Efficient Adaptively-Secure Byzantine Agreement for Long Messages

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Abstract. We investigate the communication complexity of Byzantine agreement protocols for long messages against an adaptive adversary. In this setting, prior results either achieved a communication complexity of $O(nl \cdot \mathsf{poly}(\kappa))$ or $O(nl + n^2 \cdot \mathsf{poly}(\kappa))$ for l-bit long messages. We improve the state of the art by presenting protocols with communication complexity $O(nl + n \cdot \mathsf{poly}(\kappa))$ in both the synchronous and asynchronous communication models. The synchronous protocol tolerates $t \leq (1-\varepsilon)\frac{n}{2}$ corruptions and assumes a VRF setup, while the asynchronous protocol tolerates $t \leq (1-\varepsilon)\frac{n}{3}$ corruptions under further cryptographic assumptions. Our protocols are very simple and combine subcommittee election with the recent approach of Nayak et al. (DISC '20). Surprisingly, the analysis of our protocols is *all but simple* and involves an interesting new application of Mc Diarmid's inequality to obtain *optimal* corruption thresholds.

1 Introduction

Byzantine agreement (BA) is a fundamental problem in distributed computing. In a Byzantine agreement protocol consisting of n parties, each party starts with an input value, and at the end of the protocol, all honest (non-faulty) parties output a value. Byzantine agreement protocols guarantee that if all honest parties input the same value v, then they must output v; otherwise, they output any agreed upon value. Moreover, this holds even if some threshold t out of n parties are Byzantine (arbitrarily malicious).

Byzantine agreement forms a core abstraction for many blockchains where consensus is required on large values among a large number of parties. Moreover, due to the value of the transactions contained in these blockchains, they need to tolerate strong adaptive adversaries who are capable of corrupting any party based on the state of the protocols subject to the Byzantine threshold constraint. These requirements lead to the following natural question: *What is the lowest communication complexity possible for Byzantine agreement protocols on large values tolerating an adaptive adversary?*

This question has been partially answered in the literature. For instance, it has been shown that BA can be solved with $o(n^2)$ communication complexity against an adaptive adversary [11, 1, 6]. At a high level, these protocols take the approach of electing committees of size κ (where κ is a security parameter) and only the committee members send messages to all parties. This allows achieving a communication complexity of $O(n \cdot \mathsf{poly}(\kappa))$. However, this computation implicitly assumes inputs with a constant number of bits. If the inputs are of size l bits, the communication complexity is $O(nl \cdot \mathsf{poly}(\kappa))$.

A different line of work seeks to achieve the optimal communication complexity of $O(nl)$ for *long messages*, i.e., where $l \gg n$ [17, 9, 16, 19, 10]. More precisely, the best known protocols in this area achieve a communication complexity of $O(nl + \kappa n^2)$ [17] and the main goal of these works is to further reduce the latter term as much as possible. At a high level, these protocols take the approach of agreeing on the hash of an input value with $O(\kappa n^2)$ communication (κ is the size of a hash) assuming appropriate BA protocols for κ-sized inputs and then use erasure coding techniques to distribute the l-bit long blocks with communication $O(nl)$. In this work, we ask whether we can achieve the best of both approaches. In particular, *Does there exist a Byzantine agreement protocol for l bit values tolerating an adaptive adversary with $O(nl + n \cdot \mathsf{poly}(\kappa))$ communication complexity?*

We answer this question positively. Surprisingly, the techniques from the two lines of work do not compose in a straightforward manner to achieve the desired communication complexity. In fact, Nayak et al. [17] present a lower bound of $\Omega(nl + A(\kappa) + n^2)$ where $A(\kappa)$ is the communication complexity of Byzantine agreement on κ bit inputs. However, the bound holds only for deterministic protocols. For the first time, we use randomization in the extension part (as well as the underlying protocol) to circumvent the lower bound and achieve $O(nl + n \cdot \mathsf{poly}(\kappa))$ complexity. We present two protocols one assuming synchronous network and another assuming asynchronous network, that achieve these guarantees.

1.1 Simple Adaptively Secure BA Protocols for Long Messages

Our first result is a synchronous, adaptively secure BA protocol tolerating $t \leq (1 - \varepsilon) \cdot \frac{n}{2}$ Byzantine parties, for some arbitrary constant $\varepsilon > 0$. The second result is asynchronous and tolerates $t \leq (1 - \varepsilon) \cdot \frac{n}{3}$ corruptions.

Theorem 1. *For all constants $\varepsilon > 0$, assuming appropriate cryptographic setup assumptions, there exists an adaptively secure synchronous Byzantine agreement protocol achieving a communication complexity of $O(nl + \mathsf{poly}(\kappa)n)$ for l-bit values for*

1. *$t \leq (1 - \varepsilon) \cdot \frac{n}{2}$ Byzantine parties under a synchronous network, and*
2. *$t \leq (1 - \varepsilon) \cdot \frac{n}{3}$ Byzantine parties under an asynchronous network.*

We describe the intuition behind the synchronous protocol. Using an adaptively-secure subquadratic 1-bit BA protocol from [1], all parties can agree on the κ-bit accumulator value with a communication of $O(\kappa^3 n)$. Thus the key challenge is to distribute the l-bit value to all parties with linear communication while tolerating an adaptive adversary. Typically, distributing a large value to n parties using erasure codes is performed in two steps. First, create n encoded shares of the value, one for each party, of size $O(\frac{l}{n})$, and send the shares to the respective parties. Then, each party sends its own share to all other parties. If every party receives sufficiently many shares (Byzantine parties may not send shares), they can reconstruct the l-bit value. Observe that the latter step incurs $\Omega(n^2)$ communication, thus dominating the $n \cdot \mathsf{poly}(\kappa)$ term of the desired communication complexity. To make this approach efficient, we have to find the right amount of shares to create and the right parties to share them with. If we naïvely create one share per party, we will need all parties to speak so that we can reconstruct the long message. Clearly, this results in poor communication complexity. On the other hand, if we share the messages with only a small committee C, an adaptive adversary can corrupt all the parties in C and prevent reconstruction of the long message.

To address these concerns, our solution relies on a public partition of parties into one of κ buckets such that each bucket holds n/κ parties. We then elect κ-sized committees at random (using the standard VRF approach for cryptographic sortition) to perform each of the two steps described earlier. In the first step, the value is encoded into κ shares of size $O(\frac{l}{\kappa})$ and the j-th share is sent to parties in bucket j. In the second step, the elected committee members from each of the κ buckets send their share to all parties. This incurs an $O(\kappa n \cdot \frac{l}{\kappa}) = O(ln)$ bits of communication. The crux of our argument lies in showing that when $t \leq (1-\varepsilon) \cdot \frac{n}{2}$, a majority of buckets contains an honest party who is also elected as a committee member. Thus, the shares that these honest parties send are sufficient to reconstruct the initial value. There are several subtleties involved in correlating the committee members chosen to agree via 1-bit BA with the committee chosen to distribute the l bit message. If we elect parties to the committee C using the common approach of verifiable random functions, it is not possible to argue via standard Chernoff-type bounds that sufficiently many of the buckets will be covered by members of C.

This is because the number of committee members across buckets are correlated and a rushing adaptive adversary can observe the number of committee members for any subset of the buckets before corrupting others. Instead, our argument relies on a subtle application of Mc Diarmid's inequality, which, to the best of our knowledge, has not been explored in this type of protocol.

Using our insights from the synchronous setting, we also obtain a protocol for the asynchronous setting by substituting the 1-bit agreement protocol with the recent (asynchronous) BA construction of Blum et al. [3].

1.2 Related Work

Work related to extension protocols. In the following, we denote as $\mathcal{A}(1), \mathcal{A}(\kappa)$ the communication complexity of a BA protocol with input domain of size 1 and κ bits, respectively. The problem of extending the domain of Byzantine agreement protocols is a well-studied one in the literature. To the best of our knowledge, the first work that considered this problem is that of Turpin and Coan [19] who showed how to reach agreement on messages from arbitrary domains given agreement on binary values in the corruption regime $t < n/3$ with synchrony. The problem has also been considered for other related primitives such as Byzantine broadcast [12, 7] or reliable broadcast [4, 17]. Previous works that focus on this problem are the works by Fitzi and Hirt [9], and that of Liang and Vaidya [16]. In the synchronous setting with $t < n/3$ and error-freeness, the protocol of Ganesh and Patra [10] previously provided the best known protocol which achieves $O(nl + n^2 \cdot \mathcal{A}(1))$ communication complexity. For the computational setting with $t < n/2$, the protocols of Ganesh and Patra [10] previously provided the best known solution achieving $O(nl + n\mathcal{A}(\kappa) + \kappa n^3)$. These complexities were recently further improved by the protocols of Nayak et al. [17] who gave protocols that achieve $O(nl + \mathcal{A}(\kappa) + n^2\kappa)$ communication complexity for the computational setting when $t < n/3$ or $t < n/2$. Nayak et al. also improved on error-free protocols in the $t < n/3$ setting, giving a protocol that achieves $O(nl+n\mathcal{A}(1)+n^3)$ communication complexity.

Work related to adaptively secure sub-quadratic communication protocols. Dolev and Reischuk [8] first showed that deterministic Byzantine agreement protocols incur $\Omega(t^2)$ communication complexity when tolerating $t < n$ Byzantine faults. King et al. [15, 13, 14] presented the first Byzantine agreement protocols that can be solved with subquadratic communication complexity under inverse polynomial in n error probability. More recently, Algorand [11, 6] showed constructions with $O(n \cdot \mathsf{poly}(\kappa))$ communication complexity for adaptively secure Byzantine agreement tolerating $t < (1 - \varepsilon)n/3$ Byzantine parties in the synchronous setting assuming memory erasures. This was further improved by Abraham et al. [1] in the synchronous and partially synchronous network setting tolerating $t < (1-\varepsilon)n/2$ and $t < (1-\varepsilon)n/3$ respectively without assuming memory erasures. Finally, Blum et al. [3] presented a subquadratic communication protocol in the asynchronous setting tolerating $t < (1 - \varepsilon)n/3$ faults. As discussed above, these protocols achieve subquadratic communication complexity, but fail to provide the asymptotically optimal complexity $O(nl)$ when l grows beyond n. Nonetheless, these protocols do serve as important building blocks in extension protocols such as the ones presented here (i.e., to agree efficiently on the short message shares).

2 Model and Preliminaries

We consider a setting with n parties $P_1, \ldots, P_n$ that have access to a complete network or pairwise authenticated channels. The adversary is adaptive, and can corrupt up to t parties at any point of the protocol execution in an arbitrary manner. However, we make two standard assumptions on the capability of the adversary (see, e.g., [5, 3]. First, parties can perform an *atomic send operation*, i.e., they can send a message to any number of parties *simultaneously* and without the adversary corrupting them in between (different) sends. Second, the adversary cannot perform *after-the-fact* removal, i.e., cannot take back messages sent by parties while they were still honest. We consider protocols in the synchronous and asynchronous network settings. In a synchronous network, we assume communication in lock-step rounds where messages sent by a party at the start of a round arrives at its destination by the end of that round. On the other hand, in an asynchronous network, messages are assumed to arrive at their destination eventually.

2.1 Definitions

Let us recap the definition of Byzantine agreement.

Definition 1 (Byzantine Agreement). *Let Π be a protocol executed by parties $P_1, \ldots, P_n$, where each party P_i starts with an input x_i and parties terminate upon generating output. We say that Π is an t-secure Byzantine agreement protocol if the following properties hold when up to t parties are corrupted:*

- *Validity: If all honest parties start with the same input x, then every honest party outputs x.*
- *Consistency: All honest parties output the same value.*

2.2 Primitives

Our protocols will make use of standard linear error correcting codes and cryptographic accumulators.

Linear error correcting code. We use standard Reed-Solomon (RS) codes with parameters (κ, b). The codewords are elements in a Galois Field $GF(2^a)$ with $\kappa \leq 2^a - 1$. There are two algorithms:

- Encoding. Given inputs $m_1, \ldots, m_b$, the encoding function outputs κ codewords (a.k.a. shares) $(s_1, \ldots, s_\kappa)$ of length κ, such that any b codewords uniquely determine the input message and the other codewords.
- Decoding. Given κ codewords $(s_1, \ldots, s_\kappa)$, one can reconstruct the original message $(m_1, \ldots, m_b)$ even when $\kappa - b$ values are erased.

Looking ahead in our protocols, we will choose random committee subsets of κ parties out of the n parties, and we will set the parameter to $b = t_\kappa$, which will correspond to a lower bound on the number of honest parties in a committee.

Cryptographic accumulators. We recall the definition of cryptographic accumulators [2]. Given a set of values, the primitive can produce an accumulated value and a witness for each element in the set. Then, given the accumulated value and a witness, one can verify that a particular element is in the set.

Definition 2. *A cryptographic accumulator consists of algorithms* $(\mathsf{Gen}, \mathsf{Eval}, \mathsf{CreateWit}, \mathsf{Verify})$*, where:*

- $\mathsf{Gen}(1^\kappa, T)$*: It takes a parameter κ and an accumulation threshold T and returns an accumulator key* $\mathtt{ak}$.
- $\mathsf{Eval}(\mathtt{ak}, \mathcal{D})$*: It takes an accumulator key* $\mathtt{ak}$ *and a set of values to accumulate $\mathcal{D}$ and returns an accumulated value z for $\mathcal{D}$.*
- $\mathsf{CreateWit}(\mathtt{ak}, z, d_i)$*: It takes an accumulator key* $\mathtt{ak}$*, an accumulated value z for $\mathcal{D}$ and a value d_i, and returns $\perp$ if $d_i \notin \mathcal{D}$ or a witness w_i otherwise.*
- $\mathsf{Verify}(\mathtt{ak}, z, w_i, d_i)$*: It takes an accumulator key, accumulated value z for $\mathcal{D}$, witness w_i, value d_i, and returns 1 if w_i is a witness for $d_i \in \mathcal{D}$ and 0 otherwise.*

We require our accumulator to satisfy standard collision-free properties [18].

2.3 Concentration Bounds I

We recall the Chernoff concentration bound.

Lemma 1 (Homogenous Chernoff Bound). *Let $X_1, ..., X_n$ be i.i.d. Bernoulli random variables with parameter p. Let $X := \sum_i X_i$, so $\mu := E[X] = p \cdot n$. Then, for $\delta \in [0,1]$,*

$$\Pr[X \geq (1+\delta)\cdot\mu] \leq e^{-\delta^2\mu/(2+\delta)} \qquad \text{and} \qquad \Pr[X \leq (1-\delta)\cdot\mu] \leq e^{-\delta^2\mu/2}.$$

Let $\chi_{s,n}$ denote the distribution that samples a subset of the n parties, where each party is included independently with probability s/n. The following lemma will be useful in our analysis.

Corollary 1. *Fix $\kappa \leq s \leq n$ and $0 < \varepsilon < \frac{1}{4}$, and let $t = (1-3\varepsilon)n/2$ be the number of corrupted parties. If $C \leftarrow \chi_{s,n}$, then C contains less than $(1-2\varepsilon)s/2$ corrupted parties except with negligible probability.*

Proof. Let $H \subseteq [n]$ be the indices of the honest parties. Let X_j be the Bernoulli random variable indicating if $P_j \in C$, so $\Pr[X_j = 1] = s/n$. Define $Z := \sum_{j \notin H} X_j$. Then, since $E[Z] = t \cdot s/n = (1-3\varepsilon)s/2$, setting $\delta = \frac{\varepsilon}{1-3\varepsilon}$ in Lemma 1 yields

$$\Pr\left[Z \geq (1-2\varepsilon)s/2\right] \leq \texttt{neg}(\kappa).$$

(Almost) the same proof yields:

Corollary 2. *Fix $\kappa \leq s \leq n$ and $0 < \varepsilon < \frac{1}{4}$, and let $t = (1-3\varepsilon)n/3$ be the number of corrupted parties. If $C \leftarrow \chi_{s,n}$, then C contains less than $(1-2\varepsilon)s/3$ corrupted parties except with negligible probability.*

Corollary 3. *Fix $s \leq n$ and $0 < \varepsilon < 1$. If $C \leftarrow \chi_{s,n}$, then C contains more than $(1-\varepsilon)\cdot s$ many parties except with probability at most $O(e^{-\varepsilon^2 s})$.*

Proof. Let $H \subseteq [n]$ be the indices of the honest parties. Let X_j be the Bernoulli random variable indicating if $P_j \in C$, so $\Pr[X_j = 1] = s/n$. Define $Z := \sum_{j \notin H} X_j$. Then, since $E[Z] = s$, setting $\delta = \varepsilon$ in Lemma 1 yields

$$\Pr\left[Z \leq (1-\varepsilon)\cdot s\right] \leq e^{\varepsilon^2 \cdot s/2}.$$

3 Balls and Buckets Analysis for Throwing *ck* Balls in *k* Buckets

In this section, we present the technical inequality that will be used in our protocols in subsequent sections. We will start with the following concentration bound:

Theorem 2. *(McDiarmid's Inequality) Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $X_j \in \mathcal{K}_j$, for some measurable set $\mathcal{K}_j$. Suppose $f : \prod_{i=1}^n \mathcal{K}_j \to R$ is 'Lipschitz' in the following sense: for each $k \leq n$ and any two input sequence $x, x \in \prod_j \mathcal{K}_j$, that differ only in the k^{th} coordinate,*

$$|f(x) - f(x')| \leq \sigma_k.$$

Let $Y = f(X_1, X_2, \ldots, X_n)$. Then for any $\alpha > 0$,

$$\mathbf{Pr}[|Y - \mathbf{E}[Y]| \geq \alpha] \leq 2 \cdot \exp\left(-\frac{2\alpha^2}{\sum_{j=1}^n \sigma_j^2}\right).$$

The binomial distribution with parameters n and p is the discrete probability distribution of the number of successes in a sequence of n independent experiments, each asking a yes-no question, and each with its own Boolean-valued outcome: success (with probability p) or failure (with probability $1-p$).

Let $c \geq 1$ and $k \geq 1$ be the parameters where k is the number of buckets and ck is the number of balls (committee members). Consider the following random experiment: We throw ck balls in k buckets independently and uniformly at random. Let b_i be the expected number of buckets with exactly i balls.

Let X_j^i be the indicator random variable that the j^{th} bucket has exactly i balls. Thus, we can write b_i as:

$$b_i = \sum_{j=1}^{k} \mathbf{E}[X_j^i].$$

We also have,

$$\mathbf{E}[X_j^i] = \binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i}.$$

By linearity of expectation,

$$\begin{aligned} b_i &= \sum_{j=1}^{k} \binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i} \\ &= k \cdot \binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i}. \end{aligned}$$

The following lemma shows that the number of buckets with exactly i balls is concentrated around b_i.

Lemma 2. *For* $0 \leq i \leq c$, $\mathbf{Pr}\left[\left|\begin{matrix}\textit{number of buckets} \\ \textit{with exactly } i \textit{ balls}\end{matrix} - b_i\right| \geq \varepsilon \cdot b_i\right] \leq 2\exp(-\frac{\varepsilon^2}{e^{3c}} \cdot k)$.

Proof. Let $m = ck$ and define a function $f : [k]^{ck} \to R$ as follows. $f(a_1, a_2, \ldots, a_m)$ is the number of j such that $a_j = i$. We are interested in the random variable $Y = f(x_1, x_2, \ldots, x_m)$ where each x_i is independent and uniform in $[k]$. It is clear that f is Lipschitz with a Lipschitz constant of 1, i.e, if you change only one input coordinate, then the function value changes by at most 1. Towards applying Theorem 2, we have $\sigma_j = 1$ for all $j \in [m]$ and hence $\sum_j \sigma_j^2 = m$. Using McDiarmid's inequality 2 and using a loose bound of $b_i \geq \frac{c^i}{i^i e^c} k$.

$$\begin{aligned} \mathbf{Pr}[|Y - b_i| \geq \varepsilon \cdot b_i] &\leq 2\exp\left(-\frac{2\varepsilon^2 b_i^2}{m}\right) \\ &\leq 2\exp\left(-\frac{2\varepsilon^2 b_i^2}{ck}\right) \\ &\leq 2\exp\left(-\frac{2\varepsilon^2 c^{2i}}{i^{2i} e^{2c} \cdot c} \cdot k\right) \\ &\leq 2\exp\left(-\frac{\varepsilon^2}{e^{3c}} \cdot k\right). \end{aligned}$$

We only need concentration for $= 0, 1, \ldots, c-1$ for the overall argument that follows next. Since each holds with probability $1 - \exp(-\varepsilon^2 k / e^{O(c)})$, by union bound, we have that the number of buckets with i balls is concentrated around it's expectation for $i = 0, 1, \ldots, c-1$ happens with probability at least $1 - c \cdot \exp(-\varepsilon^2 k / e^{O(c)})$.

Claim. Let $\tau \in (0, 1/2]$ be any constant. There exists a constant $0 \leq c_\tau \leq c$ such that the following two inequalities hold simultaneously. We have,

1. $\sum_{i=0}^{c_\tau} b_i \leq \tau k$.
2. $\sum_{i=1}^{c_\tau} i \cdot b_i \geq (\tau - o_k(1)) \cdot ck$.

Proof. Let c_τ be the largest constant such that (1) holds. The sum $\sum_{i=0}^{c_\tau} b_i/k$ is the cumulative density of the binomial distribution with parameters ck and $\frac{1}{k}$ at c_τ. As the median of the binomial distribution with parameters ck and $\frac{1}{k}$ is c, we have $c_\tau \leq c$ for $\tau \in (0, 1/2]$. We will show that, for this constant c_τ, the inequality (2) holds.

$$\begin{aligned}
\sum_{i=1}^{c_\tau} i \cdot b_i &= \sum_{i=0}^{c_\tau} i \cdot k\binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i} \\
&= k \cdot \sum_{i=0}^{c_\tau} i \cdot \binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i} \\
&= k \cdot \sum_{i=1}^{c_\tau} ck \cdot \binom{ck-1}{i-1} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i} && \left(k\tbinom{n}{k} = n\tbinom{n-1}{k-1}\right) \\
&= k \cdot ck \cdot \frac{1}{k} \cdot \sum_{i=1}^{c_\tau} \binom{ck-1}{i-1} \cdot \left(\frac{1}{k}\right)^{i-1} \cdot \left(1 - \frac{1}{k}\right)^{(ck-1)-(i-1)} \\
&= ck \cdot \sum_{i=0}^{c_\tau - 1} \binom{ck-1}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{(ck-1)-i}. && (\text{Setting } i \leftarrow i-1)
\end{aligned}$$

Now, the summation is precisely the cumulative density of the binomial distribution with parameters $ck - 1$ and $\frac{1}{k}$ at $c_\tau - 1$. We now rearrange the terms to get the cumulative density of the binomial distribution with parameters ck and $\frac{1}{k}$ at $c_\tau + 1$ in the summation. This way we can relate it to the constant τ.

$$\begin{aligned}
\sum_{i=1}^{c_\tau} i \cdot b_i &= ck \cdot \sum_{i=0}^{c_\tau - 1} \binom{ck-1}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{(ck-1)-i} \\
&= ck \cdot \sum_{i=0}^{c_\tau - 1} \frac{\frac{ck-1-i}{ck-1}}{(1-1/k)} \binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i} \\
&\geq (1 - o_k(1)) \cdot ck \cdot \sum_{i=0}^{c_\tau - 1} \binom{ck}{i} \cdot \left(\frac{1}{k}\right)^i \cdot \left(1 - \frac{1}{k}\right)^{ck-i} \\
&= (1 - o_k(1)) \cdot ck \cdot \sum_{i=0}^{c_\tau - 1} \frac{b_i}{k} \\
&= (1 - o_k(1)) \cdot ck \cdot \left(\left(\sum_{i=0}^{c_\tau + 1} \frac{b_i}{k} \right) - \frac{b_{c_\tau}}{k} - \frac{b_{c_\tau + 1}}{k} \right) \\
&\geq (1 - o_k(1)) \cdot ck \cdot \left(\tau - \frac{b_{c_\tau}}{k} - \frac{b_{c_\tau + 1}}{k} \right) \\
&= ck\,(\tau - o_k(1)).
\end{aligned}$$

Here, in the first inequality, we used the fact that c_τ is at most c. The second inequality uses the fact that the constant c_τ is the largest constant that satisfies inequality (1). Therefore, $\sum_{i=0}^{c_\tau+1} \frac{b_i}{k} \geq \tau$.

Now, $c_\tau \leq c$ for every $\tau \in (0, 1/2]$. Using this, we combine Lemma 2 and Claim 3 along with a simple application of union bound and the fact that the sums are natural numbers, to get the following Corollary.

Corollary 4. *For every $\tau \in (0, 1/2]$ there exists a constant c_τ such that the following holds. Suppose we throw ck balls in k buckets, each uniformly and independently at random. Let b'_i be the number of buckets with exactly i balls. Then the following two inequalities hold with probability at least $1 - 2 \cdot c \cdot \exp(-\frac{\varepsilon^2}{e^{3c}} \cdot k)$.*

1. $\sum_{i=0}^{c_\tau} b'_i \leq \lfloor (1+\varepsilon)\tau k \rfloor$.
2. $\sum_{i=1}^{c_\tau} i \cdot b'_i \geq (1-\varepsilon)(\tau - o_k(1)) \cdot ck$.

4 Adaptively Secure Synchronous Communication-Efficient Protocol for Long Messages

4.1 Protocol Description

We begin by recalling the adaptively-secure sub-quadratic BA protocol of Abraham et al. [1]. In this protocol, the step of each round i is performed by a randomly chosen committee C_i^b (each committee is tied to a value $b \in \{0, 1\}$), who reveals itself only when it is their turn to speak in the protocol. We assume that parties have available via a trusted setup efficient algorithms $\mathsf{ComProve}$ and ComVer that allow them to prove and verify membership of a committee, and we do not make this explicit in our protocol (this can typically be achieves via a VRF setup). Once members of the committee send their messages for round i, it is too late for the adversary to corrupt them, as they can not take back messages that were previously sent by honest parties[5]. We run two versions of the protocol, the first is for κ-valued messages and denoted as $\mathsf{BA}(\kappa)$, the other for binary-valued messages, and denoted as $\mathsf{BA}(1)$. Since the protocol in [1] is binary, we simply run it κ many times in parallel to agree on a κ bit message.

We assume that our protocol specifies a (arbitrary) partition of the n parties into κ buckets $B_1, \ldots, B_\kappa$ of n/κ parties each. We begin by explaining the setup of our protocol. The setup consists of an honest dealer that honestly chooses/distributes the accumulator keys.

Protocol Setup for $\mathit{\Pi}_{\mathtt{sprBA}}$

Accumulators

Generate the accumulator key $\mathtt{ak} = \mathsf{Gen}(1^\kappa, \kappa)$ and give it to all parties.

We now describe the protocol. In our description, we refer to C_i^b as the committee for the i-th round of an execution of $\mathsf{BA}(1)$ for the bit b. We denote as C^* a special committee (also selected at random $\mathsf{ComProve}$) whose members are designated to perform the forwarding step in our protocol.

In the following description, we will use two sub-protocols, Encode and Rec, which are based on RS codes. We specify them relative to t_κ which in our protocol is set as $t_\kappa = (1+\varepsilon)\frac{\kappa}{2}$.

- $\mathsf{Encode}(m)$. Given a message of size l, it divides the message into b blocks, and computes n codewords $(s_1, \ldots, s_n)$ using RS codes, such that even when t_κ values are erased, one can recover the original message.
- $\mathsf{Rec}(\mathcal{S}_i, \mathtt{ak}, z, t_\kappa)$ removes incorrect values s_j in $\mathcal{S}_i$ that cannot be verified by the witness w_j and accumulation value z. And then reconstructs the message using RS code, where at most t_κ values are removed.

[5] This is a somewhat simplified discussion; the adversary may still equivocate a message on behalf of the newly corrupted party. The protocol in [1] introduces a special technique to get around this issue.

Protocol $\Pi_{\texttt{sprBA}}$

Let $t_\kappa = \lfloor (1+\varepsilon)\frac{\kappa}{2} \rfloor$. The protocol is described from the point of view of party P_i who holds an l-bit input message m_i.

1: Compute $\mathcal{D}_i \coloneqq (s_1, \ldots, s_\kappa) = \mathsf{Encode}(m_i)$, the accumulation value $z_i = \mathsf{Eval}(\mathsf{ak}, \mathcal{D}_i)$. Input z_i to $\mathsf{BA}(\kappa)$.
2: When the above BA outputs z, if $z = z_i$ and $P_i \in C_1^1$, input 1 to $\mathsf{BA}(1)$. Moreover, distribute the long block as follows. Compute a witness $w_j = \mathsf{CreateWit}(\mathsf{ak}, z, s_j)$ for each share s_j in Step 1 and send the tuple (s_j, w_j) to each party $P_k \in B_j$. Otherwise, if $z \neq z_i$ and $P_i \in C_1^0$, input 0 to $\mathsf{BA}(1)$.
3: If the output of the above BA is 0, output $\perp$ and abort. Otherwise, if $P_i \in C^* \cap B_j$: For the set of tuples $\{(s_j, w_j)\}$ received in the previous step from parties in C_1^1, if there exists an (s_j, w_j) such that $\mathsf{Verify}(\mathsf{ak}, z, w_j, s_j) = 1$, then send (s_j, w_j) to all parties.
4: Let $\mathcal{S}_i \coloneqq \{(s_j, w_j)\}$ be the set of messages received from the previous step from parties in C^*. If there are messages from parties belonging to at least $(1-\varepsilon)\frac{\kappa}{2}$ different buckets, output the reconstructed value $\mathsf{Reconstruct}(\mathcal{S}_i, \mathsf{ak}, z, t_\kappa)$. Otherwise, output $\perp$.

The following theorem will be proven in a sequence of lemmas.

Theorem 3. *Let $0 < \varepsilon < 1/4$. Assuming a setup for VRFs, $\Pi_{\texttt{sprBA}}$ is a synchronous Byzantine agreement protocol secure up to $t \leq (1-3\varepsilon)n/2$ adaptive corruptions. The communication complexity is $O(nl + \kappa^3 n)$ for l-bit values.*

In the proofs, we will need that the sub-protocol $\mathsf{BA}(1)$ satisfies the following somewhat stronger *committee-based* notion of validity described in the lemma below.

Lemma 3. *If all honest parties in C_1^b input b to $\mathsf{BA}(1)$, and no honest party in C_1^{1-b} inputs $1-b$ to $\mathsf{BA}(1)$, then the output of $\mathsf{BA}(1)$ is b.*

Proof. This follows from the fact that in protocol $\mathsf{BA}(1)$ only parties in the committee for the first round, which is C_1^b or C_1^{1-b}, speak and send their input to all other parties. Hence, if only honest parties in C_1^b input to $\mathsf{BA}(1)$ and no honest party from C_1^{1-b} inputs to $\mathsf{BA}(1)$, then it follows immediately from the validity proof given in [1] that the protocol should output b.

Lemma 4. *$\Pi_{\texttt{sprBA}}$ satisfies validity.*

Proof. If all honest parties have the same input message $m_i = m$, then all honest parties input the same accumulated value $z = z_i$ to $\mathsf{BA}(\kappa)$ in Step 1. By validity of $\mathsf{BA}(\kappa)$, all honest parties receive z as output. Hence, all honest parties in C_1^1 input 1 to $\mathsf{BA}(1)$ in Step 2 and distribute the shares of m. By Lemma 3, they receive 1 as output from $\mathsf{BA}(1)$.

Each honest party $P_j \in C^* \cap B_j$ receives a valid share s_j^i from each honest party $P_i \in C_1^1$, and forwards one of these shares to all parties. Parties are added to C^* uniformly at random, each with probability $c\kappa/n$. Denote E_0 the event that fewer than $c\kappa$ parties are in C^*.

Whenever E_0 does not occur, we can map the process of adding parties to C^* to the process of throwing $c\kappa$ or more balls at κ buckets. By Corollary 3, we have that $\mathbf{Pr}[E_0]$ is negligible. Moreover, the optimal strategy for the adversary to minimize the number of buckets in which an honest party sends a share is clearly to corrupt the buckets that contain smaller amount of parties from C^*.

Let us denote E_1 the event that $\frac{c\kappa}{2}(1-2\varepsilon)$ or more parties in C^* are corrupted. By Corollary 1, $\mathbf{Pr}[E_1]$ is negligible. Therefore, by a union bound, $\mathbf{Pr}[E_0 \cup E_1]$ is also negligible.

In the following, we condition on the event $\neg E_0 \wedge \neg E_1$ (which by the above occurs with overwhelming probability).

By Corollary 4, and choosing $\tau = 1/2$, there is a constant $c_{1/2}$ such that $\sum_{i=1}^{c_{1/2}} i \cdot b_i \geq (1-\varepsilon)\,(1/2 - o_\kappa(1)) \cdot c\kappa \geq \frac{c\kappa}{2}(1-2\varepsilon)$, where the last inequality holds as long as $o_\kappa(1) \leq \frac{\varepsilon}{2(1-\varepsilon)}$. Therefore, the adversary can not corrupt all committee members in the buckets that contain up to $c_{1/2}$ or less committee members. These amounts of buckets correspond to at most $\lfloor (1+\varepsilon)\kappa/2 \rfloor$ buckets, by Corollary 4.

Putting things together, at Step 4, at least $\kappa - t_\kappa \geq (1-\varepsilon)\frac{\kappa}{2}$ honest parties in C^* send a share, and thus every honest party receives at least that many valid shares. This way, all honest parties can reconstruct and output the long message m.

Lemma 5. *$\Pi_{\mathtt{sprBA}}$ satisfies consistency.*

Proof. If $\mathsf{BA}(1)$ outputs 0, all honest parties output $\perp$. If $\mathsf{BA}(1)$ outputs 1, then by Lemma 3, there must exist an honest party $P_i \in C_1^1$ that input 1 to $\mathsf{BA}(1)$. First, this party P_i distributes its long messages m_i. Second, by Step 2 of the protocol, it must be the case that this honest party has received $z = z_i$. Using the consistency property of $\mathsf{BA}(\kappa)$, all honest parties must have delivered $z = z_i$. Thus, every honest party $P_j \in C^*$ obtains a valid tuple (s_j, w_j) from P_i and can verify its correctness using the accumulator value z and forward it. Hence, in Step 4, we can use the same argumentation as in the previous lemma to establish that at least $\kappa - t_\kappa$ honest parties in C^* send a share and every honest party can subsequently reconstruct m_i. Note that no other value can be reconstructed, because security of the accumulator and consistency of $\mathsf{BA}(\kappa)$ ensures that all honest parties share the same long message, and dishonest parties cannot compute valid pairs of share-witness different from those received by honest parties.

Communication complexity. The most expensive steps in the protocol are the run of $\mathsf{BA}(\kappa)$ in Step 1 (which itself consists of κ parallel runs of $\mathsf{BA}(1)$) and the distribution of the long blocks in Step 2. The costs for Step 1 are bounded as $O(\kappa^3 \cdot n)$ since every run of $\mathsf{BA}(1)$ costs $O(\kappa^2 \cdot n)$. The costs for Step 2 are bounded by $O(l \cdot n)$. Overall, we obtain a complexity of $O(n \cdot l + \kappa^3 \cdot n)$.

5 Adaptively Secure Asynchronous Communication-Efficient Protocol for Long Messages

We briefly recall the asynchronous adaptively-secure BA protocol of Blum et al. [3]. As for the previous protocol, the step of each round i is performed by a randomly chosen committee C_i, who reveals itself only when it is their turn to speak in the protocol. Again, we assume that parties are endowed (via some trusted setup) with efficient routines $\mathsf{ComProve}$ and ComVer that allow to prove and verify committee membership. The remaining accumulator setup is as for Π_{sprABA} and we also reuse the routines Encode and Rec introduced in the previous section.

Again, we run two versions of the protocol, the first is for κ-valued messages and denoted as $\mathsf{ABA}(\kappa)$, the other for binary-valued messages, and denoted as $\mathsf{ABA}(1)$. Since the protocol in [3] is binary, we simply run it κ many times in parallel to agree on a κ bit message. As before, we choose the committees with expected size $c\kappa$.

Protocol Π_{sprABA}

Let $t_\kappa = \lfloor (1+\varepsilon) \cdot \frac{\kappa}{3} \rfloor$. The protocol is described from the point of view of party P_i who holds an l-bit input message m_i.

1: Compute $\mathcal{D}_i \coloneqq (s_1, \ldots, s_\kappa) = \mathsf{Encode}(m_i)$, the accumulation value $z_i = \mathsf{Eval}(\mathsf{ak}, \mathcal{D}_i)$. Input z_i to $\mathsf{ABA}(\kappa)$.

2: When the above BA outputs z, if $z = z_i$ and $P_i \in C_1$, input 1 to $\mathsf{ABA}(1)$. Moreover, distribute the long block as follows. Compute a witness $w_j = \mathsf{CreateWit}(\mathsf{ak}, z, s_j)$ for each share s_j in Step 1 and send the tuple (s_j, w_j) to each party $P_k \in B_j$. Otherwise, if $z \neq z_i$ and $P_i \in C_1$, input 0 to $\mathsf{ABA}(1)$.

3: If the output of the above BA is 0, output $\perp$ and abort. Otherwise, if $P_i \in C^* \cap B_j$: For the set of tuples $\{(s_j, w_j)\}$ received in the previous step from parties in C_1, if there exists an (s_j, w_j) such that $\mathsf{Verify}(\mathsf{ak}, z, w_j, s_j) = 1$, then send (s_j, w_j) to all parties.

4: Let $\mathcal{S}_i \coloneqq \{(s_j, w_j)\}$ be the set of messages received from the previous step from parties in C^*. If there are messages from parties belonging to at least $\frac{2\kappa}{3} \cdot (1-\varepsilon)$ different buckets, output the reconstructed value $\mathsf{Reconstruct}(\mathcal{S}_i, \mathsf{ak}, z, t_\kappa)$. Otherwise, output $\perp$.

We follow a very similar strategy as in the previous section. In our main theorem statement, we include the cryptographic setup required to run the protocol of Blum et al. [3] without going in to much details as to how they work. Roughly speaking, their protocol starts from an initial setup provided by a trusted dealer. This initial setup allows parties to run a fixed number of multi-party computations (MPCs) and BAs with subquadratic communication complexity. The parties use these cheap (in terms of communication) MPCs to emulate the trusted dealer and refresh the setup for future cheap MPCs and BAs for any number of times. To run MPC with these complexities, their protocol requires strong setup assumptions including threshold fully homomorphic encryption, non-interactive zero knowledge, and anonymous public key encryption (where a ciphertext can not be linked to a public key without knowing the secret key).

Theorem 4. *Let $0 < \varepsilon < 1/4$. Assuming a setup for non-interactive zero-knowledge, threshold fully homomorphic encryption, and anonymous public key encryptions, Π_{sprABA} is an asynchronous Byzantine agreement protocol secure up to $t \leq (1-3\varepsilon)n/3$ adaptive corruptions. The communication complexity is $O(nl + \kappa^6 n)$ for l-bit values.*

The proof of the following lemma is almost identical to that of Lemma 3.

Lemma 6. *If all honest parties in C_1 input b to $\mathsf{ABA}(1)$ then the output of $\mathsf{ABA}(1)$ is b.*

Lemma 7. *Π_{sprABA} satisfies validity if $t \leq (1-3\varepsilon)n/3$ parties are corrupted.*

Proof. If all honest parties have the same input message $m_i = m$, then all honest parties input the same accumulated value $z = z_i$ to $\mathsf{ABA}(\kappa)$ in Step 1. By validity of $\mathsf{ABA}(\kappa)$, all honest parties receive z as output. Hence, all honest parties in C_1 input 1 to $\mathsf{ABA}(1)$ in Step 2 and distribute the shares of m. By Lemma 6, they receive 1 as output from $\mathsf{ABA}(1)$.

Each honest party $P_j \in C^* \cap B_j$ receives a valid share s_j^i from each honest party $P_i \in C_1$, and forwards one of these shares to all parties. Parties are added to C^* uniformly at random via $\mathsf{ComProve}$ with probability $c\kappa/n$. Denote E_0 the event that fewer than $c\kappa$ parties are in C^*.

Whenever E_0 does not occur, we can map the process of adding parties to C^* (via $\mathsf{ComProve}$) to the process of throwing $c\kappa$ or more balls at κ buckets. By Corollary 3, we have that $\mathbf{Pr}[E_0]$ is negligible. Moreover, the optimal strategy for the adversary to minimize the number of buckets in which an honest party sends a share is clearly to corrupt the buckets that contain smaller amount of parties from C^*.

Let us denote E_1 the event that $\frac{c\kappa}{3}(1-2\varepsilon)$ or more parties in C^* are corrupted. By Corollary 2, $\mathbf{Pr}[E_1]$ is negligible. Therefore, by a union bound, $\mathbf{Pr}[E_0 \cup E_1]$ is also negligible.

In the following, we condition on the event $\neg E_0 \wedge \neg E_1$ (which by the above occurs with overwhelming probability).

By Corollary 4, and choosing $\tau = 1/3$, there is a constant $c_{1/3}$ such that $\sum_{i=1}^{c_{1/3}} i \cdot b_i \geq (1-\varepsilon)\cdot(1/3 - o_\kappa(1))\cdot c\kappa \geq \frac{c\kappa}{3}(1-2\varepsilon)$, where the last inequality holds as long as $o_\kappa(1) \leq \frac{\varepsilon}{3(1-\varepsilon)}$. Therefore, the adversary can not corrupt all committee members in the buckets that contain up to $c_{1/3}$ many committee members. These amounts of buckets correspond to at most $\lfloor(1+\varepsilon)\kappa/3\rfloor$ buckets, by Corollary 4.

Putting things together, at Step 4, at least $\kappa - t_\kappa \geq (1-\varepsilon)\frac{2\kappa}{3}$ honest parties in C^* send a share, and thus every honest party receives at least that many valid shares. This way, all honest parties can reconstruct and output the long message m.

The proof of the following lemma is identical as for the synchronous case.

Lemma 8. *Π_{sprABA} satisfies consistency.*

Communication complexity. The most expensive steps in the protocol are the run of $\mathsf{BA}(\kappa)$ in Step 1 (which itself consists of κ parallel runs of $\mathsf{BA}(1)$) and the distribution of the long blocks in Step 2. The costs for Step 1 are bounded as $O(\kappa^6 \cdot n)$ since every run of $\mathsf{BA}(1)$ costs $O(\kappa^5 \cdot n)$. The costs for Step 2 are bounded by $O(l \cdot n)$. Overall, we obtain a complexity of $O(n \cdot l + \kappa^6 \cdot n)$.

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