Bandersnatch: a fast elliptic curve built over the BLS12-381 scalar field

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Abstract

In this short note, we introduce Bandersnatch, a new elliptic curve built over the BLS12-381 scalar field. The curve is equipped with an efficient endomorphism, allowing a fast scalar multiplication algorithm. Our benchmark shows that the multiplication is 42% faster, compared to another curve, called Jubjub, having similar properties. Nonetheless, Bandersnatch does not provide any performance improvement for either rank 1 constraint systems (R1CS) or multi scalar multiplications, compared to the Jubjub curve.

1 Introduction

BLS12-381 is a pairing-friendly curve created by Sean Bowe here in 2017. Currently, BLS12-381 is undergoing a standardization process from the IRTF Crypto Forum Research Group, and is universally used for digital signatures and zero-knowledge proofs by many projects orbiting in the blockchain universe: Zcash, Ethereum 2.0, Anoma, Skale, Algorand, Dfinity, Chia, and more. The ZCash team introduced Jubjub here, an elliptic curve built over the BLS12-381 scalar field $\mathbb{F}_{r_{\mathrm{BLS}}}$. This curve is not pairing-friendly, but leads to constructions where $\mathbb{F}_{r_{\mathrm{BLS}}}$ arithmetic circuits can be manipulated using the BLS12-381 curve. The Jubjub curve can be represented in the twisted Edwards coordinates, allowing efficiency inside zk-SNARK circuits. In order for some cryptographic applications to scale, it is necessary to have efficient scalar multiplication on the non-pairing-friendly curve. The main drawback of Jubjub is the slow scalar multiplication algorithm compared, for example, with the “Bitcoin curve” (SECP256k1). It comes from the fact that the curve does not have an efficiently computable endomorphism, necessary for computing scalar multiplications using the GLV method [1] (a technique protected by a US patent until Sep 2020 [2], but that expired and is freely usable now).

Our contribution. The Jubjub curve is a curve with a large discriminant, meaning that the GLV method is not possible on this curve. We performed an exhaustive search of curves of small discriminant, defined over the BLS12-381 scalar field. This way, we obtain an elliptic curve using the Complex Multiplication method [3], where the scalar multiplication algorithm is efficient thanks to the GLV method [1].

We implement this curve in Rust, using the Arkworks framework, and release our code to the open domain [4]. Table 1 shows a comparison of Bandersnatch curve and Jubjub curve. Details deferred to Section 4.

	multiplication cost
Jubjub	75 μs
Bandersnatch	44 μs
Improvement	42%

Table 1: Bandersnatch vs Jubjub

We also report the number of constraints one needs to express a group multiplication in the form of rank one constraint system (R1CS), a common approach for expressing circuits for zero-knowledge proof systems. A group multiplication takes 3325 constraints when the point is in affine form over the twisted Edwards curve. This matches what we have for Jubjub curve.

Organization of the paper. In Section 2, we describe how we obtained several curves allowing the GLV method together with cryptographic security. Then, we introduce in Section 3 the Bandersnatch curve in different models (in Weierstrass, Montgomery and twisted Edwards coordinates). Finally, we compare the scalar multiplication algorithm over the Bandersnatch and the Jubjub curves in Section 4 from a practical point of view.

2 Small discriminant curves

The GLV method [1] is a well known trick for accelerating scalar multiplication over particular curve. In a nutshell, it applies to elliptic curves where an endomorphism ψ can be efficiently computed. The GLV method applies in particular for j-invariant $j = 0$ (resp. $j = 1728$) curves because a non-trivial automorphism can be computed using only one modular multiplication. The method also applies for other curves where the endomorphism is slightly more expensive, called *small discriminant* curves.

Let E be an elliptic curve defined over $\mathbb{F}_p$ of trace t. E and its quadratic twist E^t are $\mathbb{F}_{p^2}$-isomorphic curves and their orders over $\mathbb{F}_p$ are closely related with the trace t:

$$\#E(\mathbb{F}_p) = p + 1 - t \qquad \#E^t(\mathbb{F}_p) = p + 1 + t.$$

See [5] for a complete introduction to elliptic curves. In this work, we are looking for cryptographic applications based on ordinary elliptic curves, meaning that we look for $t \not\equiv 0 \bmod p$. The endomorphism ring of these curves have a particular structure: $\mathrm{End}(E)$ is an order of the imaginary quadratic field $\mathbb{Q}(\sqrt{t^2 - 4p})$. From now, we denote $-D$ to be the discriminant of $\mathrm{End}(E)$, and $\{\mathrm{Id}, \psi\}$ a basis of the endomorphism ring. The fundamental discriminant corresponds to the discriminant of the maximal order containing $\mathrm{End}(E)$. This way, ψ is of degree $\frac{D+1}{4}$ or $D/4$ depending on the value of D modulo 4, and ψ can be defined using polynomials of degree $O(D)$ thanks to the Vélu's formulas [6]. Thus, the evaluation of ψ is efficient only for curves of small discriminant.

In this work, we restrict to curves defined over the BLS12-381 scalar field $\mathbb{F}_{r_{\mathrm{BLS}}}$. From now, we denote $p = r_{\mathrm{BLS}}$ and we look for curves with a 128-bit cryptographic security. Curves with $-D = -3$ and -4 do not have a large prime subgroup defined over $\mathbb{F}_p$. Hence, we look for small discriminant $-D < -4$ curves with subgroup and twist-subgroup security. It means that $\#E(\mathbb{F}_p)$ has a roughly 256-bit prime factor, as well as $\#E^t(\mathbb{F}_p)$.

As the endomorphism cost is closely related to the discriminant, we restrict to $-D \geq -388$ so that ψ can be efficiently computed. Moreover, we restrict on fundamental discriminants (discriminants of the maximal orders of imaginary quadratic fields). We denote $\mathcal{O}_{-D}$ the maximal order of discriminant $-D$. Elliptic curves with $\mathrm{End}(E) \subset \mathcal{O}_{-D}$ are isogenous curves, meaning that there is a rational map between them. Isogenous curves have the same order so that we can restrict on fundamental discriminants for our search.

We compute an exhaustive search among all the possible (fundamental) discriminants ($-292 \leq -D \leq -3$). Given a discriminant $-D$, roughly half of the curves are supersingular and hence not relevant to our cryptographic applications. We list in Table 2 the ordinary curves we obtained. In this table, p_n denotes a prime of n bits. The generation of these curves is reproducible using this file. We finally obtain an interesting curve for $-D = -8$ with large prime order subgroups on both the curve and its twist. We present in Section 3 the curve in several models, together with the endomorphism in order to apply the GLV scalar multiplication algorithm.

3 Bandersnatch

The Bandersnatch is obtained from a discriminant $-D = -8$, meaning that the endomorphism ring is $\mathbb{Z}[\sqrt{-2}]$. We obtain the curve j-invariant using the Complex Multiplication method, based on the Hilbert class polynomial $H_{-D}(X)$. The roots of H_{-D} are j-invariants of elliptic curves whose endomorphism ring is of discriminant $-D$. From a j-invariant, we obtain the curve equation in different models. Before looking into the details of three different representations, we briefly recall how to exhibit the degree 2 endomorphism ψ.

Degree 2 endomorphism. The endomorphism ψ has a kernel generated by a 2-torsion point. Hence, we can obtain the rational maps defining ψ by

$-$D	Curve sec.	Curve order
-3	65-bit	$2^2 \cdot 3 \cdot 97 \cdot 19829809 \cdot 2514214987 \cdot 423384683867248993 \cdot p_{131}$
	14-bit	$2^{64} \cdot 906349^4 \cdot p_{28}^4$
	77-bit	$7 \cdot 43 \cdot 1993 \cdot 2137 \cdot 43558993 \cdot 69032539613749 \cdot p_{154}$
	41-bit	$3 \cdot 7 \cdot 13 \cdot 79 \cdot 2557 \cdot 33811 \cdot 1645861201 \cdot 75881076241177 \cdot$ $86906511869757553 \cdot p_{82}$
	13-bit	$3^2 \cdot 11^2 \cdot 19^2 \cdot 10177^2 \cdot 125527^2 \cdot 859267^2 \cdot 2508409^2 \cdot 2529403^2 \cdot p_{26}^2$
	118-bit	$836509 \cdot p_{236}$
-4	59-bit	$2^{32} \cdot 5 \cdot 73 \cdot 906349^2 \cdot 254760293^2 \cdot p_{119}$
	37-bit	$2^2 \cdot 29 \cdot 233 \cdot 34469 \cdot 1327789373 \cdot 19609848837063073 \cdot$ $159032890827948314857 \cdot p_{74}$
	37-bit	$2 \cdot 3^2 \cdot 11^2 \cdot 13 \cdot 1481 \cdot 10177^2 \cdot 859267^2 \cdot 52437899^2 \cdot 346160718017 \cdot p_{74}$
	57-bit	$2 \cdot 5 \cdot 19^2 \cdot 1709 \cdot 125527^2 \cdot 2508409^2 \cdot 2529403^2 \cdot p_{114}$
-8	**122-bit**	$\mathbf{2^7 \cdot 3^3 \cdot p_{244}}$
	126-bit	$\mathbf{2^2 \cdot p_{253}}$
-11	69-bit	$5 \cdot 191 \cdot 5581 \cdot 18793 \cdot 48163 \cdot 46253594704380463613 \cdot p_{138}$
	73-bit	$3^3 \cdot 11^2 \cdot 9269797 \cdot 17580060420191283788101 \cdot p_{147}$
-19	110-bit	$7 \cdot 11^2 \cdot 19 \cdot 23 \cdot 397 \cdot 419 \cdot p_{220}$
	74-bit	$3^2 \cdot 5 \cdot 503 \cdot 10779490483 \cdot 433275286013779991 \cdot p_{149}$
-24	53-bit	$2^2 \cdot 3^2 \cdot 7 \cdot 19^2 \cdot 127 \cdot 29402034080953 \cdot 2970884754778276642175743 \cdot p_{106}$
	86-bit	$2^5 \cdot 5 \cdot 39628279 \cdot 1626653036429383 \cdot p_{172}$
-51	112-bit	$3^2 \cdot 5 \cdot 61223923 \cdot p_{224}$
	120-bit	$23^2 \cdot 41 \cdot p_{241}$
-67	67-bit	$3479887483 \cdot 56938338857 \cdot 8474085246072233 \cdot p_{135}$
	79-bit	$3^2 \cdot 8478452882270519617659314063 \cdot p_{159}$
-88	61-bit	$2^2 \cdot 11 \cdot 16984307 \cdot 24567897636186592260640293583411 \cdot p_{122}$
	66-bit	$2^9 \cdot 3^2 \cdot 31 \cdot 6133 \cdot 116471 \cdot 69487476515565975361139 \cdot p_{133}$
-132	73-bit	$2 \cdot 1753 \cdot 101235113104036296384208928969 \cdot p_{147}$
	92-bit	$2 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 23 \cdot 587 \cdot 701 \cdot 32299799971 \cdot p_{184}$
-136	62-bit	$2^3 \cdot 7^3 \cdot 19^3 \cdot 10939 \cdot 11131315086725327441688173207 \cdot p_{125}$
	87-bit	$2^2 \cdot 5 \cdot 5741 \cdot 30851 \cdot 533874022134253 \cdot p_{175}$
-228	114-bit	$2 \cdot 3^2 \cdot 19 \cdot 89 \cdot 5189 \cdot p_{228}$
	81-bit	$2 \cdot 947 \cdot 277603 \cdot 28469787063396608749 \cdot p_{162}$
-244	89-bit	$2^2 \cdot 13 \cdot 523 \cdot 1702319 \cdot 2827715661581 \cdot p_{179}$
	88-bit	$2^8 \cdot 3^2 \cdot 5 \cdot 71 \cdot 907 \cdot 2749 \cdot 146221 \cdot 2246269 \cdot p_{176}$
-264	83-bit	$2^3 \cdot 11 \cdot 131 \cdot 12543757399 \cdot 2818746796297 \cdot p_{167}$
	82-bit	$2^2 \cdot 3 \cdot 5^2 \cdot 2287 \cdot 2134790941497418864559 \cdot p_{165}$
-276	70-bit	$2 \cdot 11^2 \cdot 8839 \cdot 78797899 \cdot 323360863688748558301 \cdot p_{140}$
	88-bit	$2 \cdot 3 \cdot 5 \cdot 6197 \cdot 138617 \cdot 16664750312513 \cdot p_{177}$
-292	92-bit	$2 \cdot 54983 \cdot 5220799 \cdot 2671917733 \cdot p_{185}$
	86-bit	$2 \cdot 11^2 \cdot 149 \cdot 354689 \cdot 24012883 \cdot 32483123 \cdot p_{172}$

Table 2: Curves for discriminants $-3 \geq -D \geq -292$.

looking at the curves 2-isogenous to Bandersnatch. Only one has the same j-invariant, meaning that up to an isomorphism, the Vélu's formulas [6] let us obtain compute ψ. For cryptographic use-cases, we are interested in computing ψ on the p_{253}-order subgroup of the curve. On these points, ψ acts as a scalar multiplication by the eigenvalue

$$\lambda = \texttt{0x13b4f3dc4a39a493edf849562b38c72bcfc49db970a5056ed13d21408783df05}.$$

By construction, ψ is the endomorphism $\sqrt{-2} \in \mathcal{O}_{-8}$. Thus, λ satisfies $\lambda^2 + 2 = 0 \bmod p_{253}$. In the following sections, we provide details on the curve equation, the ψ rational maps, and a generator of the p_{253}-order subgroup in the case of the affine Weierstrass, projective Montgomery and projective twisted Edwards representations. The parameters are reproducible using the script of this file.

3.1 Weierstrass curve

Curve equation. The Bandersnatch curve can be represented in the Weierstrass model using the equation

$$E_W : y^2 = x^3 - 3763200000x - 78675968000000.$$

Endomorphism. The endomorphism ψ can obtained using the method detailed above. We obtain the following expression:

$$\psi_{\mathrm{W}}(x, y) = \left(u^2 \cdot \frac{x^2 + 44800x + 2257920000}{x + 44800}, u^3 \cdot y \cdot \frac{x^2 + 2 \cdot 44800x + t_0}{(x + 44800)^2} \right).$$

```
u=0x23c58c92306dbb96236140669daf1e2420ffd8fc8de2036c69307ddaa306c7d4
t0=0x73eda753299d7d483339d80809a1d80553bda402fffe5bfeffffffffef10be001.
```

Subgroup generator. The generator of the p_{253}-order subgroup is computed by finding the lexicographically smallest valid x-coordinate of a point of the curve, and scaling it by the cofactor 4 such that the result is not the point at infinity. From a point with $x = 2$, we obtain a generator $E_W(x_W, y_W)$ where:

```
xW=a76451786f95a802c0982bbd0abd68e41b92adc86c8859b4f44679b21658710
yW=44d150c8b4bd14f79720d021a839e7b7eb4ee43844b30243126a72ac2375490a.
```

3.2 Twisted Edwards curve

Curve equation. Bandersnatch can also be represented in twisted Edwards coordinates, where the group law is complete. In this model, the Bandersnatch curve can be defined by the equation

$$E_{\mathrm{TE}} : -5x^2 + y^2 = 1 + dx^2y^2, d = \frac{138827208126141220649022263972958607803}{171449701953573178309673572579671231137}.$$

Twisted Edwards group law is more efficient with a coefficient $a = -1$ (see [7] for details). In our case, -5 is not a square in $\mathbb{F}_p$. Thus, the curve with equation $-x^2 + y^2 = 1 - dx^2y^2/5$ is the quadratic twist of Bandersnatch. We provide a representation with $a = -5$, leading to a slightly more expensive group law because multiplying by -5 is more expensive than a multiplication by -1, but this cost will be neglected compared to the improvement of the GLV method. See Section 4 for details.

Endomorphism. From this representation, we exhibit the degree 2 endomorphism in twisted Edwards coordinates:

$$\psi_{\mathrm{TE}}(x,y,z) = (xa_1(y+a_2z)(y+a_3z), b_1(y+b_2z)(y+b_3z)yz^2, (y+c_1z)(y+c_2z)yz^2)$$

```
a1=0x23c58c92306dbb95960f739827ac195334fcd8fa17df036c692f7ddaa306c7d4
a2=0x52c9f28b828426a561f00d3a63511a882ea712770d9af4d6ee0f014d172510b4
a3=0x2123b4c7a71956a2d149cacda650bd7d2516918bf263672811f0feb1e8daef4d
b1=0x52c9f28b828426a561f00d3a63511a882ea712770d9af4d6ee0f014d172510b4
b2=0x50d06958b6e8ce1ab1b2745bd377e5bde07e867f02611eae1c098cd1519b574a
b3=0x231d3dfa72b4af2d818763ac3629f247733f1d83fd9d3d50e3f6732dae64a8b7
c1=0x5ede5fd005b839be71b70d491ebfddeff693de40b4c002a7fc1ae7171cc9f7b5
c2=0x150f478323e54389c182cabeeae1fa155d29c5c24b3e595703e518e7e336084c.
```

This map can be computed in 17 multiplications and 6 additions modulo p.

Subgroup generator. The generator of the p_{253}-order subgroup obtained in Section 3.1 has twisted coordinates of the form $E_{\mathrm{TE}}(x_{\mathrm{TE}}, y_{\mathrm{TE}}, 1)$ with

```
xTE=29c132cc2c0b34c5743711777bbe42f32b79c022ad998465e1e71866a252ae18
yTE=2a6c669eda123e0f157d8b50badcd586358cad81eee464605e3167b6cc974166.
```

3.3 Montgomery curve

Curve equation. A twisted Edwards curve is always birationally equivalent to a Montgomery curve. We obtain the mapping between these two representations following [8]. While the twisted Edwards model fits better for $\mathbb{F}_p$ circuit arithmetic and more generally for the zero-knowledge proof context, we provide here the Montgomery version because the scalar multiplication is more efficient in this model:

$$E_M : By^2 = x^3 + Ax^2 + x$$

```
B=0x300c3385d13bedb7c9e229e185c4ce8b1dd3b71366bb97c30855c0aa41d62727
A=0x4247698f4e32ad45a293959b4ca17afa4a2d2317e4c6ce5023e1fd63d1b5de98.
```

Endomorphism. Montgomery curves allow efficient scalar multiplication using the Montgomery ladder. We provide here the endomorphism ψ in this model:

$$\psi_{\mathrm{M}}(x,-,z) = (-(x-z)^2 - cxz, -, 2xz)$$

```
c=0x4247698f4e32ad45a293959b4ca17afa4a2d2317e4c6ce5023e1fd63d1b5de9a.
```

Subgroup generator. The generator of the p_{253}-order subgroup given above is of the form $E_M(x_M, -, 1)$ with:

```
xM=67c5b5fed18254e8acb66c1e38f33ee0975ae6876f9c5266a883f4604024b3b8.
```

3.4 Security of Bandersnatch

The Bandersnatch curve order is $2^2 \cdot r$ for a 253-bit long prime r. Its quadratic twist has order $2^7 \cdot 3^3 \cdot r'$, where r' is another prime of 244 bits. Hence, the Bandersnatch curve satisfies twist security after a quick cofactor check. We estimate that the Bandersnatch curve (resp. its quadratic twist) has 126 bits of security (resp. 122 bits of security).

4 Comparison

The twisted Edwards representation is mostly used in practice, and we now focus on the comparison between Jubjub and Bandersnatch in this model.

4.1 Scalar multiplications for a variable base point

Because of its large discriminant, the scalar multiplication on Jubjub is a basic double-and-add algorithm, meaning that it computes a multiplication by n in $\log n$ doublings and $\log n/2$ additions (in average) on the curve.

The endomorphism ψ lets us compute the scalar multiplication faster than a double-and-add algorithm with few precomputations. For a point P and a scalar n, we first evaluate ψ at P and decompose $n = n_1 + \lambda n_2$. Then a multi scalar multiplication is computed in $\log n/2$ doublings and $3 \log n/8$ additions (in average) on the curve.

We benchmarked our implementation with both GLV enabled and disabled, and compared it with Arkworks' own Jubjub implementation. Our benchmark is conducted over an AMD 5900x CPU, with Ubuntu 20.04, rust 1.52 stable version, and Arkwork 0.3.0 release version. We used `criterion` micro-benchmark toolchain, version 0.3.0, for data collection. We compile Arkworks with two options, namely `default` and `asm`, respectively. The `default` setup relies on `num_bigint` crate for large integer arithmetics, while `asm` turns on assembly for finite field multiplication.

Arkworks use the aforementioned double-and-add multiplication methodology, without side channel protections such as Montgomery ladders. Our non-GLV implementation also follows this design. For our GLV implementation, there are three components, namely, the endomorphism, the scalar decomposition, and the multi scalar multiplication (MSM). We implement those schemes and present the micro-benchmarks in Table 3. Specifically, we do not use the MSM implementation in Arkworks: our scalars, after the decomposition, contain roughly 128 bits of leading zeros, and our own MSM implementation is optimized for this setting.

Table 3 presents the full picture of the benchmark. When GLV is disabled, we observe a similar but a little worse performance for Bandersnatch curve, compared to the Jubjub curve, due to the slightly larger coefficient $a = -5$ and a larger scalar field of 253 bits (Jubjub curve has $a = -1$ and a scalar field of 252 bits). When GLV is enabled, we report a 45% improvement of the Bandersnatch curve, and a 42% improvement over the Jubjub curve.

	`default`	`asm`
Jubjub	75 μs	69 μs
Bandersnatch without GLV	78 μs	72 μs
Bandersnatch with GLV	44 μs	42 μs
Endomorphism	2.4 μs	1.8 μs
Scalar decomposition	0.75 μs	0.7 μs
multi scalar multiplication	42 μs	40.8 μs
Overall Improvement	42%	39%

Table 3: Bandersnatch vs Jubjub: Performance

To make a meaningful comparison, we benchmark the cost of the group multiplication over the default generators. Note that Arkworks do not implement optimizations for fixed generators nonetheless. We then sample field elements uniformly at random, for each new iteration, and the benchmark result is consolidated over 100 iterations.

4.2 Multi scalar multiplications

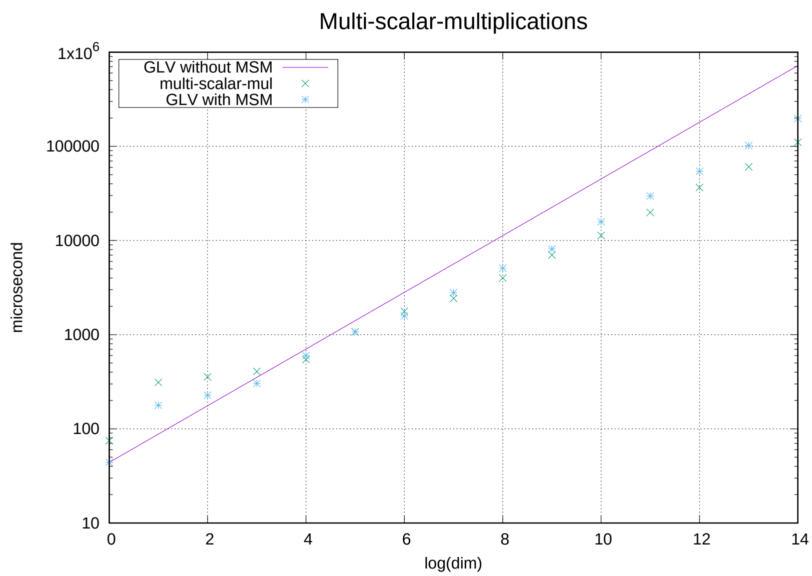
This section reports the performance of variable base multi scalar multiplications (MSM). Note that this MSM is compatible, but different from the MSM inside the GLV. In particular, for a sum of k scalar multiplications, we report the data point for:

- invoking the MSM over the k base scalars randomly sampled, expected to be around 256 bits;
- using GLV endomorphism to break the k base scalars into $2k$ new base scalars, of halved size, i.e. of 128 bits.

The data is presented in Figure 4.2. Specifically, as a baseline, the trivial solution, captained by *GLV without MSM*, is the product of the number of bases and the cost of doing a single GLV multiplication. Note that the MSM algorithms incur an overhead to build some tables, which make them less favorable compared to the trivial solution when dimension is really small. For a dimension greater than 4, MSM algorithms begin to out-perform trivial solutions. For dimension greater than 128, it is more efficient to invoke the MSM directly, rather than doing it over the GLV basis. The reason is that the size of the basis becomes too large, so that the gain we get from halving the scalars is offset from the gain we get from halving the basis. We remark that this threshold point is platform dependent.

4.3 R1CS constraints

The Bandersnatch curve is zk-SNARK friendly: its base field matches the scalar field for the BLS12-381 curve, a pairing-friendly curve, on top of which people



build zk-SNARK systems, such as Groth16 [9] or Plonk [10]. In such a setting, the prover can sufficiently argue certain relationships over arithmetic circuits rather than binary circuits. The circuit is expressed in a form of Rank-1 constraint system (R1CS), and in general, the complexity is determined by the number of constraints in an R1CS.

We evaluate the number of constraints for a variable base group multiplication. For a double-and-add algorithm, our code reports 3325 constraints in total. As a sanity check, within the core logic, it takes 6 constraints per addition, 5 constraints per doubling and 2 constraints per bit selection. This adds up to 13 constraints per bit, or 3315 constraints per group multiplication (and we reasonably assume some system overhead consumes another 10 constraints).

5 Conclusion

Tne last decade has seen great improvements on practical zk-SNARK systems. An essential stepping stone of these schemes is an efficient elliptic curve whose base field matches the scalar field for some pairing-friendly curve. On this note, we present Bandersnatch as an alternative to the commonly used base curve Jubjub. Due to the existence of an efficiently computable endomorphism, the scalar multiplication over this curve is 42% times faster than the Jubjub curve. For multi scalar multiplications, we report a narrowed advantage over Jubjub curve when the dimension is small, but it vanishes for larger dimensions. We also do not observe any improvement in terms of number of constraints in the corresponding R1CS circuit.

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