b_{65}b_{69}b_{116} + b_{72}b_{88}b_{116} + b_6b_{116}$ $+ b_{19}b_{116} + b_{40}b_{116} + b_{49}b_{116} + b_{68}b_{116} + b_{77}b_{116} + b_{93} + b_{93}b_{116} + s_{12}b_{16}b_{116} + b_{16}b_{99}b_{116}$ $+ b_{116} + s_{46}b_{99}b_{116} + b_{50} + b_{63} + b_{121} + s_{13} + b_{15} + b_{28} + b_{49} + b_{58} + b_{77} + b_{86} + s_{21}b_{25}$ $+ b_{25}b_{108} + s_{73}s_{92}$	$b_{23}, b_{111}, b_{16}b_{116},$ $b_{25}, b_{99}b_{116}, *$
$b_{170}$	$s_{42} = b_{110}b_{126} + b_{103}b_{107} + b_{82}b_{90} + b_{69}b_{101} + b_{59}b_{60} + b_{53}b_{55} + b_{45}b_{109} + s_{55}s_{62} + b_{54}s_{50}$ $+ b_{115} + b_{106} + b_{98} + b_{87} + b_{78} + b_{68} + b_{57} + b_{44} + b_{42} + 1$	*
$b_{159}$	$s_{31} = b_{43}b_{126} + b_{126}s_{73} + b_{99}b_{115} + s_{91} + b_{92}b_{96} + b_{71}b_{79} + b_{58}b_{90} + b_{48}b_{49} + b_{42}b_{44}$ $+ b_{34}b_{98} + s_{44}s_{51} + b_{43}s_{39} + b_{127} + b_{122} + b_{120} + b_{104} + b_{95} + b_{87} + b_{76} + b_{67} + b_{57} + b_{46}$ $+ b_{33} + b_{31} + 1$	$b_{43}, b_{126}, *$
$b_{153}$	$s_{25} = b_{37}b_{120} + b_{120}s_{67} + b_{93}b_{109} + s_{85} + b_{86}b_{90} + b_{65}b_{73} + b_{52}b_{84} + b_{42}b_{43} + b_{36}b_{38} + b_{121}$ $+ b_{28}b_{92} + s_{38}s_{45} + b_{37}s_{33} + b_{116} + b_{114} + b_{98} + b_{89} + b_{81} + b_{70} + b_{61} + b_{51} + b_{40} + b_{27}$ $+ b_{25} + 1$	*
$b_{145}$	$s_{17} = b_{29}b_{112} + b_{112}s_{59} + b_{85}b_{101} + s_{77} + b_{78}b_{82} + b_{57}b_{65} + b_{44}b_{76} + b_{34}b_{35}$ $+ b_{28}b_{30} + b_{20}b_{84} + s_{30}s_{37} + b_{29}s_{25} + b_{113} + b_{108} + b_{106} + b_{90}$ $+ b_{81} + b_{73} + b_{62} + b_{53} + b_{43} + b_{32} + b_{19} + b_{17} + 1$	$b_{29}, b_{112}, *$
$b_{143}$	$s_{15} = b_{27}b_{110} + b_{110}s_{57} + b_{83}b_{99} + s_{75}s_{94} + b_{76}b_{80} + b_{55}b_{63} + b_{42}b_{74} + b_{32}b_{33}$ $+ b_{26}b_{28} + b_{18}b_{82} + s_{28}s_{35} + b_{27}s_{23} + b_{111} + b_{106} + b_{104} + b_{88} + b_{79}$ $+ b_{71} + b_{60} + b_{51} + b_{41} + b_{30} + b_{17} + b_{15} + 1$	$b_{27}, b_{110}, *$
$b_{138}$	$s_{10} = b_{22}b_{105} + b_{105}s_{52} + b_{78}b_{94} + s_{70}s_{89} + b_{71}b_{75} + b_{50}b_{58} + b_{37}b_{69} + b_{27}b_{28}$ $+ b_{21}b_{23} + b_{13}b_{77} + s_{23}s_{30} + b_{22}s_{18} + b_{106} + b_{101} + b_{99} + b_{83} + b_{74}$ $+ b_{66} + b_{55} + b_{46} + b_{36} + b_{25} + b_{12} + b_{10} + 1$	$b_{22}, b_{105}, *$
$b_{137}$	$s_9 = b_{21}b_{104} + b_{104}s_{51} + b_{77}b_{93} + s_{69}s_{88} + b_{70}b_{74} + b_{49}b_{57} + b_{36}b_{68} + b_{26}b_{27}$ $+ b_{20}b_{22} + b_{12}b_{76} + s_{22}s_{29} + b_{21}s_{17} + b_{105} + b_{100} + b_{98} + b_{82} + b_{73}$ $+ b_{65} + b_{54} + b_{45} + b_{35} + b_{24} + b_{11} + b_9 + 1$	$b_{21}, *$
$b_{136}$	$s_8 = b_{20}b_{103} + b_{103}s_{50} + b_{76}b_{92} + s_{68}s_{87} + b_{69}b_{73} + b_{48}b_{56} + b_{35}b_{67} + b_{25}b_{26}$ $+ b_{19}b_{21} + b_{11}b_{75} + s_{21}s_{28} + b_{20}s_{16} + b_{104} + b_{99} + b_{97} + b_{81} + b_{72} + b_{64}$ $+ b_{53} + b_{44} + b_{34} + b_{23} + b_{10} + b_8 + 1$	*
$s_{135}$	$s_{77} = b_{19}b_{102} + b_{102}s_{49} + s_{67}s_{86} + s_{20}s_{27} + b_{19}s_{15} + b_{96} + s_{88}$ $+ b_{80} + s_7 + b_{71} + b_{52} + s_{45} + b_{43} + b_{22} + s_{14} + b_9$	$b_{19}, b_{102}, *$
$b_{134}$	$s_6 = b_{18}b_{101} + b_{101}s_{48} + b_{74}b_{90} + s_{66}s_{85} + b_{67}b_{71} + b_{46}b_{54} + b_{33}b_{65} + b_{23}b_{24}$ $+ b_{17}b_{19} + b_9b_{73} + s_{19}s_{26} + b_{18}s_{14} + b_{102} + b_{97} + b_{95} + b_{79} + b_{70} + b_{62}$ $+ b_{51} + b_{42} + b_{32} + b_{21} + b_8 + b_6 + 1$	$b_{101}, *$
$b_{133}$	$s_5 = b_{17}b_{100} + b_{100}s_{47} + b_{73}b_{89} + s_{65}s_{84} + b_{66}b_{70} + b_{45}b_{53} + b_{32}b_{64} + b_{22}b_{23}$ $+ b_{16}b_{18} + b_8b_{72} + s_{18}s_{25} + b_{17}s_{13} + b_{101} + b_{96} + b_{94} + b_{78} + b_{69} + b_{61}$ $+ b_{50} + b_{41} + b_{31} + b_{20} + b_7 + b_5 + 1$	$b_{17}, b_{100}, *$
$b_{131}$	$s_3 = b_{15}b_{98} + b_{98}s_{45} + b_{71}b_{87} + s_{63}s_{82} + b_{64}b_{68} + b_{43}b_{51} + b_{30}b_{62} + b_{20}b_{21}$ $+ b_{14}b_{16} + b_6b_{70} + s_{16}s_{23} + b_{15}s_{11} + b_{99} + b_{94} + b_{92} + b_{76} + b_{67}$ $+ b_{59} + b_{48} + b_{39} + b_{29} + b_{18} + b_5 + b_3 + 1$	$b_{15}, b_{98}, *$

<sup>1</sup> Each \* in this table indicates a different expression of partial key bits.

Table 8. IV terms of degree 43 in  $z_{257}$ -part 1

FFFFFDFB6	FBFFFFFB6	FFFFFDBB7	FFFFFDFA7	FFDFFDFB7	FBFFFFB7
FBFFFFFA7	FF9FFFFB7	BFFFFDFB7	DFDFFDFB7	BFBFFFFB7	FDBFFFFB7
FFFFFD7B7	FBFFFF7B7	FFFFBDFB7	FFFFDDFB7	FBFBFFFFB7	FBFFDFFB7
FBFFDFB7	FFFFFDFB3	F7FFFFDFB7	FEFFFFDFB7	FBFFFFFB3	F7BFFFFB7
FBFFFFFB7	EFFFFDFB7	EBFFFFFB7	FFFFDF97	FFDFDFB7	FEFFFFDFB7
FFFFFDEB7	7FFFFDFB7	FBFFFFF97	FBFDFFFB7	FAFFFFFB7	FBFFFFFB7
7BFFFFFB7	BFFFFF7	9FFFFF7	9EFFFFF7	DFFFFFF7	BFDFFF7
BFEDFFFF7	9FDFFF7	DFDFFF7	DFEDFFFF7	FEFFFFF6	FEFFFFCF7
FFFFF6F6	FFFFBF6	FFFFDF6	FFFFF6F2	F7FFFFF6	FEFFFFF6
FFFFFEEF6	FFFFFEE6	FFDFFF6	FBFFFFF6	FFFFFED6	DFFFFFF6
FFFFFDEF6	FFFFDF6	FFFF7F6	FFFFF6F3	FFFFF6F7	FFFFFCE7
FFDFFFCF7	F7FFFFCF7	FEFFFFCF7	FFFFF4F7	FBFBFFCF7	FFDFFFCF7
FFFFDFCF7	FBFFFFCF7	FFDFFF7	FFFFF7	FFDFDF7	FFFFDF3
7EFFFFDF7	FEFFFF9F7	EFDFDF7	F7EFFFFDF7	FEFFFF5F7	FEFBFFDF7
FEFFFFDF7	FEDFDF7	FAFFFFDF7	FEFFFFDE7	FCFFFFDF7	FEFFDF7
7FFFFDF3	FFFFF9F3	EFFFFDF3	FFFFF5F3	FBFBFFDF3	FFDFDF3
FFFFDF3	FEFFFFDF3	FFFFFDE3	FFDFDF3	FBFFFFDF3	FFDFDF3
7FFFFDE7	7FDFDF7	7FFFFDF7	7EFFFFDF7	FFFFF9F7	FFDFF9F7
F7FFFF9F7	FEFFFF9F7	EFFFFDE7	EFFFFDF7	E7FFFFDF7	EEFFFFDF7
FFFFF5F7	FFDFDF7	FFDFDF7	7FFFF5F7	FFFFF1F7	EFFFF5F7
7FFBFFDF7	FFFFBF9F7	EBFFBDF7	7FFDFDF7	7FFDFDF7	7FBFFDF7
FFFFDF9F7	FFFFDF9F7	FBFFFF9F7	EFFDFDF7	EFFDFDF7	EBFFDF7
F7FFFF5F7	FEFFFF5F7	F7FBFFDF7	FEFFBDF7	FEFFDFE7	FDFFDFE7
FDDDFDF7	FBDBFFDF7	F6FFFFDF7	F7DFDF7	FEFFDF7	F7BFFDF7
FBFFDF7	F7FFFFDE7	FEFFFFDE7	F7DFFDF7	FEFFDF7	F7FFDF7
FEFFDFDF7	F7FFDFDF7	FFFFDF5F7	FFFFF5E7	FFDFDF5F7	FBFFDF5F7
FFFF9FDF7	FFFFBFDE7	FFDFBDF7	FBFBFFDF7	FEFFDFDF7	FFDDFFDF7
FBFDFFDF7	FFFFDFDE7	FFDFDFDF7	FFFFDDDF7	FFFFDFDE7	FFDFDFDF7
FBFFFFDE7	F9FFFFDF7	FBFFDFDF7	FEFFFFEF7	FEFFDFDF7	FFDFFEF7
DFEFFFEF7	DEFFFFEF7	DFDFFFEF7	FBFFDFEF7	FBFFFFEF7	FBFFFFEF7
FBDFDFEF7	FBFFFFFA7	EBFFFFEF7	FBFFFF6F7	FBFFFFF3	FBFFFFEF7
DBFFFFEF7	F9FFFFEF7	FBFFDFEF7	FBFFFFED7	FBFF7F7	BFFDFEF7
BFFFFFEF7	BFFFFEF7	BFFDFEF7	BFFFFFA7	AFFFFEF7	BFFFF6F7
BFFFFF3	BFFFFEF7	BFDFDFEF7	BFFDFEF7	BFBFFFFEF7	B7FFFFEF7
BEFFFFEF7	BFFBFFEF7	9FFFFEF7	BDFFFFEF7	BFFDFEF7	BFFDFEF7
BFFF7FEF7	FEFFDFEF7	FFFFDFEF7	FFDDFFEF7	DEFFDFEF7	FFFFFE76
FFFF7F76	7EFFFFEF7	7FFFFEF7	7FDFFEF7	FEFFFAF7	FFFFFAF7
FFDFFAF7	7FFFFFAF7	EFFFFFEF7	EFFFFEF7	EFFDFEF7	6FFFFEF7
EFFFFFAF7	FEFFFF6F7	FFFFF6F7	FFDFF6F7	FBFBFFEF7	FFFFBEEF7
FFDFBFFEF7	FFDFBFF3	FFFFDFEF3	FEFFFFF3	FFFFFEEF3	FFFFFEE3
FFDFFFEF3	FEFFFFEF7	FFDFDFEF7	FCFFFFEF7	FEFFDFEF7	FFFFFEE7
FFDFFEF7	FFFFDEEF7	FFDFFEF7	FFDFFEF7	FFFFDFEF7	FFDFFEF7
FFDFDFEF7	FFFFF6F3	FFFFF6E7	FFDFF6F7	FFFFDF6F7	7FFFF6F7
FFFFF2F7	EFFFFF6F7	7FFFFF3	7FFFFEF7	7FDFFEF7	7FFFFDF7
FFFFF2F3	FFFFFAF7	FFDFFAF7	FFFFFAF7	EFFFFF3	EFFFFEF7
FDFFFFEF7	EFFFFDFEF7	FFFFF6	FAFFFFEF7	FBFFFFEF7	FBDFFEF7
F6FFFFEF7	F7EFFFFEF7	FEFFFFEF7	F7FFFFEF7	FEFFFFEF7	F7DFFEF7
FEFFDFEF7	F7BFFFFEF7	EBFFFFEF7	7BFFFFEF7	77FFFFEF7	7EFFFFEF7
FBFFFFAF7	F7FFFFAF7	FEFFFFAF7	EBFFFFEF7	E7FFFFEF7	EEFFFFEF7
FBFBFF6F7	F7FFFF6F7	FEFFFF6F7	FBFFFFF3	FBFFFFEF7	FF9FFFFEF7
FBFFDFEF7	F7FFFFF3	EFFFFF3	F7FFFFEF7	FEFFFFEF7	F7DFFEF7
FDFFFEF7	F7FFDFEF7	FEFFDFEF7	7FFBFFEF7	FFFFBFAF7	EFFBFFEF7
5FFFFEF7	DEFFFAF7	CFFFFEF7	FFFFB6F7	DEFF6F7	FFFFF3
FFFFBFEF7	FFDFBFEF7	FFFFBDFEF7	DFFBFFEF7	DEFFFEF3	DEFFFEF7
DEFFFEF7	DEFFDFEF7	FBFBFFEF7	F7FBFFEF7	FEFFFEF7	DFBFFFEF7
D7FFFFEF7	DEFFFEF7	FEFFFEF7	FDFFFEF7	FDFFFEF7	FEFFDFEF7
FFFFFCE7	FFDFDFEF7	FEFFDFEF7	FFFFDFEF7	FFDFDFEF7	FEFF7FEF7
FFFF7EF7	FFDF7FEF7	DDFFFFEF7	DEFFDFEF7	DEFFDFEF7	DEFF7FEF7
7DFFFFEF7	FDFFFAF7	EDFFFEF7	FDFF6F7	FDFFBFEF7	FDFFFEF3
FDFFFEF7	FDFFFEF7	FDFFDFEF7	FDFFFCF7	FFFFDFCF7	FFFFFC7
FFFF7FCF7	FDBFFFEF7	F5FFFFEF7	FCFFFFEF7	FEFFFEF6	FEFFDFEF6
FFDFFEF6	DEFFFFF6	DEFFFEF6	DEFFFEF6	7FFDFEF7	7FFFFDF7
7FFFFED7	FFFFDFAF7	FFFFDAF7	FFFFFAD7	EFFDFEF7	EFFDFEF7
EFFFFED7	7FFFF7FEF7	FFFF7FAF7	EFF7FEF7	FFDF6F7	FFFF6F7
FFFFF6D7	FFFF9FEF7	FFFFBDEF7	FFFFBFEF7	FFDFFEF3	FFDFFEF7
FFDFFEF7	FFFDDFEF7	FFFFDFEF3	FFFFFED3	FFFFDEE7	FFDFDFEF7
FFFFDDEF7	FFFFFEC7	FFDFFEED7	FFFFDFED7	FFFF7F6F7	FFFFB7FEF7
FFFF7FEF3	FFFF7FE7	FFDF7FEF7	FFFF5FEF7	FBFBFFDEF7	FBFBFFEF7
FBFBFFED7	F7FFDFEF7	FEFFDFEF7	F7FFDFEF7	FEFFDFEF7	F7FFDFEF7
FEFFFEED7	FBFB7FEF7	F7FF7FEF7	FEFF7FEF7	FDFFDFEF7	DEFFDFEF7
FDFFFEED7	FFFFDFEF7	FFFFDFED7	FFFFDFED7	FFFF7FEF7	FDFF7FEF7









