

# Post-Quantum Security of the Even-Mansour Cipher

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**Abstract.** The Even-Mansour cipher is a simple method for constructing a (keyed) pseudorandom permutation  $E$  from a public random permutation  $P : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . It is a core ingredient in a wide array of symmetric-key constructions, including several lightweight cryptosystems presently under consideration for standardization by NIST. It is secure against classical attacks, with optimal attacks requiring  $q_E$  queries to  $E$  and  $q_P$  queries to  $P$  such that  $q_E \cdot q_P \approx 2^n$ . If the attacker is given *quantum* access to both  $E$  and  $P$ , however, the cipher is completely insecure, with attacks using  $q_E, q_P = O(n)$  queries known.

In any plausible real-world setting, however, a quantum attacker would have only *classical* access to the keyed permutation  $E$  implemented by honest parties, while retaining quantum access to  $P$ . Attacks in this setting with  $q_E \cdot q_P^2 \approx 2^n$  are known, showing that security degrades as compared to the purely classical case, but leaving open the question as to whether the Even-Mansour cipher can still be proven secure in this natural “post-quantum” setting.

We resolve this question, showing that any attack in that setting requires  $q_E \cdot q_P^2 + q_P \cdot q_E^2 \approx 2^n$ . Our results apply to both the two-key and single-key variants of Even-Mansour. Along the way, we establish several generalizations of results from prior work on quantum-query lower bounds that may be of independent interest.

## 1 Introduction

The Even-Mansour cipher [11] is a well-known approach for constructing a block cipher  $E$  from a public random permutation  $P : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . The cipher  $E : \{0, 1\}^{2n} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  is defined as

$$E_{k_1, k_2}(x) = P(x \oplus k_1) \oplus k_2$$

where, at least in the original construction,  $k_1, k_2$  are uniform and independent. Security in the standard (classical) setting is well understood [11, 9]: roughly, an unbounded attacker with access to  $P$  and  $P^{-1}$  cannot distinguish whether it is

interacting with  $E_{k_1, k_2}$  and  $E_{k_1, k_2}^{-1}$  (for uniform  $k_1, k_2$ ) or  $R$  and  $R^{-1}$  (for an independent, random permutation  $R$ ) unless it makes  $\approx 2^{n/2}$  queries to its oracles. A variant where  $k_1$  is uniform and  $k_2 = k_1$  has the same security [9]. These bounds are tight, and key-recovery attacks using  $O(2^{n/2})$  queries are known [11,9].

Unfortunately, the Even-Mansour construction is insecure against a fully quantum attack in which the attacker is given *quantum* access to its oracles [19,16]. In such a setting, the adversary can evaluate the unitary operators

$$\begin{aligned} U_P &: |x\rangle|y\rangle \mapsto |x\rangle|y \oplus P(x)\rangle \\ U_{E_{k_1, k_2}} &: |x\rangle|y\rangle \mapsto |x\rangle|y \oplus E_{k_1, k_2}(x)\rangle \end{aligned}$$

(and the analogous unitaries for  $P^{-1}$  and  $E_{k_1, k_2}^{-1}$ ) on any quantum state it prepares. Here, Simon’s algorithm [21] can be applied to  $E_{k_1, k_2} \oplus P$  to give a key-recovery attack using only  $O(n)$  queries.

To place this seemingly devastating attack in context, it is worth recalling that the original motivation for considering unitary oracles of the form above in quantum-query complexity was that one can always transform a classical circuit for a function  $f$  into a reversible (and hence unitary) quantum circuit for  $U_f$ . In a cryptographic context, it is thus reasonable (indeed, necessary) to allow adversaries to use  $U_f$  whenever  $f$  is a function whose circuit they know. On the other hand, if the circuit for  $f$  is *not* known to the adversary, then there is no mechanism by which it can implement  $U_f$  on its own. In particular, if  $f$  involves a private key, then the only way an adversary could possibly obtain quantum access to  $f$  would be if there were an explicit interface granting such access. In many (if not most) real-world applications, however, the honest parties using the keyed function  $f$  will be using classical computers, and it is then implausible that the adversary could force such devices to behave like quantum computers—especially if the only interface with  $f$  is via a classical communication channel.

In most real-world applications of Even-Mansour, therefore, an attacker would have only *classical* access to the keyed permutation  $E_{k_1, k_2}$  and its inverse, while retaining quantum access to  $P$  and  $P^{-1}$ . In particular, this seems to be the “right” attack model for most applications of the resulting block cipher, e.g., to constructing a secure encryption scheme using some mode of operation. The setting in which the attacker is given classical oracle access to keyed primitives but quantum access to public primitives is sometimes called the “Q1 setting” [5]; we will refer to it simply as the *post-quantum* setting. In this setting, Kuwakado and Morii [19] showed a key-recovery attack on Even-Mansour using the BHT collision-finding algorithm [7] that requires only  $\approx 2^{n/3}$  oracle queries. Their attack uses exponential memory but this was improved in subsequent work [14,5], culminating in an attack using the same number of queries but with polynomial memory complexity. While these results demonstrate that the Even-Mansour construction is *quantitatively* less secure in the post-quantum setting than in the classical setting, they do not answer the more important *qualitative* question of whether the Even-Mansour construction remains secure in the post-quantum setting, or whether attacks using polynomially many queries might be possible.

## 1.1 Our Results

As our main result, we prove a lower bound showing that  $\approx 2^{n/3}$  queries are *necessary* for attacking the Even-Mansour cipher in the post-quantum setting. In more detail, if  $q_P$  denotes the number of (quantum) queries to  $P, P^{-1}$  and  $q_E$  denotes the number of (classical) queries to  $E_{k_1, k_2}, E_{k_1, k_2}^{-1}$ , we show that any attack succeeding with constant probability requires either  $q_P^2 \cdot q_E = \Omega(2^n)$  or  $q_P \cdot q_E^2 = \Omega(2^n)$ . (Equating  $q_P$  and  $q_E$  gives the claimed result.) Formally:

**Theorem 1.** *Let  $\mathcal{A}$  be a quantum algorithm making  $q_E$  classical queries to its first oracle (including forward and inverse queries) and  $q_P$  quantum queries to its second oracle (including forward and inverse queries.) Then*

$$\left| \Pr_{k_1, k_2, P} [\mathcal{A}^{E_{k_1, k_2}, P}(1^n) = 1] - \Pr_{R, P} [\mathcal{A}^{R, P}(1^n) = 1] \right| \leq 10 \cdot 2^{-n/2} \cdot (q_E \sqrt{q_P} + q_P \sqrt{q_E}),$$

where  $P, R$  are uniform  $n$ -bit permutations, and the marginal distributions of  $k_1, k_2 \in \{0, 1\}^n$  are uniform.

The above applies, in particular, to the two-key and one-key variants of the cipher. A simplified version of the proof works also for the case where  $P$  is a random function,  $E_k(x) = P(x \oplus k)$  with  $k$  uniform, and  $\mathcal{A}$  is given forward-only access to both  $P$  and  $E$ .

Real-world attackers are usually assumed to make far fewer queries to keyed, “online” primitives than to public, “offline” primitives. (Indeed, while an offline query is just a local computation, an online query requires, e.g., causing an honest user to encrypt a certain message.) In such a regime, where  $q_E \ll q_P$ , the bound on the adversary’s advantage in [Theorem 1](#) simplifies to  $O(q_P \sqrt{q_E} / 2^{n/2})$ . In that case  $q_P^2 q_E = \Omega(2^n)$  is necessary for constant success probability, which matches the BHT and offline Simon algorithms [\[19, 5\]](#).<sup>1</sup>

**Techniques and new technical results.** Proving [Theorem 1](#) required us to develop new techniques that we believe are interesting beyond our immediate application. We describe the main challenge and its resolution in what follows.

As we have already discussed, in the setting of post-quantum security adversaries may have a combination of classical and quantum oracles. In particular, such situations arise when a post-quantum security notion that involves keyed oracles is analyzed in the quantum random oracle model (QROM), such as when analyzing the Fujisaki-Okamoto transform [\[22, 13, 4, 25, 18, 8\]](#) or the Fiat-Shamir transform [\[23, 17, 12\]](#). In general, this presents a problem: quantum-query lower bounds typically begin by “purifying” the adversary and postponing all measurements to the end of its execution; this does not work if the adversary may decide what query to make to a classical oracle (or even whether to query that

<sup>1</sup> While our bound is tight with respect to the number of queries, it is loose with regard to the attacker’s advantage, as both the BHT and offline Simon algorithms achieve advantage  $\Theta(q_P^2 q_E / 2^n)$ . Reducing this gap is an interesting open question.

oracle at all) *based on the outcome* of an intermediate measurement. The proofs cited above address this problem in various ways, e.g., by simulating the classical oracle or by relaxing the problem and allowing quantum access to *all* oracles.

In our case, in contrast, mixed classical/quantum access to different oracles appears inherent. (In particular, if we allow quantum access to all oracles in our case, an efficient attack becomes possible!) We deal with the problem by dividing the execution of an algorithm that has classical access to  $O_1$  and quantum access to  $O_2$  into *stages*, where a stage corresponds to a period between classical queries to  $O_1$ , and then analyzing the algorithm stage-by-stage. In doing so, however, we introduce another problem: the adversary may adaptively choose the number of queries to  $O_2$  in each stage based on outcomes of intermediate measurements. While it is possible to upper bound the number of queries to  $O_2$  in each stage by the number of queries made to  $O_2$  overall, this will (in general) result in a very loose security bound. To avoid such a loss, we extend the “blinding lemma” of Alagic et al. [1] so that in addition to accommodating two-way accessible permutations (and some other generalizations) we also obtain a bound in terms of the *expected* number of queries made by a distinguisher.

**Lemma 1 (Arbitrary reprogramming, informal).** *Consider the following game played by a distinguisher  $\mathcal{D}$  making at most  $q$  queries in expectation.*

**Phase 1:**  $\mathcal{D}$  outputs a function  $F$  and a randomized algorithm  $\mathcal{B}$  that specifies how to reprogram  $F$ .

**Phase 2:** Randomness  $r$  is sampled and  $\mathcal{B}(r)$  is run to reprogram  $F$ , giving  $F'$ . A uniform  $b \in \{0, 1\}$  is chosen, and  $\mathcal{D}$  receives oracle access to either  $F$  (if  $b = 0$ ) or  $F'$  (if  $b = 1$ ).

**Phase 3:**  $\mathcal{D}$  loses access to its oracle and receives  $r$ ;  $\mathcal{D}$  outputs a bit  $b'$ .

Then  $|\Pr[\mathcal{D} \text{ outputs } 1 \mid b = 0] - \Pr[\mathcal{D} \text{ outputs } 1 \mid b = 1]| \leq 2q \cdot \sqrt{\epsilon}$ , where  $\epsilon$  is an upper bound on the probability that any given input is reprogrammed.

The name “arbitrary reprogramming” is motivated by the fact that  $F$  is completely arbitrary (and known), and the fact that the adversary can reprogram  $F$  arbitrarily—so long as some bound on the probability of reprogramming each individual input exists.

We also extend the “adaptive reprogramming lemma” [12] to the case of two-way-accessible, random permutations:

**Lemma 2 (Resampling for permutations, informal).** *Consider the following game played by a distinguisher  $\mathcal{D}$  making at most  $q$  queries.*

**Phase 1:**  $\mathcal{D}$  makes at most  $q$  (forward or inverse) quantum queries to a uniform permutation  $P : \{0, 1\}^n \rightarrow \{0, 1\}^n$ .

**Phase 2:** A uniform  $b \in \{0, 1\}$  is chosen, and  $\mathcal{D}$  is allowed to make an arbitrary number of queries to an oracle that is either equal to  $P$  (if  $b = 0$ ) or  $P'$  (if  $b = 1$ ), where  $P'$  is obtained from  $P$  by swapping the output values at two uniform points (which are given to  $\mathcal{D}$ .) Then  $\mathcal{D}$  outputs a bit  $b'$ .

Then  $|\Pr[\mathcal{D} \text{ outputs } 1 \mid b = 0] - \Pr[\mathcal{D} \text{ outputs } 1 \mid b = 1]| \leq 4\sqrt{q} \cdot 2^{-n/2}$ .

This is tight up to a constant factor (cf. [12, Theorem 7]). The name “re-sampling lemma” is motivated by the fact that reprogramming is restricted to resampling output values from the *same* distribution used to initially sample outputs of  $P$ . Note that, while Lemma 1 allows for much more general distributions, Lemma 2 gives a bound that is independent of the number of queries  $\mathcal{D}$  makes after the reprogramming occurs.

**Implications for a variant of the Hidden Shift problem.** In the well-studied Hidden Shift problem [24], one is asked to find an unknown shift  $s$  by querying an oracle for a (typically injective) function  $f$  on a group  $G$  and an oracle for its shift  $f_s(x) = f(x \cdot s)$ . If both oracles are classical, this problem has query complexity superpolynomial in  $\log |G|$ . If both oracles are quantum, then the query complexity is polynomial [10] but the algorithmic difficulty appears to depend critically on the structure of  $G$  (e.g., while  $G = \mathbb{Z}_2^n$  is easy [21],  $G = S_n$  appears to be intractable [2]).

The obvious connection between the Hidden Shift problem and Even-Mansour in general groups has been considered before [2,15,6]. In our case, it leads us to define two natural variants of the Hidden Shift problem:

1. “post-quantum” Hidden Shift: the oracle for  $f$  is quantum while the oracle for  $f_s$  is classical;
2. “two-sided” Hidden Shift: in place of  $f_s$ , use  $f_{s_1, s_2}(x) = f(x \cdot s_1) \cdot s_2$ ; if  $f$  is a permutation, grant access to  $f^{-1}$  and  $f_{s_1, s_2}^{-1}$  as well.

These two variants can be considered jointly or separately and, for either variant, one can consider worst-case or average-case settings [2]. Our main result implies:

**Theorem 2 (informal).** *Solving the post-quantum Hidden Shift problem on any group  $G$  requires a number of queries that is superpolynomial in  $\log |G|$ . This holds for both the one-sided and two-sided versions of the problem, and for both the worst-case and the average-case settings.*

Theorem 2 follows from the proof of Theorem 1 via a few straightforward observations. First, an inspection of the proof shows that the particular structure of the underlying group (i.e., the XOR operation on  $\{0, 1\}^n$ ) is not relevant; the proof works identically for any group, simply replacing  $2^n$  with  $|G|$  in the bounds. The two-sided case of Theorem 2 then follows almost immediately: worst-case search is at least as hard as average-case search, and average-case search is at least as hard as average-case decision, which is precisely Theorem 1 (with the appropriate underlying group). Finally, as noted earlier, an appropriate analogue of Theorem 1 also holds in the “forward-only” case where  $E(x) = P(x \oplus k)$  and  $P$  is a random function. This yields the one-sided case of Theorem 2.

**Paper organization.** In Section 2 we state the technical lemmas needed for our main result. In Section 3 we give the full proof of Theorem 1, i.e., post-quantum security of the Even-Mansour cipher (both the two-key and one-key variants), based on the technical lemmas stated informally above (and formally in Section 2). In Section 4 we prove the technical lemmas themselves. Finally, in

Appendix A, we give a proof of post-quantum security for the one-key, “forward-only” variant of Even-Mansour. While this is a relatively straightforward adaptation of the proof of our main result, it does not follow directly from it; moreover, it is substantially simpler and so may serve as a good warm-up for the reader before tackling our main result.

## 2 Reprogramming Lemmas

In this section we collect some technical lemmas that we will need for the proof of Theorem 1. We first discuss a particular extension of the “blinding lemma” of Alagic et al. [1, Theorem 11], which formalizes Lemma 1. We then state a generalization of the “reprogramming lemma” of Grilo et al. [12], which formalizes Lemma 2. The complete proofs of these technical results are given in Section 4.

We frequently consider adversaries with “quantum access” to some function  $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ . This means the adversary is given access to a black-box gate implementing the  $(n + m)$ -qubit unitary operator  $|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle$ .

### 2.1 Arbitrary Reprogramming

Consider a reprogramming experiment that proceeds as follows. First, a distinguisher  $\mathcal{D}$  specifies an arbitrary function  $F$  along with a probabilistic algorithm  $\mathcal{B}$  which describes how to reprogram  $F$ . Specifically, the output of  $\mathcal{B}$  is a set of points  $B_1$  at which  $F$  may be reprogrammed, along with the values the function should take at those potentially reprogrammed points. Then  $\mathcal{D}$  is given quantum oracle access to either  $F$  or the reprogrammed version of  $F$ , its goal being to determine which is the case. Finally, when  $\mathcal{D}$  is done making its oracle queries,  $\mathcal{D}$  is also given the randomness that was used to run  $\mathcal{B}$ . Intuitively, the only way  $\mathcal{D}$  can tell if its oracle has been reprogrammed is by querying with significant amplitude on some point in  $B_1$ . We bound  $\mathcal{D}$ ’s advantage in terms of the probability that any particular value lies in the set  $B_1$  defined by  $\mathcal{B}$ ’s output.

By suitably modifying the proof of Alagic et al. [1, Theorem 11], one can show that the distinguishing probability of  $\mathcal{D}$  in the game described above is at most  $2q \cdot \sqrt{\epsilon}$ , where  $q$  is an upper bound on the number of oracle queries and  $\epsilon$  is an upper bound on the probability that any given input  $x$  is reprogrammed (i.e., that  $x \in B_1$ ). However, that result is only proved for distinguishers with a fixed upper bound on the number of queries they make. To obtain a tighter bound for our application, we need a version of the result for distinguishers that may *adaptively* choose how many queries they make based on outcomes of intermediate measurements. We recover the aforementioned bound if we let  $q$  denote the number of queries made by  $\mathcal{D}$  *in expectation*.

For a set  $B \subset \{0, 1\}^m \times \{0, 1\}^n$  such that each  $w \in \{0, 1\}^m$  is the first element of at most one tuple in  $B$ , and a function  $F : \{0, 1\}^m \rightarrow \{0, 1\}^n$ , define

$$F^{(B)}(w) := \begin{cases} y & \text{if } (w, y) \in B \\ F(w) & \text{otherwise.} \end{cases}$$

We prove the following in [Section 4.1](#):

**Lemma 3 (Formal version of Lemma 1).** *Let  $\mathcal{D}$  be a distinguisher in the following game:*

**Phase 1:**  $\mathcal{D}$  outputs descriptions of a function  $F_0 = F : \{0, 1\}^m \rightarrow \{0, 1\}^n$  and a randomized algorithm  $\mathcal{B}$  whose output is a set  $B \subset \{0, 1\}^m \times \{0, 1\}^n$  where each  $x \in \{0, 1\}^m$  is the first element of at most one tuple in  $B$ . Let  $B_1 \stackrel{\text{def}}{=} \{x \mid \exists y : (x, y) \in B\}$  and  $\epsilon = \max_{x \in \{0, 1\}^m} \{\Pr_{B \leftarrow \mathcal{B}}[x \in B_1]\}$ .

**Phase 2:**  $\mathcal{B}$  is run to obtain  $B$ . Let  $F_1 = F^{(B)}$ . A uniform bit  $b$  is chosen, and  $\mathcal{D}$  is given quantum access to  $F_b$ .

**Phase 3:**  $\mathcal{D}$  loses access to  $F_b$ , and receives the randomness  $r$  used to invoke  $\mathcal{B}$  in phase 2. Then  $\mathcal{D}$  outputs a guess  $b'$ .

For any  $\mathcal{D}$  making  $q$  queries in expectation when its oracle is  $F_0$ , it holds that

$$|\Pr[\mathcal{D} \text{ outputs } 1 \mid b = 1] - \Pr[\mathcal{D} \text{ outputs } 1 \mid b = 0]| \leq 2q \cdot \sqrt{\epsilon}.$$

## 2.2 Resampling

Next, we consider the following experiment: first, a distinguisher  $\mathcal{D}$  is given quantum access to an oracle for a random function  $F$ ; then, in the second stage,  $F$  may be “reprogrammed” so its value on a single, uniform point  $s$  is changed to an independent, uniform value. Because the distribution of  $F(s)$  is the same both before and after any reprogramming, we refer to this as “resampling.” The goal for  $\mathcal{D}$  is to determine whether or not its oracle was reprogrammed. Intuitively, the only way  $\mathcal{D}$  can tell if this is the case—even if it is given  $s$  and unbounded access to the oracle in the second stage—is if  $\mathcal{D}$  happened to put a large amplitude on  $s$  in some query to the oracle in the first stage. The lemmas we state here formalize this intuition.

We begin by establishing notation and recalling a result of Grilo et al. [12]. Given a function  $F : \{0, 1\}^m \rightarrow \{0, 1\}^n$  and  $s \in \{0, 1\}^m$ ,  $y \in \{0, 1\}^n$ , define the “reprogrammed” function  $F_{s \rightarrow y} : \{0, 1\}^m \rightarrow \{0, 1\}^n$  as

$$F_{s \rightarrow y}(w) = \begin{cases} y & \text{if } w = s \\ F(w) & \text{otherwise.} \end{cases}$$

The following is a special case of [12, Prop. 1]:

**Lemma 4 (Resampling for random functions).** *Let  $\mathcal{D}$  be a distinguisher in the following game:*

**Phase 1:** A uniform  $F : \{0, 1\}^m \rightarrow \{0, 1\}^n$  is chosen, and  $\mathcal{D}$  is given quantum access to  $F_0 = F$ .

**Phase 2:** Uniform  $s \in \{0, 1\}^m$ ,  $y \in \{0, 1\}^n$  are chosen, and we let  $F_1 = F_{s \rightarrow y}$ . A uniform bit  $b$  is chosen, and  $\mathcal{D}$  is given  $s$  and quantum access to  $F_b$ . Then  $\mathcal{D}$  outputs a guess  $b'$ .

For any  $\mathcal{D}$  making at most  $q$  queries to  $F_0$  in Phase 1, it holds that

$$|\Pr[\mathcal{D} \text{ outputs } 1 \mid b = 1] - \Pr[\mathcal{D} \text{ outputs } 1 \mid b = 0]| \leq 1.5\sqrt{q/2^m}.$$

We extend the above to the case of two-way accessible, random *permutations*. Now, a random permutation  $P$  is chosen in the first phase; in the second phase,  $P$  may be reprogrammed by swapping the outputs corresponding to two random inputs. For  $a, b \in \{0, 1\}^n$ , let  $\text{swap}_{a,b} : \{0, 1\}^n \rightarrow \{0, 1\}^n$  denote the permutation that maps  $a \mapsto b$  and  $b \mapsto a$  but is otherwise equal to the identity.

**Lemma 5 (Formal version of Lemma 2).** *Let  $\mathcal{D}$  be a distinguisher in the following game:*

**Phase 1:** *A uniform permutation  $P : \{0, 1\}^n \rightarrow \{0, 1\}^n$  is chosen, and  $\mathcal{D}$  is given quantum access to  $P_0 = P$  and  $P_0^{-1} = P^{-1}$ .*

**Phase 2:** *Uniform  $s_0, s_1 \in \{0, 1\}^n$  are chosen, and we let  $P_1 = P \circ \text{swap}_{s_0, s_1}$ . Uniform  $b \in \{0, 1\}$  is chosen, and  $\mathcal{D}$  is given  $s_0, s_1$ , and quantum access to  $P_b, P_b^{-1}$ . Then  $\mathcal{D}$  outputs a guess  $b'$ .*

*For any  $\mathcal{D}$  making at most  $q$  queries (combined) to  $P_0, P_0^{-1}$  in the first phase,  $|\Pr[\mathcal{D} \text{ outputs } 1 \mid b = 1] - \Pr[\mathcal{D} \text{ outputs } 1 \mid b = 0]| \leq 4\sqrt{q/2^n}$ .*

The proof of this lemma is given in [Section 4.2](#).

### 3 Post-Quantum Security of Even-Mansour

We now establish the post-quantum security of the Even-Mansour cipher based on the lemmas from the previous section. Recall that the Even-Mansour cipher is defined as  $E_k(x) := P(x \oplus k_1) \oplus k_2$ , where  $P : \{0, 1\}^n \rightarrow \{0, 1\}^n$  is a public random permutation and  $k = (k_1, k_2) \in \{0, 1\}^{2n}$  is a key. Our proof assumes only that the marginal distributions of  $k_1$  and  $k_2$  are each uniform, i.e., that

$$\Pr[k_1 = \kappa] = \Pr[k_2 = \kappa] = 2^{-n} \quad \forall \kappa.$$

This covers the original Even-Mansour cipher [11] where  $k$  is uniform over  $\{0, 1\}^{2n}$  and the one-key variant [9] where  $k_1$  is uniform and then  $k_2$  is set equal to  $k_1$ .

For  $E_k$  to be efficiently invertible, the permutation  $P$  must itself support efficient inversion; that is, the oracle for  $P$  must be accessible in both the forward and inverse directions. We thus consider adversaries  $\mathcal{A}$  who can access both the cipher  $E_k$  and the permutation  $P$  in both the forward and inverse directions. The goal of  $\mathcal{A}$  is to distinguish this world from the ideal world in which it interacts with independent random permutations  $R, P$ . In this section, it will be implicit in our notation that all oracles are two-way accessible.

In the following, we let  $\mathcal{P}_n$  be the set of all permutations of  $\{0, 1\}^n$ . We write  $E_k[P]$  to denote the Even-Mansour cipher using permutation  $P$  and key  $k$ ; we do this both to emphasize the dependence on  $P$ , and to enable references to Even-Mansour with a permutation other than  $P$ . Our main result is as follows:

**Theorem 3 (Theorem 1, restated).** *Let  $\mathcal{A}$  be an adversary making  $q_E$  classical queries to its first oracle and  $q_P$  quantum queries to its second oracle. Then*

$$\left| \Pr_{\substack{k \leftarrow D \\ P \leftarrow \mathcal{P}_n}} [\mathcal{A}^{E_k[P], P}(1^n) = 1] - \Pr_{R, P \leftarrow \mathcal{P}_n} [\mathcal{A}^{R, P}(1^n) = 1] \right| \leq 10 \cdot 2^{-n/2} (q_E \sqrt{q_P} + q_P \sqrt{q_E}),$$

where  $D$  is any distribution over  $k = (k_1, k_2)$  such that the marginal distributions of  $k_1$  and  $k_2$  are each uniform.

*Proof.* Without loss of generality, we assume  $\mathcal{A}$  never makes a redundant classical query: once it learns an input/output pair  $(x, y)$  by making a query to its classical oracle, it never again submits the query  $x$  (respectively,  $y$ ) to the forward (respectively, inverse) direction of that oracle.

We divide an execution of  $\mathcal{A}$  into  $q_E + 1$  stages  $0, \dots, q_E$ , where the  $j$ th stage corresponds to the time between the  $j$ th and  $(j + 1)$ st classical queries of  $\mathcal{A}$ . In particular, the 0th stage corresponds to the period of time before  $\mathcal{A}$  makes its first classical query, and the  $q_E$ th stage corresponds to the period of time after  $\mathcal{A}$  makes its last classical query. We allow  $\mathcal{A}$  to adaptively distribute its  $q_P$  quantum queries between these stages arbitrarily, provided the total is  $q_P$ . We let  $q_{P,j}$  denote the expected number of queries  $\mathcal{A}$  makes in the  $j$ th stage in the ideal world  $\mathcal{A}^{R,P}$  and we observe that  $\sum_{j=0}^{q_E} q_{P,j} = q_P$ .

Recall that  $\text{swap}_{a,b}$  swaps  $a$  and  $b$  and fixes all other inputs. Given a permutation  $P$ , an ordered list of pairs  $T = ((x_1, y_1), \dots, (x_t, y_t))$ , and a key  $k = (k_1, k_2)$ , define

$$P_{T,k} = \text{swap}_{P(x_1 \oplus k_1), y_1 \oplus k_2} \circ \dots \circ \text{swap}_{P(x_t \oplus k_1), y_t \oplus k_2} \circ P. \quad (1)$$

(If  $T$  is empty, then  $P_{T,k} = P$ .) Note that  $E_k[P_{T,k}](x_i) = y_i$  for all  $i$ . The intuition is as follows: if we first fix a  $P$ , and then select a set  $T$  of observed cipher input-output pairs, then (except for some edge cases)  $P_{T,k}$  is a permutation which appears to be consistent with that cipher and is otherwise nearly equal to  $P$ .

We now define a sequence of experiments  $\mathbf{H}_j$ , for  $j = 0, \dots, q_E$ .

**Experiment  $\mathbf{H}_j$ .** Sample  $R, P \leftarrow \mathcal{P}_n$  and  $k \leftarrow D$ . Then:

1. Run  $\mathcal{A}$ , answering its classical queries using  $R$  and its quantum queries using  $P$ , stopping immediately *before* its  $(j + 1)$ st classical query. Let  $T_j = ((x_1, y_1), \dots, (x_j, y_j))$  be the ordered list of all input/output pairs that  $\mathcal{A}$  received from its classical oracle.
2. For the remainder of the execution of  $\mathcal{A}$ , answer its classical queries using  $E_k[P]$  and its quantum queries using  $P_{T_j,k}$ .

We can compactly represent  $\mathbf{H}_j$  as the experiment in which  $\mathcal{A}$ 's queries are answered using the oracle sequence

$$\underbrace{P, R, P, \dots, R, P}_j \text{ classical queries}, \underbrace{E_k[P], P_{T_j,k}, \dots, E_k[P], P_{T_j,k}}_{q_E - j \text{ classical queries}}.$$

Each appearance of  $R$  or  $E_k[P]$  indicates a single classical query. Each appearance of  $P$  or  $P_{T_j,k}$  indicates a stage during which  $\mathcal{A}$  makes multiple (quantum) queries to that oracle but no queries to its classical oracle. Observe that  $\mathbf{H}_0$  corresponds to the execution of  $\mathcal{A}$  in the real world, i.e.,  $\mathcal{A}^{E_k[P],P}$ , and that  $\mathbf{H}_{q_E}$  is the execution of  $\mathcal{A}$  in the ideal world, i.e.,  $\mathcal{A}^{R,P}$ .

For  $j = 0, \dots, q_E - 1$ , we introduce additional experiments  $\mathbf{H}'_j$ :

**Experiment  $\mathbf{H}'_j$ .** Sample  $R, P \leftarrow \mathcal{P}_n$  and  $k \leftarrow D$ . Then:

1. Run  $\mathcal{A}$ , answering its classical queries using  $R$  and its quantum queries using  $P$ , stopping immediately *after* its  $(j+1)$ st classical query. Let  $T_{j+1} = ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))$  be the ordered list of all input/output pairs that  $\mathcal{A}$  learned from its classical oracle.
2. For the remainder of the execution of  $\mathcal{A}$ , answer its classical queries using  $E_k[P]$  and its quantum queries using  $P_{T_{j+1},k}$ .

Thus,  $\mathbf{H}'_j$  corresponds to running  $\mathcal{A}$  using the oracle sequence

$$\underbrace{P, R, P, \dots, R, P}_j \text{ classical queries}, R, P_{T_{j+1},k}, \underbrace{E_k[P], P_{T_{j+1},k} \dots, E_k[P], P_{T_{j+1},k}}_{q_E - j - 1 \text{ classical queries}}.$$

In Lemmas 6 and 7, we establish bounds on the distinguishability of  $\mathbf{H}'_j$  and  $\mathbf{H}_{j+1}$ , as well as  $\mathbf{H}_j$  and  $\mathbf{H}'_j$ . For  $0 \leq j < q_E$  these give:

$$\begin{aligned} |\Pr[\mathcal{A}(\mathbf{H}'_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}_{j+1}) = 1]| &\leq 2 \cdot q_{P,j+1} \cdot \sqrt{\frac{2 \cdot (j+1)}{2^n}}. \\ |\Pr[\mathcal{A}(\mathbf{H}_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}'_j) = 1]| &\leq 8 \cdot \sqrt{\frac{q_P}{2^n}} + 2q_E \cdot 2^{-n} \end{aligned}$$

Using the above, we have

$$\begin{aligned} &|\Pr[\mathcal{A}(\mathbf{H}_0) = 1] - \Pr[\mathcal{A}(\mathbf{H}_{q_E}) = 1]| \\ &\leq \sum_{j=0}^{q_E-1} \left( 8 \cdot \sqrt{\frac{q_P}{2^n}} + 2q_E \cdot 2^{-n} + 2 \cdot q_{P,j+1} \sqrt{\frac{2 \cdot (j+1)}{2^n}} \right) \\ &\leq 2q_E^2 \cdot 2^{-n} + \sum_{j=0}^{q_E-1} \left( 8 \cdot \sqrt{\frac{q_P}{2^n}} + 2 \cdot q_{P,j+1} \sqrt{\frac{2q_E}{2^n}} \right) \\ &\leq 2q_E^2 \cdot 2^{-n} + 2^{-n/2} \cdot (8q_E \sqrt{q_P} + 2 \cdot q_P \sqrt{2q_E}). \end{aligned}$$

We now simplify the bound further. If  $q_P = 0$ , then  $E_k$  and  $R$  are perfectly indistinguishable and the theorem holds; thus, we may assume  $q_P \geq 1$ . We can also assume  $q_E < 2^{n/2}$  since otherwise the bound is larger than 1. Under these assumptions, we have  $q_E^2 \cdot 2^{-n} \leq q_E \cdot 2^{-n/2} \leq q_E \sqrt{q_P} \cdot 2^{-n/2}$  and so

$$\begin{aligned} &2q_E^2 \cdot 2^{-n} + 2^{-n/2} (8q_E \sqrt{q_P} + 2q_P \sqrt{2q_E}) \\ &\leq 2 \cdot q_E \sqrt{q_P} \cdot 2^{-n/2} + 2^{-n/2} (8q_E \sqrt{q_P} + 2q_P \sqrt{2q_E}) \\ &\leq 10 \cdot 2^{-n/2} (q_E \sqrt{q_P} + q_P \sqrt{q_E}), \end{aligned}$$

as claimed.  $\square$

To complete the proof of [Theorem 3](#), we now establish the two lemmas showing that  $\mathbf{H}'_j$  is close to  $\mathbf{H}_{j+1}$  and  $\mathbf{H}_j$  is close to  $\mathbf{H}'_j$  for  $0 \leq j < q_E$ .

**Lemma 6.** For  $j = 0, \dots, q_E - 1$ ,

$$\left| \Pr[\mathcal{A}(\mathbf{H}'_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}_{j+1}) = 1] \right| \leq 2 \cdot q_{P,j+1} \sqrt{2 \cdot (j+1)/2^n},$$

where  $q_{P,j+1}$  is the expected number of queries  $\mathcal{A}$  makes to  $P$  in the  $(j+1)$ st stage in the ideal world (i.e., in  $\mathbf{H}_{q_E}$ .)

*Proof.* Recall we can write the oracle sequences defined by  $\mathbf{H}'_j$  and  $\mathbf{H}_{j+1}$  as

$$\begin{aligned} \mathbf{H}'_j &: P, R, P, \dots, R, P, \quad R, P_{T_{j+1},k}, \quad E_k[P], P_{T_{j+1},k}, \dots, E_k[P], P_{T_{j+1},k} \\ \mathbf{H}_{j+1} &: \underbrace{P, R, P, \dots, R, P}_j \text{ classical queries}, \quad R, P, \quad \underbrace{E_k[P], P_{T_{j+1},k}, \dots, E_k[P], P_{T_{j+1},k}}_{q_E - j - 1 \text{ classical queries}}. \end{aligned}$$

Let  $\mathcal{A}$  be a distinguisher between  $\mathbf{H}'_j$  and  $\mathbf{H}_{j+1}$ . We construct from  $\mathcal{A}$  a distinguisher  $\mathcal{D}$  for the blinding experiment from [Lemma 3](#):

**Phase 1:**  $\mathcal{D}$  samples  $P, R \leftarrow \mathcal{P}_n$ . It then runs  $\mathcal{A}$ , answering its quantum queries using  $P$  and its classical queries using  $R$ , until after it responds to  $\mathcal{A}$ 's  $(j+1)$ st classical query. Let  $T_{j+1} = ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))$  be the list of input/output pairs  $\mathcal{A}$  received from its classical oracle thus far.  $\mathcal{D}$  defines  $F(t, x) := P^t(x)$  for  $t \in \{1, -1\}$ . It also defines a randomized algorithm  $\mathcal{B}$ , which depends on  $P$  and  $T_{j+1}$ . Given randomness  $k = (k_1, k_2)$ , the algorithm  $\mathcal{B}$  computes the set  $B$  of input-output pairs such that the result of reprogramming  $F$  with  $B$  (i.e.,  $F^{(B)}(t, x)$ ) is equal to  $P_{T_{j+1},k}^t$ .

**Phase 2:**  $\mathcal{B}$  is run using randomness  $k = (k_1, k_2)$  to generate  $B$ , and  $\mathcal{D}$  is given quantum access to an oracle  $F_b$ .  $\mathcal{D}$  resumes running  $\mathcal{A}$ , answering its quantum queries using  $P^t = F_b(t, \cdot)$ . Phase 2 ends when  $\mathcal{A}$  makes its next (i.e.,  $(j+2)$ nd) classical query.

**Phase 3:**  $\mathcal{D}$  is given the randomness used by  $\mathcal{B}$  to generate  $k$ . It resumes running  $\mathcal{A}$ , answering its classical queries using  $E_k[P]$  and its quantum queries using  $P_{T_{j+1},k}$ . Finally, it outputs whatever  $\mathcal{A}$  outputs.

Observe that  $\mathcal{D}$  is a valid distinguisher for the arbitrary reprogramming experiment of [Lemma 3](#). It is immediate that if  $b = 0$  (i.e.,  $\mathcal{D}$ 's oracle in phase 2 is  $F_0 = F$ ), then  $\mathcal{A}$ 's output is identically distributed to its output in  $\mathbf{H}_{j+1}$ , whereas if  $b = 1$  (i.e.,  $\mathcal{D}$ 's oracle in phase 2 is  $F_1 = F^{(B)}$ ), then  $\mathcal{A}$ 's output is identically distributed to its output in  $\mathbf{H}'_j$ . It follows that  $|\Pr[\mathcal{A}(\mathbf{H}'_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}_{j+1}) = 1]|$  is equal to the distinguishing advantage of  $\mathcal{D}$  in the reprogramming experiment. To bound this quantity using [Lemma 3](#), we bound the reprogramming probability and the expected number of queries of  $\mathcal{D}$  in Phase 2 (when  $F = F_0$ .)

The reprogramming probability  $\epsilon$  can be bounded using the definition (1) of  $P_{T_{j+1},k}$  and the fact that  $F^{(B)}(t, x) = P_{T_{j+1},k}^t$ . Fixing  $P$  and  $T_{j+1}$ , the probability

that any given  $(t, x)$  is reprogrammed is at most the probability (over  $k$ ) that it is in the set

$$\{(1, x_i \oplus k_1), (1, P^{-1}(y_i \oplus k_2)), (-1, P(x_i \oplus k_1)), (-1, y_i \oplus k_2)\}_{1 \leq i \leq j+1}.$$

Taking a union bound and applying the fact that the marginal distributions of  $k_1$  and  $k_2$  are both uniform, we get  $\epsilon \leq 2(j+1)/2^n$ .

The expected number of queries made by  $\mathcal{D}$  in Phase 2 when  $F = F_0$  is equal to the expected number of queries made by  $\mathcal{A}$  in its  $(j+1)$ st stage in  $\mathbf{H}_{j+1}$ . Since  $\mathbf{H}_{j+1}$  and  $\mathbf{H}_{q_E}$  are identical until after the  $(j+1)$ st stage is complete, this is precisely  $q_{P,j+1}$ .  $\square$

**Lemma 7.** For  $j = 0, \dots, q_E$ ,

$$|\Pr[\mathcal{A}(\mathbf{H}_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}'_j) = 1]| \leq 8 \cdot \sqrt{\frac{q_P}{2^n}} + 2q_E \cdot 2^{-n}.$$

*Proof.* Recall that we can write the oracle sequences defined by  $\mathbf{H}_j$  and  $\mathbf{H}'_j$  as

$$\begin{aligned} \mathbf{H}_j &: P, R, P, \dots, R, P, \quad E_k[P], P_{T_j, k}, \quad E_k[P], P_{T_j, k}, \dots, E_k[P], P_{T_j, k} \\ \mathbf{H}'_j &: \underbrace{P, R, P, \dots, R, P}_{j \text{ classical queries}}, R, \quad P_{T_{j+1}, k}, \quad \underbrace{E_k[P], P_{T_{j+1}, k}, \dots, E_k[P], P_{T_{j+1}, k}}_{q_E - j - 1 \text{ classical queries}}. \end{aligned}$$

Let  $\mathcal{A}$  be a distinguisher between  $\mathbf{H}_j$  and  $\mathbf{H}'_j$ . We construct from  $\mathcal{A}$  a distinguisher  $\mathcal{D}$  for the reprogramming experiment of [Lemma 5](#):

**Phase 1:**  $\mathcal{D}$  is given quantum access to a permutation  $P$ . It samples  $R \leftarrow \mathcal{P}_n$  and then runs  $\mathcal{A}$ , answering its quantum queries with  $P$  and its classical queries with  $R$  (in the appropriate directions), until<sup>2</sup>  $\mathcal{A}$  submits its  $(j+1)$ st classical query  $x_{j+1}$ . At that point,  $\mathcal{D}$  has a list  $T_j = ((x_1, y_1), \dots, (x_j, y_j))$  of the input/output pairs  $\mathcal{A}$  has received from its classical oracle thus far.

**Phase 2:** Now  $\mathcal{D}$  receives  $s_0, s_1 \in \{0, 1\}^n$  and quantum oracle access to a permutation  $P_b$ . Then  $\mathcal{D}$  sets  $k_1 := s_0 \oplus x_{j+1}$ , chooses  $k_2 \leftarrow D_{|k_1}$  (where this represents the conditional distribution on  $k_2$  given  $k_1$ ), and sets  $k := (k_1, k_2)$ .  $\mathcal{D}$  continues running  $\mathcal{A}$ , answering its remaining classical queries (including the  $(j+1)$ st one) using  $E_k[P_b]$ , and its remaining quantum queries using

$$(P_b)_{T_j, k} = \text{swap}_{P_b(x_1 \oplus k_1), y_1 \oplus k_2} \circ \dots \circ \text{swap}_{P_b(x_j \oplus k_1), y_j \oplus k_2} \circ P_b.$$

Finally,  $\mathcal{D}$  outputs whatever  $\mathcal{A}$  outputs.

Note that although  $\mathcal{D}$  needs to make additional queries to  $P_b$  at the start of Phase 2 (to determine  $P_b(x_1 \oplus k_1), \dots, P_b(x_j \oplus k_1)$ ), the bound of [Lemma 5](#) only depends on the number of quantum queries  $\mathcal{D}$  makes in Phase 1, which is at most  $q_P$ .

<sup>2</sup> We assume for simplicity that this query is in the forward direction, but the case where it is in the inverse direction can be handled entirely symmetrically (using the fact that the marginal distribution of  $k_2$  is uniform).

We now analyze the execution of  $\mathcal{D}$  in the two cases of the game of [Lemma 5](#):  $b = 0$  (no reprogramming) and  $b = 1$  (reprogramming). In both cases,  $P$  and  $R$  are independent, uniform permutations, and  $\mathcal{A}$  is run with quantum oracle  $P$  and classical oracle  $R$  until it makes its  $(j + 1)$ st classical query; thus, through the end of Phase 1, the above execution of  $\mathcal{A}$  is consistent with both  $\mathbf{H}_j$  and  $\mathbf{H}'_j$ .

At the start of Phase 2, uniform  $s_0, s_1 \in \{0, 1\}^n$  are chosen. Since  $\mathcal{D}$  sets  $k_1 := s_0 \oplus x_{j+1}$ , the distribution of  $k_1$  is uniform and hence  $k$  is distributed according to  $D$ . The two cases ( $b = 0$  and  $b = 1$ ) now begin to diverge.

**Case  $b = 0$  (no reprogramming).** In this case,  $\mathcal{A}$ 's remaining classical queries (including its  $(j + 1)$ st classical query) are answered using  $E_k[P_0] = E_k[P]$ , and its remaining quantum queries are answered using  $(P_0)_{T_j, k} = P_{T_j, k}$ . The output of  $\mathcal{A}$  is thus distributed identically to its output in  $\mathbf{H}_j$  in this case.

**Case  $b = 1$  (reprogramming).** In this case, we have

$$P_b = P_1 = P \circ \text{swap}_{s_0, s_1} = \text{swap}_{P(s_0), P(s_1)} \circ P = \text{swap}_{P(x_{j+1} \oplus k_1), P(s_1)} \circ P. \quad (2)$$

The response to  $\mathcal{A}$ 's  $(j + 1)$ st classical query is thus

$$y_{j+1} \stackrel{\text{def}}{=} E_k[P_1](x_{j+1}) = P_1(x_{j+1} \oplus k_1) \oplus k_2 = P_1(s_0) \oplus k_2 = P(s_1) \oplus k_2. \quad (3)$$

The remaining classical queries of  $\mathcal{A}$  are then answered using  $E_k[P_1]$ , while its remaining quantum queries are answered using  $(P_1)_{T_j, k}$ . If we let  $\text{Expt}_j$  refer to the experiment in which  $\mathcal{D}$  executes  $\mathcal{A}$  as a subroutine when  $b = 1$ , it follows from [Lemma 5](#) that

$$|\Pr[\mathcal{A}(\mathbf{H}_j) = 1] - \Pr[\mathcal{A}(\text{Expt}_j) = 1]| \leq 4\sqrt{q_P/2^n}. \quad (4)$$

We now define three events:

1.  $\text{bad}_1$  is the event that  $y_{j+1} \in \{y_1, \dots, y_j\}$ .
2.  $\text{bad}_2$  is the event that  $s_1 \oplus k_1 \in \{x_1, \dots, x_j\}$ .
3.  $\text{bad}_3$  is the event that, in phase 2,  $\mathcal{A}$  queries its classical oracle in the forward direction on  $s_1 \oplus k_1$ , or the inverse direction on  $P(s_0) \oplus k_2$  (with result  $s_1 \oplus k_1$ ).

Since  $y_{j+1} = P(s_1) \oplus k_2$  is uniform (because  $k_2$  is uniform and independent of  $P$  and  $s_1$ ), it is immediate that  $\Pr[\text{bad}_1] \leq j/2^n$ . Similarly,  $s_1 \oplus k_1 = s_1 \oplus s_0 \oplus x_{j+1}$  is uniform, and so  $\Pr[\text{bad}_2] \leq j/2^n$ . As for the last event, we have:

*Claim.*  $\Pr[\text{bad}_3] \leq (q_E - j)/2^n + 4\sqrt{q_P/2^n}$ .

*Proof.* Consider the algorithm  $\mathcal{D}'$  that behaves identically to  $\mathcal{D}$  in phases 1 and 2, but then when  $\mathcal{A}$  terminates outputs 1 iff event  $\text{bad}_3$  occurred. When  $b = 0$  (no reprogramming), the execution of  $\mathcal{A}$  is independent of  $s_1$ , and so the probability that  $\text{bad}_3$  occurs is at most  $(q_E - j)/2^n$ . Now observe that  $\mathcal{D}'$  is a distinguisher for the reprogramming game of [Lemma 5](#), with advantage  $|\Pr[\text{bad}_3|b = 1] - (q_E - j)/2^n|$ . The claim then follows from [Lemma 5](#).  $\square$

<pre> 1 <math>P, R \leftarrow \mathcal{P}_n</math> 2 Run <math>\mathcal{A}</math> with quantum access to <math>P</math> and classical access to <math>R</math>, until <math>\mathcal{A}</math> makes its    (<math>j+1</math>)st classical query <math>x_{j+1}</math>; let <math>T_j</math> be as in the text 3 <math>s_0, s_1 \leftarrow \{0, 1\}^n</math>, <math>P_1 := P \circ \text{swap}_{s_0, s_1}</math> 4 <math>k_1 := s_0 \oplus x_{j+1}</math>, <math>k_2 \leftarrow D_{ k_1}</math>, <math>k := (k_1, k_2)</math> 5 <math>y_{j+1} := E_k[P_1](x_{j+1})</math> 6 <math>Q := (P_1)_{T_j, k}</math> 7 <b>if</b> <math>y_{j+1} \in \{y_1, \dots, y_j\}</math> <b>then</b> <math>\text{bad}_1 := \text{true}</math>, <span style="border: 1px solid black; padding: 2px;"><math>y_{j+1} \leftarrow \{0, 1\}^n \setminus \{y_1, \dots, y_j\}</math></span> 8 Give <math>y_{j+1}</math> to <math>\mathcal{A}</math> as the answer to its (<math>j+1</math>)st classical query 9 <math>T_{j+1} := ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))</math> 10 <b>if</b> <math>s_1 \oplus k_1 \in \{x_1, \dots, x_j\}</math> <b>then</b> <math>\text{bad}_2 := \text{true}</math> 11 <b>if</b> <math>\text{bad}_1 = \text{true}</math> <b>or</b> <math>\text{bad}_2 = \text{true}</math> <b>then</b> <span style="border: 1px solid black; padding: 2px;"><math>Q := P_{T_{j+1}, k}</math></span> 12 Continue running <math>\mathcal{A}</math> with quantum access to <math>Q</math> and classical access to <math>\mathcal{O}/\mathcal{O}^{-1}</math> </pre>	<pre> 13 <math>\mathcal{O}(x)</math> 14 <math>y := E_k[P_1](x)</math> 15 <b>if</b> <math>x = s_1 \oplus k_1</math> <b>then</b> 16 <span style="border: 1px solid black; padding: 2px;"><math>\text{bad}_3 := \text{true}</math>, <math>y := E_k[P](x)</math></span> 17 <b>return</b> <math>y</math> </pre>
<pre> 18 <math>\mathcal{O}^{-1}(y)</math> 19 <math>x := E_k^{-1}[P_1](y)</math> 20 <b>if</b> <math>x = s_1 \oplus k_1</math> <b>then</b> 21 <span style="border: 1px solid black; padding: 2px;"><math>\text{bad}_3 := \text{true}</math>, <math>x := E_k^{-1}[P](y)</math></span> 22 <b>return</b> <math>x</math> </pre>	

**Fig. 1.**  $\text{Expt}'_j$  includes the boxed statements, whereas  $\text{Expt}_j$  does not.

In Figure 1, we show code for  $\text{Expt}_j$  and a related experiment  $\text{Expt}'_j$ . Note that  $\text{Expt}_j$  and  $\text{Expt}'_j$  are identical until either  $\text{bad}_1$ ,  $\text{bad}_2$ , or  $\text{bad}_3$  occur, and so by the fundamental lemma of game playing [3]<sup>3</sup> we have

$$\begin{aligned}
|\Pr[\mathcal{A}(\text{Expt}'_j) = 1] - \Pr[\mathcal{A}(\text{Expt}_j) = 1]| &\leq \Pr[\text{bad}_1 \vee \text{bad}_2 \vee \text{bad}_3] \\
&\leq 2q_E/2^n + 4\sqrt{q_P/2^n}. \quad (5)
\end{aligned}$$

We complete the proof by arguing that  $\text{Expt}'_j$  is identical to  $\mathbf{H}'_j$ :

1. In  $\text{Expt}'_j$ , the oracle  $Q$  used in line 12 is always equal to  $P_{T_{j+1}, k}$ . When  $\text{bad}_1$  or  $\text{bad}_2$  occurs this is immediate (since then  $Q$  is set to  $P_{T_{j+1}, k}$  in line 11). But if  $\text{bad}_1$  does not occur then Equation (3) holds, and if  $\text{bad}_2$  does not occur then for  $i = 1, \dots, j$  we have  $x_i \oplus k_1 \neq s_0$  and  $x_i \oplus k_1 \neq s_1$  (where the former is because  $x_{j+1} \oplus k_1 = s_0$  but  $x_i \neq x_{j+1}$  by assumption, and the latter is by definition of  $\text{bad}_2$ ). Thus  $P_1(x_i \oplus k_1) = P(x_i \oplus k_1)$  for  $i = 1, \dots, j$ ,

<sup>3</sup> Note that the fundamental lemma of game playing is an information-theoretic result. As  $\text{bad}_1$ ,  $\text{bad}_2$  and  $\text{bad}_3$  are classical probabilistic events in the execution of  $\mathcal{D}$ , we can thus apply this result in our (quantum) setting.

and so

$$\begin{aligned}
Q &= (P_1)_{T_j, k} = \text{swap}_{P_1(x_1 \oplus k_1), y_1 \oplus k_2} \circ \cdots \circ \text{swap}_{P_1(x_j \oplus k_1), y_j \oplus k_2} \circ P_1 \\
&= \text{swap}_{P(x_1 \oplus k_1), y_1 \oplus k_2} \circ \cdots \circ \text{swap}_{P(x_{j+1} \oplus k_1), y_{j+1} \oplus k_2} \circ P \\
&= P_{T_{j+1}, k},
\end{aligned}$$

using Equations (2) and (3).

2. In  $\text{Expt}'_j$ , the value  $y_{j+1}$  is uniformly distributed in  $\{0, 1\}^n \setminus \{y_1, \dots, y_j\}$ . Indeed, we have already argued above that the value  $y_{j+1}$  computed in line 14 is uniform in  $\{0, 1\}^n$ . But if that value lies in  $\{y_1, \dots, y_j\}$  (and so  $\text{bad}_1$  occurs) then  $y_{j+1}$  is re-sampled uniformly from  $\{0, 1\}^n \setminus \{y_1, \dots, y_j\}$  in line 7.
3. In  $\text{Expt}'_j$ , the response from oracle  $\mathcal{O}(x)$  is always equal to  $E_k[P](x)$ . When  $\text{bad}_3$  occurs this is immediate. But if  $\text{bad}_3$  does not occur then  $x \neq s_1 \oplus k_1$ ; we also know that  $x \neq s_0 \oplus k_1 = x_{j+1}$  by assumption. But then  $P_1(x \oplus k_1) = P(x \oplus k_1)$  and so  $E_k[P_1](x) = E_k[P](x)$ . A similar argument shows that the response from  $\mathcal{O}^{-1}(y)$  is always  $E_k^{-1}[P](y)$ .

Syntactically rewriting  $\text{Expt}'_j$  using the above observations yields an experiment that is identical to  $\mathbf{H}'_j$ . (See Appendix B for further details.) Lemma 7 thus follows from Equations (4) and (5).  $\square$

## 4 Proofs of the Technical Lemmas

In this section, we give the proofs of our technical lemmas: the “arbitrary reprogramming lemma” (Lemma 3) and the “resampling lemma” (Lemma 5).

### 4.1 Proof of the Arbitrary Reprogramming Lemma

Lemma 3 allows for distinguishers that choose the number of queries they make adaptively, e.g., depending on the oracle provided and the outcomes of any measurements, and the bound is in terms of the number of queries  $\mathcal{D}$  makes *in expectation*. As discussed in Section 1.1, the ability to directly handle adaptive distinguishers is necessary for our proof, and to our knowledge has not been addressed before. To formally reason about adaptive distinguishers, we model the intermediate operations of the distinguisher and the measurements it makes as *quantum channels*. With this as our goal, we first recall some necessary background and establish some notation.

Recall that a density matrix  $\rho$  is a positive semidefinite matrix with unit trace. A quantum channel—the most general transformation between density matrices allowed by quantum theory—is a completely positive, trace-preserving, linear map. The quantum channel corresponding to the unitary operation  $U$  is the map  $\rho \mapsto U\rho U^\dagger$ . Another type of quantum channel is a *pinching*, which corresponds to the operation of making a measurement. Specializing to the only kind of pinching needed in our proof, consider the measurement of a single-qubit

register  $C$  given by the projectors  $\{\Pi_0, \Pi_1\}$  with  $\Pi_b = |b\rangle\langle b|_C$ . This corresponds to the pinching  $\mathcal{M}_C$  where

$$\mathcal{M}_C(\rho) = \Pi_0\rho\Pi_0 + \Pi_1\rho\Pi_1.$$

Observe that a pinching only produces the post-measurement state, and does not separately give the outcome (i.e., the result 0 or 1).

Consider a quantum algorithm  $\mathcal{D}$  with access to an oracle  $\mathcal{O}$  operating on registers  $X, Y$  (so  $\mathcal{O}|x\rangle|y\rangle = |x\rangle|y \oplus \mathcal{O}(x)\rangle$ ). We define the unitary  $c\mathcal{O}$  for the *controlled* version of  $\mathcal{O}$ , operating on registers  $C, X$ , and  $Y$  (with  $C$  a single-qubit register), as

$$c\mathcal{O}|c\rangle|x\rangle|y\rangle = |c\rangle|x\rangle|y \oplus c \cdot \mathcal{O}(x)\rangle.$$

With this in place, we may now view an execution of  $\mathcal{D}^{\mathcal{O}}$  as follows. The algorithm uses registers  $C, X, Y$ , and  $E$ . Let  $q_{\max}$  be an upper bound on the number of queries  $\mathcal{D}$  ever makes. Then  $\mathcal{D}$  applies the quantum channel

$$(\Phi \circ c\mathcal{O} \circ \mathcal{M}_C)^{q_{\max}} \tag{6}$$

to some initial state  $\rho = \rho_0^{(0)}$ . That is, for each of  $q_{\max}$  iterations,  $\mathcal{D}$  applies to its current state the pinching  $\mathcal{M}_C$  followed by the controlled oracle  $c\mathcal{O}$  and then an arbitrary quantum channel  $\Phi$  (that we take to be the same in all iterations without loss of generality<sup>4</sup>) operating on all its registers. Finally,  $\mathcal{D}$  applies a measurement to produce its final output. If we let  $\rho_{i-1}^{(0)}$  denote the intermediate state immediately before the pinching is applied in the  $i$ th iteration, then  $p_{i-1} = \text{Tr} \left[ |1\rangle\langle 1|_C \rho_{i-1}^{(0)} \right]$  represents the probability that the oracle is applied (or, equivalently, that a query is made) in the  $i$ th iteration, and so  $q = \sum_{i=1}^{q_{\max}} p_{i-1}$  is the expected number of queries made by  $\mathcal{D}$  when interacting with oracle  $\mathcal{O}$ .

**Proof of Lemma 3.** An execution of  $\mathcal{D}$  takes the form of Equation (6) up to a final measurement. For some fixed value of the randomness  $r$  used to run  $\mathcal{B}$ , set  $\Upsilon_b = \Phi \circ c\mathcal{O}_{F_b} \circ \mathcal{M}_C$ , and define

$$\rho_k \stackrel{\text{def}}{=} \left( \Upsilon_1^{q_{\max}-k} \circ \Upsilon_0^k \right) (\rho),$$

so that  $\rho_k$  is the final state if the first  $k$  queries are answered using a (controlled)  $F_0$  oracle and then the remaining  $q_{\max} - k$  queries are answered using a (controlled)  $F_1$  oracle. Furthermore, we define  $\rho_i^{(0)} = \Upsilon_0^i(\rho)$ . Note also that  $\rho_{q_{\max}}$  (resp.,  $\rho_0$ ) is the final state of the algorithm when the  $F_0$  oracle (resp.,  $F_1$  oracle) is used the entire time. We bound  $\mathbb{E}_r \left[ \delta \left( |r\rangle\langle r| \otimes \rho_{q_{\max}}, |r\rangle\langle r| \otimes \rho_0 \right) \right]$ , where  $\delta(\cdot, \cdot)$  denotes the trace distance.

Define  $\tilde{F}^{(B)}(x) = F(x) \oplus F^{(B)}(x)$ , and note that  $\tilde{F}^{(B)}(x) = 0^n$  for  $x \notin B_1$ . Since the trace distance is non-increasing under quantum channels, for any  $r$  we

<sup>4</sup> This can be done by having a register serve as a counter that is incremented with each application of  $\Phi$ .

have

$$\begin{aligned}\delta(|r\rangle\langle r| \otimes \rho_k, |r\rangle\langle r| \otimes \rho_{k-1}) &\leq \delta\left(c\mathcal{O}_{F_0} \circ \mathcal{M}_C\left(\rho_{k-1}^{(0)}\right), c\mathcal{O}_{F_1} \circ \mathcal{M}_C\left(\rho_{k-1}^{(0)}\right)\right) \\ &= \delta\left(\mathcal{M}_C\left(\rho_{k-1}^{(0)}\right), c\mathcal{O}_{\tilde{F}(B)} \circ \mathcal{M}_C\left(\rho_{k-1}^{(0)}\right)\right).\end{aligned}$$

By definition of a controlled oracle,

$$\begin{aligned}c\mathcal{O}_{\tilde{F}(B)} \circ \mathcal{M}_C\left(\rho_{k-1}^{(0)}\right) &= c\mathcal{O}_{\tilde{F}(B)}\left(|1\rangle\langle 1|_C \rho_{k-1}^{(0)} |1\rangle\langle 1|_C + |0\rangle\langle 0|_C \rho_{k-1}^{(0)} |0\rangle\langle 0|_C\right) \\ &= \mathcal{O}_{\tilde{F}(B)}\left(|1\rangle\langle 1|_C \rho_{k-1}^{(0)} |1\rangle\langle 1|_C\right) + |0\rangle\langle 0|_C \rho_{k-1}^{(0)} |0\rangle\langle 0|_C,\end{aligned}$$

and thus

$$\begin{aligned}\delta\left(\mathcal{M}_C\left(\rho_{k-1}^{(0)}\right), c\mathcal{O}_{\tilde{F}(B)} \circ \mathcal{M}_C\left(\rho_{k-1}^{(0)}\right)\right) &= \delta\left(|1\rangle\langle 1|_C \rho_{k-1}^{(0)} |1\rangle\langle 1|_C, \mathcal{O}_{\tilde{F}(B)}\left(|1\rangle\langle 1|_C \rho_{k-1}^{(0)} |1\rangle\langle 1|_C\right)\right) \\ &= p_{k-1} \cdot \delta\left(\sigma_{k-1}, \mathcal{O}_{\tilde{F}(B)}\left(\sigma_{k-1}\right)\right)\end{aligned}$$

where, recall,  $p_{k-1} = \text{Tr}\left[|1\rangle\langle 1|_C \rho_{k-1}^{(0)}\right]$  is the probability that a query is made in the  $k$ th iteration, and we define the normalized state  $\sigma_{k-1} \stackrel{\text{def}}{=} \frac{|1\rangle\langle 1|_C \rho_{k-1}^{(0)} |1\rangle\langle 1|_C}{p_{k-1}}$ . Therefore,

$$\begin{aligned}\mathbb{E}_r\left[\delta(|r\rangle\langle r| \otimes \rho_{q_{\max}}, |r\rangle\langle r| \otimes \rho_0)\right] &\leq \sum_{k=1}^{q_{\max}} \mathbb{E}_B\left[\delta(|r\rangle\langle r| \otimes \rho_k, |r\rangle\langle r| \otimes \rho_{k-1})\right] \\ &\leq \sum_{k=1}^{q_{\max}} p_{k-1} \cdot \mathbb{E}_B\left[\delta(\sigma_{k-1}, \mathcal{O}_{\tilde{F}(B)}(\sigma_{k-1}))\right] \\ &\leq q \cdot \max_{\sigma} \mathbb{E}_B\left[\delta(\sigma, \mathcal{O}_{\tilde{F}(B)}(\sigma))\right],\end{aligned}\tag{7}$$

where we write  $\mathbb{E}_B$  for the expectation over the set  $B$  output by  $\mathcal{B}$  in place of  $\mathbb{E}_r$ .

Since  $\sigma$  can be purified to some state  $|\psi\rangle$ , and  $\delta(|\psi\rangle, |\psi'\rangle) \leq \|\psi - \psi'\|_2$  for pure states  $|\psi\rangle, |\psi'\rangle$ , we have

$$\begin{aligned}\max_{\sigma} \mathbb{E}_B\left[\delta(\sigma, \mathcal{O}_{\tilde{F}(B)}(\sigma))\right] &\leq \max_{|\psi\rangle} \mathbb{E}_B\left[\delta(|\psi\rangle, \mathcal{O}_{\tilde{F}(B)}|\psi\rangle)\right] \\ &\leq \max_{|\psi\rangle} \mathbb{E}_B\left[\|\psi - \mathcal{O}_{\tilde{F}(B)}|\psi\rangle\|_2\right].\end{aligned}$$

Because  $\mathcal{O}_{\tilde{F}(B)}$  acts as the identity on  $(\mathbb{I} - \Pi_{B_1})|\psi\rangle$  for any  $|\psi\rangle$ , we have

$$\begin{aligned}\mathbb{E}_B\left[\|\psi - \mathcal{O}_{\tilde{F}(B)}|\psi\rangle\|_2\right] &= \mathbb{E}_B\left[\|\Pi_{B_1}|\psi\rangle - \mathcal{O}_{\tilde{F}(B)}\Pi_{B_1}|\psi\rangle + (\mathbb{I} - \mathcal{O}_{\tilde{F}(B)})(\mathbb{I} - \Pi_{B_1})|\psi\rangle\|_2\right] \\ &\leq \mathbb{E}_B\left[\|\Pi_{B_1}|\psi\rangle\|_2\right] + \mathbb{E}_B\left[\|\mathcal{O}_{\tilde{F}(B)}\Pi_{B_1}|\psi\rangle\|_2\right] \\ &= 2 \cdot \mathbb{E}_B\left[\|\Pi_{B_1}|\psi\rangle\|_2\right] \\ &\leq 2\sqrt{\mathbb{E}_B\left[\|\Pi_{B_1}|\psi\rangle\|_2^2\right]},\end{aligned}\tag{8}$$

using Jensen's inequality in the last step. Let  $|\psi\rangle = \sum_{x \in \{0,1\}^m, y \in \{0,1\}^n} \alpha_{x,y} |x\rangle |y\rangle$  where  $\|\psi\|_2^2 = \sum_{x,y} \alpha_{x,y}^2 = 1$ . Then

$$\begin{aligned} \mathbb{E}_B [\|\Pi_{B_1} |\psi\rangle\|_2^2] &= \mathbb{E}_B \left[ \sum_{x,y: x \in B_1} \alpha_{x,y}^2 \right] \\ &= \sum_{x,y} \alpha_{x,y}^2 \cdot \Pr[x \in B_1] \leq \epsilon. \end{aligned} \quad (9)$$

Together with Equations (7) and (8), this gives the desired result.  $\square$

## 4.2 Proof of the Resampling Lemma

We begin by introducing a superposition-oracle technique based on the one by Zhandry [25], but different in that our oracle represents a two-way accessible, uniform permutation (rather than a uniform function). We also do not need to “compress” the oracle, as an inefficient representation suffices for our purposes.

For an arbitrary function  $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , define the state

$$|f\rangle_F = \bigotimes_{x \in \{0,1\}^n} |f(x)\rangle_{F_x},$$

where  $F$  is the collection of registers  $\{F_x\}_{x \in \{0,1\}^n}$ . We represent an evaluation of  $f$  via an operator  $O$  whose action on the computational basis is given by

$$O_{XYF} |x\rangle_X |y\rangle_Y |f\rangle_F = \text{CNOT}_{F_x:Y}^{\otimes n} |x\rangle_X |y\rangle_Y |f\rangle_F = |x\rangle_X |y \oplus f(x)\rangle_Y |f\rangle_F,$$

where  $X, Y$  are  $n$ -qubit registers. Handling inverse queries to  $f$  is more difficult. We want to define an inverse operator  $O^{\text{inv}}$  such that, for any permutation  $\pi$ ,

$$O_{XYF}^{\text{inv}} |\pi\rangle_F = \left( \sum_{x,y \in \{0,1\}^n} |y\rangle_Y \langle y|_Y \otimes \mathbf{X}_X^x \otimes |y\rangle_Y \langle y|_{F_x} \right) |\pi\rangle_F \quad (10)$$

(where  $\mathbf{X}$  is the Pauli-X operator, and for  $x \in \{0, 1\}^n$  we let  $\mathbf{X}^x := \mathbf{X}^{x_1} \otimes \mathbf{X}^{x_2} \otimes \dots \otimes \mathbf{X}^{x_n}$  so that  $\mathbf{X}^x |\hat{x}\rangle = |\hat{x} \oplus x\rangle$ ); then,

$$O_{XYF}^{\text{inv}} |x\rangle_X |y\rangle_Y |\pi\rangle_F = |x \oplus \pi^{-1}(y)\rangle_X |y\rangle_Y |\pi\rangle_F.$$

In order for  $O^{\text{inv}}$  to be a well-defined unitary operator, however, we must extend its definition to the entire space of functions. A convenient extension is given by the following action on arbitrary computational basis states:

$$O_{XYF}^{\text{inv}} = \prod_{x' \in \{0,1\}^n} \left( \mathbf{X}_X^{x'} \otimes |y\rangle_Y \langle y|_{F_{x'}} + (\mathbf{1} - |y\rangle_Y \langle y|_{F_{x'}}) \right),$$

so that

$$O_{XYF}^{\text{inv}} |x\rangle_X |y\rangle_Y |f\rangle_F = |x \oplus (\oplus_{x': f(x')=y} x')\rangle_X |y\rangle_Y |f\rangle_F.$$

In other words, the inverse operator XORs all preimages (under  $f$ ) of the value in register  $Y$  into the contents of register  $X$ .

We may view a uniform permutation as a uniform superposition over all permutations in  $\mathcal{P}_n$ ; i.e., we model a uniform permutation as the state

$$|\phi_0\rangle_F = (2^n!)^{-\frac{1}{2}} \sum_{\pi \in \mathcal{P}_n} |\pi\rangle_F.$$

The final state of any oracle algorithm  $\mathcal{D}$  is identically distributed whether we (1) sample uniform  $\pi \in \mathcal{P}_n$  and then run  $\mathcal{D}$  with access to  $\pi$  and  $\pi^{-1}$ , or (2) run  $\mathcal{D}$  with access to  $O$  and  $O^{\text{inv}}$  after initializing the  $F$ -registers to  $|\phi_0\rangle_F$  (and, if desired, at the end of its execution, measure the  $F$ -registers to obtain  $\pi$  and the residual state of  $\mathcal{D}$ ).

Our proof relies on the following lemma, which is a special case of the conclusion of implication ( $\diamond'$ ) in [20]. (Here and in the following, we denote the complementary projector of a projector  $P$  by  $\bar{P} \stackrel{\text{def}}{=} \mathbb{1} - P$ .)

**Lemma 8 (Gentle measurement lemma).** *Let  $|\psi\rangle$  be a quantum state and let  $\{P_i\}_{i=1}^q$  be a collection of projectors with  $\|\bar{P}_i|\psi\rangle\|_2^2 \leq \epsilon_i$  for all  $i$ . Then*

$$1 - \left| \langle \psi | \left( \prod_{i=1}^q P_i \right) | \psi \rangle \right|^2 \leq \sum_{i=1}^q \epsilon_i.$$

**Proof of Lemma 5.** We split the distinguisher  $\mathcal{D}$  into two stages  $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1)$  corresponding to the first and second phases of the experiment in Lemma 5. As discussed above, we run the experiment using the superposition oracle  $|\phi_0\rangle_F$  and then measure the  $F$ -registers at the end. Informally, our goal is to show that on average over the choice of reprogrammed positions  $s_0, s_1$ , the adversary-oracle state after  $\mathcal{D}_0$  finishes is almost invariant under the reprogramming operation (i.e., the swap of registers  $F_{s_0}$  and  $F_{s_1}$ ) unless  $\mathcal{D}_0$  makes a large number of oracle queries. This will follow from Lemma 8 because, on average over the choice of  $s_0, s_1$ , any particular query of  $\mathcal{D}_0$  (whether using  $O$  or  $O^{\text{inv}}$ ) only involves  $F_{s_0}$  or  $F_{s_1}$  with negligible amplitude.

We begin by defining the following projectors

$$(P_{s_0 s_1})_X = \begin{cases} \mathbb{1} & s_0 = s_1 \\ \mathbb{1} - |s_0\rangle\langle s_0| - |s_1\rangle\langle s_1| & s_0 \neq s_1 \end{cases}$$

$$(P_{s_0 s_1}^{\text{inv}})_{FY} = \begin{cases} \mathbb{1} & s_0 = s_1 \\ \sum_{y \in \{0,1\}^n} |y\rangle\langle y|_Y \otimes (\mathbb{1} - |y\rangle\langle y|)_{F_{s_0} F_{s_1}}^{\otimes 2} & s_0 \neq s_1. \end{cases}$$

It is straightforward to verify that for any  $s_0, s_1$ :

$$\left[ \text{Swap}_{F_{s_0} F_{s_1}}, O_{XYF} (P_{s_0 s_1})_X \right] = 0 \quad (11)$$

$$\left[ \text{Swap}_{F_{s_0} F_{s_1}}, O_{XYF}^{\text{inv}} (P_{s_0 s_1}^{\text{inv}})_{FY} \right] = 0, \quad (12)$$

where  $[\cdot, \cdot]$  denotes the commutator operation, and  $\text{Swap}_{AB}$  is the swap operator (i.e.,  $\text{Swap}_{A,B}|x\rangle_A|x'\rangle_B = |x'\rangle_A|x\rangle_B$  if the target registers  $A, B$  are distinct, and the identity if  $A$  and  $B$  refer to the same register). In words, this means that if we project a forward query to inputs other than  $s_0, s_1$ , then swapping the outputs of a function at  $s_0$  and  $s_1$  before evaluating that function has no effect; this holds also if we project an inverse query (for some associated function  $f$ ) to the set of output values that are not equal to  $f(s_0)$  nor  $f(s_1)$ .

Since  $\bar{P}_{s_0 s_1} \stackrel{\text{def}}{=} \mathbb{1} - P_{s_0 s_1} \leq |s_0\rangle\langle s_0| + |s_1\rangle\langle s_1|$  it follows that for any normalized state  $|\psi\rangle_{XE}$  of register  $X$  and an arbitrary register  $E$ ,

$$\begin{aligned} \mathbb{E}_{s_0, s_1} \left\| (\bar{P}_{s_0 s_1})_X |\psi\rangle_{XE} \right\|_2^2 &\leq \mathbb{E}_{s_0, s_1} \langle \psi | (|s_0\rangle\langle s_0| + |s_1\rangle\langle s_1|) | \psi \rangle \\ &= 2 \cdot 2^{-n}. \end{aligned} \quad (13)$$

It turns out that we can make a similar statement about  $P_{s_0 s_1}^{\text{inv}}$ . Let  $|\psi\rangle_{YXEF}$  be the query state of an adversary interacting with the inverse interface of the superposition oracle. We can write

$$|\psi\rangle_{YXEF} = \sum_{y \in \{0,1\}^n} \sum_{\pi \in \mathcal{P}_n} c_{y\pi} (2^n!)^{-\frac{1}{2}} |y\rangle_Y |\psi_{y\pi}\rangle_{XE} |\pi\rangle_F,$$

for some quantum states  $\{|\psi_{y\pi}\rangle\}_{y,\pi}$ , with  $\sum_{y \in \{0,1\}^n} |c_{y\pi}|^2 = 1$  for all  $\pi \in \mathcal{P}_n$ . Suppose  $s_0 \neq s_1$ . The projector  $P_{s_0, s_1}^{\text{inv}}$  is diagonal in the computational basis, so

$$\begin{aligned} &\left\| (\bar{P}_{s_0 s_1}^{\text{inv}})_{YF} |\psi\rangle_{YXEF} \right\|_2^2 \\ &= \langle \psi |_{YXEF} (\bar{P}_{s_0 s_1}^{\text{inv}})_{YF} |\psi\rangle_{YXEF} \\ &= \sum_{y \in \{0,1\}^n} \sum_{\pi \in \mathcal{P}_n} |c_{y\pi}|^2 (2^n!)^{-1} \langle \pi |_F \left( \mathbb{1} - (\mathbb{1} - |y\rangle\langle y|)_{F_{s_0} F_{s_1}}^{\otimes 2} \right) |\pi\rangle_F \\ &= \sum_{y \in \{0,1\}^n} \sum_{\substack{\pi \in \mathcal{P}_n: \\ y \in \{\pi(s_0), \pi(s_1)\}}} |c_{y\pi}|^2 (2^n!)^{-1} \\ &= \sum_{\substack{\pi \in \mathcal{P}_n: \\ y \in \{\pi(s_0), \pi(s_1)\}}} (2^n!)^{-1} \\ &= \frac{2(2^n - 1)!}{2^n!} = 2 \cdot 2^{-n}. \end{aligned}$$

(The fourth inequality uses the normalization of the coefficients  $c_{y\pi}$ .) If  $s_0 = s_1$ , then  $\left\| (\bar{P}_{s_0 s_1}^{\text{inv}})_{YF} |\psi\rangle_{YXEF} \right\|_2^2 = 0 \leq 2 \cdot 2^{-n}$ , so in summary we have

$$\left\| (\bar{P}_{s_0 s_1}^{\text{inv}})_{YF} |\psi\rangle_{YXEF} \right\|_2^2 \leq 2 \cdot 2^{-n} \quad (14)$$

for all  $s_0, s_1$ . Without loss of generality, we assume  $\mathcal{D}_0$  starts with initial state  $|\psi_0\rangle = |\psi'_0\rangle|\phi_0\rangle$  (which we take to include the superposition oracle's initial state  $|\phi_0\rangle$ ), computes the state

$$|\psi\rangle = U_{\mathcal{D}_0} |\psi_0\rangle = U_q O_q U_{q-1} O_{q-1} \dots U_1 O_1 |\psi_0\rangle,$$

and outputs all its registers as a state register  $E$ . Here, each  $O_i \in \{O, O^{\text{inv}}\}$  acts on registers  $XYF$ , and each  $U_j$  acts on registers  $XYE$ . To each choice of  $s_0, s_1$  we assign a decomposition  $|\psi\rangle = |\psi_{\text{good}}(s_0, s_1)\rangle + |\psi_{\text{bad}}(s_0, s_1)\rangle$  by defining

$$|\psi_{\text{good}}(s_0, s_1)\rangle = zU_q O_q P_{s_0 s_1}^q U_{q-1} O_{q-1} P_{s_0 s_1}^{q-1} \dots U_1 O_1 P_{s_0 s_1}^1 |\psi_0\rangle,$$

where  $P_{s_0 s_1}^i = P_{s_0 s_1}$  if  $O_i = O$ ,  $P_{s_0 s_1}^i = P_{s_0 s_1}^{\text{inv}}$  if  $O_i = O^{\text{inv}}$ , and  $z \in \mathbb{C}$  such that  $|z| = 1$  and  $\langle \psi | \psi_{\text{good}}(s_0, s_1) \rangle \in \mathbb{R}_{\geq 0}$ . Now note that

$$|\psi_{\text{good}}(s_0, s_1)\rangle = zU_{\mathcal{D}_0} \left( \prod_{i=1}^q Q_{s_0 s_1}^i \right) |\psi_0\rangle,$$

with  $Q_{s_0 s_1}^i = \tilde{U}_i^\dagger P_{s_0 s_1}^i \tilde{U}_i$  for  $\tilde{U}_i = U_{i-1} O_{i-1} \dots U_1 O_1$ . We would like to use [Lemma 8](#), so let

$$\epsilon_i(x_0, x_1) = \|\bar{Q}_{s_0 s_1}^i |\psi_0\rangle\|_2^2 = \|\bar{P}_{s_0 s_1}^i \tilde{U}_i |\psi_0\rangle\|_2^2.$$

Applying [Lemma 8](#) yields

$$1 - |\langle \psi | \psi_{\text{good}}(s_0, s_1) \rangle|^2 \leq \sum_{i=1}^q \epsilon_i(s_0, s_1).$$

We will now analyze the impact of reprogramming the superposition oracle after  $\mathcal{D}_0$  has finished. Recall that reprogramming swaps the values of the permutation at points  $s_0$  and  $s_1$ , which is implemented in the superposition-oracle framework by applying  $\text{Swap}_{F_{s_0} F_{s_1}}$ . Note that  $\text{Swap}_{F_{s_0} F_{s_1}} |\phi_0\rangle = |\phi_0\rangle$ . As the adversary's internal unitaries  $U_i$  do not act on  $F$ , [Equations \(11\) and \(12\)](#) then imply that

$$\text{Swap}_{F_{s_0} F_{s_1}} |\psi_{\text{good}}(s_0, s_1)\rangle = |\psi_{\text{good}}(s_0, s_1)\rangle.$$

The standard formula for the trace distance of pure states yields

$$\frac{1}{2} \left\| |\psi\rangle\langle\psi| - \text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle\langle\psi| \text{Swap}_{F_{s_0} F_{s_1}} \right\|_1 = \sqrt{1 - \left| \langle \psi | \text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle \right|^2}. \quad (15)$$

We further have

$$\begin{aligned} \left| \langle \psi | \text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle \right| &= \left| \langle \psi | \psi \rangle + \langle \psi_{\text{bad}}(s_0, s_1) | \left( \text{Swap}_{F_{s_0} F_{s_1}} - \mathbf{1} \right) |\psi_{\text{bad}}(s_0, s_1)\rangle \right| \\ &\geq 1 - 2 \|\psi_{\text{bad}}(s_0, s_1)\|_2^2 \end{aligned} \quad (16)$$

using the triangle and Cauchy-Schwarz inequalities. Combining [Equations \(15\) and \(16\)](#) we obtain

$$\frac{1}{2} \left\| |\psi\rangle\langle\psi| - \text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle\langle\psi| \text{Swap}_{F_{s_0} F_{s_1}} \right\|_1 \leq 2 \cdot \|\psi_{\text{bad}}(s_0, s_1)\|_2. \quad (17)$$

But as  $|\psi_{\text{bad}}(s_0, s_1)\rangle = |\psi\rangle - |\psi_{\text{good}}(s_0, s_1)\rangle$ , we have

$$\begin{aligned} \|\psi_{\text{bad}}(s_0, s_1)\|_2^2 &= 2 - 2 \cdot \text{Re} \langle \psi | \psi_{\text{good}}(s_0, s_1) \rangle \\ &= 2 - 2 \cdot |\langle \psi | \psi_{\text{good}}(s_0, s_1) \rangle| \\ &\leq 2 \sum_{i=1}^q \epsilon_i(s_0, s_1). \end{aligned} \quad (18)$$

Combining the last two equations we obtain

$$\frac{1}{2} \left\| |\psi\rangle\langle\psi| - \text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle\langle\psi| \text{Swap}_{F_{s_0} F_{s_1}} \right\|_1 \leq 2\sqrt{2} \sqrt{\sum_{i=1}^q \epsilon_i(s_0, s_1)}. \quad (19)$$

The remainder of the proof is the same as the analogous part of the proof of [12, Theorem 6].  $\mathcal{D}_1$ 's task boils down to distinguishing the states  $|\psi\rangle$  and  $\text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle$ , for uniform  $s_0, s_1$  that  $\mathcal{D}_1$  receives as input, using the limited set of instructions allowed by the superposition oracle. We can therefore bound  $\mathcal{D}$ 's advantage by the maximum distinguishing advantage for these two states achievable when using arbitrary quantum computation, averaged over the choice of  $s_0, s_1$ . Using the standard formula for this maximum distinguishing advantage we obtain

$$\begin{aligned} \Pr[\mathcal{D} \text{ outputs } b] - \frac{1}{2} &\leq \frac{1}{4} \mathbb{E}_{s_0, s_1} \left[ \left\| |\psi\rangle\langle\psi| - \text{Swap}_{F_{s_0} F_{s_1}} |\psi\rangle\langle\psi| \text{Swap}_{F_{s_0} F_{s_1}} \right\|_1 \right] \\ &\leq \sqrt{2} \mathbb{E}_{s_0, s_1} \left[ \sqrt{\sum_{i=1}^q \epsilon_i(s_0, s_1)} \right] \\ &\leq \sqrt{2} \sqrt{\mathbb{E}_{s_0, s_1} \left[ \sum_{i=1}^q \epsilon_i(s_0, s_1) \right]} \leq 2\sqrt{\frac{q}{2^n}}, \end{aligned}$$

where the second inequality is Equation (19), the third inequality is Jensen's inequality, and the last inequality is due to Equations (13) and (14). This implies the lemma.  $\square$

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## A Security of Forward-Only Even-Mansour

In this section we consider a simpler case, where  $E_k[F](x) := F(x \oplus k)$  for  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$  a uniform *function* and  $k$  a uniform  $n$ -bit string. Here we restrict the adversary to forward queries only, i.e., the adversary has classical access to  $E_k[F]$  and quantum access to  $F$ ; note that  $E_k^{-1}[F]$  and  $F^{-1}$  may not even be well-defined.

We let  $\mathcal{F}_n$  denote the set of all functions from  $\{0, 1\}^n$  to  $\{0, 1\}^n$ .

**Theorem 4.** *Let  $\mathcal{A}$  be a quantum algorithm making  $q_E$  classical queries to its first oracle and  $q_F$  quantum queries to its second oracle. Then*

$$\left| \Pr_{\substack{k \leftarrow \{0, 1\}^n \\ F \leftarrow \mathcal{F}_n}} \left[ \mathcal{A}^{E_k[F], F}(1^n) = 1 \right] - \Pr_{R, F \leftarrow \mathcal{F}_n} \left[ \mathcal{A}^{R, F}(1^n) = 1 \right] \right| \leq 2^{-n/2} \cdot (2q_E \sqrt{q_F} + 2q_F \sqrt{q_E}).$$

*Proof.* We make the same assumptions about  $\mathcal{A}$  as in the initial paragraphs of the proof of [Theorem 3](#). We also adopt analogous notation for the stages of  $\mathcal{A}$ , now using  $q_E$ ,  $q_F$ , and  $q_{F,j}$  as appropriate.

Given a function  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , a set  $T$  of pairs where any  $x \in \{0, 1\}^n$  is the first element of at most one pair in  $T$ , and a key  $k \in \{0, 1\}^n$ , we define the function  $F_{T,k} : \{0, 1\}^n \rightarrow \{0, 1\}^n$  as

$$F_{T,k}(x) := \begin{cases} y & \text{if } (x \oplus k, y) \in T \\ F(x) & \text{otherwise.} \end{cases}$$

Note that, in contrast to the analogous definition in [Theorem 3](#), here the order of the tuples in  $T$  does not matter and so we may take it to be a set. Note also that we are redefining the notation  $F_{T,k}$  from how it was used in [Theorem 3](#); this new usage applies to this Appendix only.

We now define a sequence of experiments  $\mathbf{H}_j$ , for  $j = 0, \dots, q_E$ :

**Experiment  $\mathbf{H}_j$ .** Sample  $R, F \leftarrow \mathcal{F}_n$  and  $k \leftarrow \{0, 1\}^n$ . Then:

1. Run  $\mathcal{A}$ , answering its classical queries using  $R$  and its quantum queries using  $F$ , stopping immediately before its  $(j + 1)$ st classical query. Let  $T_j = \{(x_1, y_1), \dots, (x_j, y_j)\}$  be the set of all classical queries made by  $\mathcal{A}$  thus far and their corresponding responses.
2. For the remainder of the execution of  $\mathcal{A}$ , answer its classical queries using  $E_k[F]$  and its quantum queries using  $F_{T_j,k}$ .

We can represent  $\mathbf{H}_j$  as the experiment in which  $\mathcal{A}$ 's queries are answered using the oracle sequence

$$\underbrace{F, R, F, \dots, R, F}_j \text{ classical queries}, \underbrace{E_k[F], F_{T_j,k}, \dots, E_k[F], F_{T_j,k}}_{q_E - j \text{ classical queries}}.$$

Note that  $\mathbf{H}_0$  is exactly the real world (i.e.,  $\mathcal{A}^{E_k[F], F}$ ) and  $\mathbf{H}_{q_E}$  is exactly the ideal world (i.e.,  $\mathcal{A}^{R, F}$ ).

For  $j = 0, \dots, q_E - 1$ , we define an additional experiment  $\mathbf{H}'_j$ :

**Experiment  $\mathbf{H}'_j$ .** Sample  $R, F \leftarrow \mathcal{F}_n$  and  $k \leftarrow \{0, 1\}^n$ . Then:

1. Run  $\mathcal{A}$ , answering its classical queries using  $R$  and its quantum queries using  $F$ , stopping immediately after its  $(j + 1)$ st classical query. Let  $T_{j+1} = ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))$  be the set of all classical queries made by  $\mathcal{A}$  thus far and their corresponding responses.
2. For the remainder of the execution of  $\mathcal{A}$ , answer its classical queries using  $E_k[F]$  and its quantum queries using  $F_{T_{j+1},k}$ .

I.e.,  $\mathbf{H}'_j$  corresponds to answering  $\mathcal{A}$ 's queries using the oracle sequence

$$\underbrace{F, R, F, \dots, R, F}_j \text{ classical queries}, \underbrace{R, F_{T_{j+1},k}, E_k[F], F_{T_{j+1},k}, \dots, E_k[F], F_{T_{j+1},k}}_{q_E - j - 1 \text{ classical queries}}.$$

We now show that  $\mathbf{H}'_j$  is close to  $\mathbf{H}_{j+1}$  and  $\mathbf{H}_j$  is close to  $\mathbf{H}'_j$  for  $0 \leq j < q_E$ .

**Lemma 9.** For  $j = 0, \dots, q_E - 1$ ,

$$|\Pr[\mathcal{A}(\mathbf{H}'_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}_{j+1}) = 1]| \leq 2 \cdot q_{F,j+1} \sqrt{(j+1)/2^n}.$$

*Proof.* Given an adversary  $\mathcal{A}$ , we construct a distinguisher  $\mathcal{D}$  for the “blinding game” of Lemma 3 that works as follows:

**Phase 1:**  $\mathcal{D}$  samples  $F, R \leftarrow \mathcal{F}_n$ . It then runs  $\mathcal{A}$ , answering its quantum queries with  $F$  and its classical queries with  $R$ , until it replies to  $\mathcal{A}$ 's  $(j+1)$ st classical query. Let  $T_{j+1} = \{(x_1, y_1), \dots, (x_{j+1}, y_{j+1})\}$  be the set of classical queries made by  $\mathcal{A}$  and their responses.  $\mathcal{D}$  defines algorithm  $\mathcal{B}$  as follows: on randomness  $k \in \{0, 1\}^n$ , output  $B = \{(x_j \oplus k, y_j)\}_{j=1}^{j+1}$ . Finally,  $\mathcal{D}$  outputs  $F$  and  $\mathcal{B}$ .

**Phase 2:**  $\mathcal{D}$  is given quantum access to a function  $F_b$ . It continues to run  $\mathcal{A}$ , answering its quantum queries with  $F_b$  until  $\mathcal{A}$  makes its next classical query.

**Phase 3:**  $\mathcal{D}$  is given the randomness  $k$  used to run  $\mathcal{B}$ . It continues running  $\mathcal{A}$ , answering its classical queries with  $E_k[F]$  and its quantum queries with  $F_{T_{j+1},k}$ . Finally,  $\mathcal{D}$  outputs whatever  $\mathcal{A}$  outputs.

When  $b = 0$  (so  $F_b = F_0 = F$ ), then  $\mathcal{A}$ 's output is identically distributed to its output in  $\mathbf{H}_{j+1}$ . On the other hand, when  $b = 1$  then  $F_b = F_1 = F^{(B)} = F_{T_{j+1},k}$  and so  $\mathcal{A}$ 's output is identically distributed to its output in  $\mathbf{H}'_j$ . The expected number of queries made by  $\mathcal{D}$  in phase 2 when  $F = F_0$  is the expected number of queries made by  $\mathcal{A}$  in stage  $(j+1)$  in  $\mathbf{H}_{j+1}$ . Since  $\mathbf{H}_{j+1}$  and  $\mathbf{H}_{q_E}$  are identical until after the  $(j+1)$ st stage, this is precisely  $q_{F,j+1}$ . Because  $k$  is uniform, we can apply Lemma 3 with  $\epsilon = (j+1)/2^n$ . The lemma follows.  $\square$

**Lemma 10.** For  $j = 0, \dots, q_E$ ,

$$|\Pr[\mathcal{A}(\mathbf{H}_j) = 1] - \Pr[\mathcal{A}(\mathbf{H}'_j) = 1]| \leq 1.5 \cdot \sqrt{q_F/2^n}.$$

*Proof.* From any adversary  $\mathcal{A}$ , we construct a distinguisher  $\mathcal{D}$  for the game of Lemma 4.  $\mathcal{D}$  works as follows:

**Phase 1:**  $\mathcal{D}$  is given quantum access to a (random) function  $F$ . It samples  $R \leftarrow \mathcal{F}_n$  and then runs  $\mathcal{A}$ , answering its quantum queries using  $F$  and its classical queries using  $R$ , until  $\mathcal{A}$  submits its  $(j+1)$ st classical query  $x_{j+1}$ . At that point, let  $T_j = \{(x_1, y_1), \dots, (x_j, y_j)\}$  be the set of input/output pairs  $\mathcal{A}$  has received from its classical oracle thus far.

**Phase 2:**  $\mathcal{D}$  is given (uniform)  $s \in \{0, 1\}^n$  and quantum oracle access to a function  $F_b$ .  $\mathcal{D}$  sets  $k := s \oplus x_{j+1}$ , and then continues running  $\mathcal{A}$ , answering its classical queries (including the  $(j+1)$ st) using  $E_k[F_b]$  and its quantum queries using the function  $(F_b)_{T_j,k}$ , i.e.,

$$x \mapsto \begin{cases} y & \text{if } (x \oplus k, y) \in T_j \\ F_b(x) & \text{otherwise.} \end{cases}$$

Finally,  $\mathcal{D}$  outputs whatever  $\mathcal{A}$  outputs.

We analyze the execution of  $\mathcal{D}$  in the two cases of the game of [Lemma 4](#). In either case, the quantum queries of  $\mathcal{A}$  in stages  $0, \dots, j$  are answered using a random function  $F$ , and  $\mathcal{A}$ 's first  $j$  classical queries are answered using an independent random function  $R$ . Note further that since  $s$  is uniform, so is  $k$ .

**Case 1:**  $b = 0$ . In this case, all the remaining classical queries of  $\mathcal{A}$  (i.e., from the  $(j + 1)$ st on) are answered using  $E_k[F]$ , and the remaining quantum queries of  $\mathcal{A}$  are answered using  $F_{T_j, k}$ . The output of  $\mathcal{A}$  is thus distributed identically to its output in  $\mathbf{H}_j$  in this case.

**Case 2:**  $b = 1$ . Here,  $F_b = F_1 = F_{s \rightarrow y}$  for a uniform  $y$ . Now, the response to the  $(j + 1)$ st classical query of  $\mathcal{A}$  is

$$E_k[F_b](x_{j+1}) = E_k[F_{s \rightarrow y}](x_{j+1}) = F_{s \rightarrow y}(k \oplus x_{j+1}) = F_{s \rightarrow y}(s) = y.$$

Since  $y$  is uniform and independent of anything else, and since  $\mathcal{A}$  has never previously queried  $x_{j+1}$  to its classical oracle, this is equivalent to answering the first  $j + 1$  classical queries of  $\mathcal{A}$  using a random function  $R$ . The remaining classical queries of  $\mathcal{A}$  are also answered using  $E_k[F_{s \rightarrow y}]$ . However, since  $E_k[F_{s \rightarrow y}](x) = E_k[F](x)$  for all  $x \neq x_{j+1}$  and  $\mathcal{A}$  never repeats the query  $x_{j+1}$ , this is equivalent to answering the remaining classical queries of  $\mathcal{A}$  using  $E_k[F]$ .

The remaining quantum queries of  $\mathcal{A}$  are answered with the function

$$x \mapsto \begin{cases} y' & \text{if } (x \oplus k, y') \in T_j \\ F_{s \rightarrow y}(x) & \text{otherwise.} \end{cases}$$

This, in turn, is precisely the function  $F_{T_{j+1}, k}$ , where  $T_{j+1}$  is obtained by adding  $(x_{j+1}, y)$  to  $T_j$  (and thus consists of the first  $j + 1$  classical queries made by  $\mathcal{A}$  and their corresponding responses). Thus, the output of  $\mathcal{A}$  in this case is distributed identically to its output in  $\mathbf{H}'_j$ .

The number of quantum queries made by  $\mathcal{D}$  in phase 1 is at most  $q_F$ . The claimed result thus follows from [Lemma 4](#).  $\square$

Using [Lemmas 9](#) and [10](#), and the fact that  $\sum_{j=1}^{q_E} q_{F,j} = q_F$ , we have

$$\begin{aligned} |\Pr[\mathcal{A}(\mathbf{H}_0) = 1] - \Pr[\mathcal{A}(\mathbf{H}_{q_E}) = 1]| &\leq 1.5q_E \sqrt{q_F/2^n} + 2 \sum_{j=1}^{q_E} q_{F,j} \sqrt{j/2^n} \\ &\leq 1.5q_E \sqrt{q_F/2^n} + 2\sqrt{q_E/2^n} \sum_{j=1}^{q_E} q_{F,j} \\ &\leq 1.5q_E \sqrt{q_F/2^n} + 2q_F \sqrt{q_E/2^n}, \end{aligned}$$

as required.  $\square$

## B Further Details for [Lemma 7](#)

The code in the top portion of [Figure 2](#) is a syntactic rewriting of  $\text{Expt}'_j$ . (Flags that have no effect on the output of  $\mathcal{A}$  are omitted.) In [line 27](#), the computation

of  $y_{j+1}$  has been expanded (note that  $E_k[P_1](x_{j+1}) = P_1(s_0) \oplus k_2 = P(s_1) \oplus k_2$ ). In line 31,  $Q$  has been replaced with  $P_{T_{j+1},k}$  and  $\mathcal{O}$  has been replaced with  $E_k[P]$  as justified in the proof of Lemma 7.

The code in the middle portion of Figure 2 results from the following changes: first, rather than sampling uniform  $s_0$  and then setting  $k_1 := s_0 \oplus x_{j+1}$ , the code now samples a uniform  $k_1$ . Similarly, rather than choosing uniform  $s_1$  and then setting  $y_{j+1} := P(s_1) \oplus k_2$ , the code now samples a uniform  $y_{j+1}$  (note that  $P$  is a permutation, so  $P(s_1)$  is uniform). Since neither  $s_0$  nor  $s_1$  is used anywhere else, each can now be omitted.

The code in the bottom portion of Figure 2 simply chooses  $k = (k_1, k_2)$  according to distribution  $D$ , and chooses uniform  $y_{j+1} \in \{0, 1\}^n \setminus \{y_1, \dots, y_j\}$ . It can be verified by inspection that this final experiment is equivalent to  $\mathbf{H}'_j$ .

<p>23 <math>P, R \leftarrow \mathcal{P}_n</math>  24 Run <math>\mathcal{A}</math> with quantum access to <math>P</math> and classical access to <math>R</math>, until <math>\mathcal{A}</math> makes its <math>(j+1)</math>st classical query <math>x_{j+1}</math>; let <math>T_j</math> be as in the text  25 <math>s_0, s_1 \leftarrow \{0, 1\}^n</math>  26 <math>k_1 := s_0 \oplus x_{j+1}</math>, <math>k_2 \leftarrow D_{ k_1}</math>, <math>k := (k_1, k_2)</math>  27 <math>y_{j+1} := P(s_1) \oplus k_2</math>  28 <b>if</b> <math>y_{j+1} \in \{y_1, \dots, y_j\}</math> <b>then</b> <math>y_{j+1} \leftarrow \{0, 1\}^n \setminus \{y_1, \dots, y_j\}</math>  29 Give <math>y_{j+1}</math> to <math>\mathcal{A}</math> as the answer to its <math>(j+1)</math>st classical query  30 <math>T_{j+1} := ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))</math>  31 Continue running <math>\mathcal{A}</math> with quantum access to <math>P_{T_{j+1},k}</math> and classical access to <math>E_k[P]</math></p>
<p>32 <math>P, R \leftarrow \mathcal{P}_n</math>  33 Run <math>\mathcal{A}</math> with quantum access to <math>P</math> and classical access to <math>R</math>, until <math>\mathcal{A}</math> makes its <math>(j+1)</math>st classical query <math>x_{j+1}</math>; let <math>T_j</math> be as in the text  34 <math>k_1 \leftarrow \{0, 1\}^n</math>, <math>k_2 \leftarrow D_{ k_1}</math>, <math>k := (k_1, k_2)</math>, <math>y_{j+1} \leftarrow \{0, 1\}^n</math>  35 <b>if</b> <math>y_{j+1} \in \{y_1, \dots, y_j\}</math> <b>then</b> <math>y_{j+1} \leftarrow \{0, 1\}^n \setminus \{y_1, \dots, y_j\}</math>  36 Give <math>y_{j+1}</math> to <math>\mathcal{A}</math> as the answer to its <math>(j+1)</math>st classical query  37 <math>T_{j+1} := ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))</math>  38 Continue running <math>\mathcal{A}</math> with quantum access to <math>P_{T_{j+1},k}</math> and classical access to <math>E_k[P]</math></p>
<p>39 <math>P, R \leftarrow \mathcal{P}_n</math>  40 Run <math>\mathcal{A}</math> with quantum access to <math>P</math> and classical access to <math>R</math>, until <math>\mathcal{A}</math> makes its <math>(j+1)</math>st classical query <math>x_{j+1}</math>; let <math>T_j</math> be as in the text  41 <math>k \leftarrow D</math>, <math>y_{j+1} \leftarrow \{0, 1\}^n \setminus \{y_1, \dots, y_j\}</math>  42 Give <math>y_{j+1}</math> to <math>\mathcal{A}</math> as the answer to its <math>(j+1)</math>st classical query  43 <math>T_{j+1} := ((x_1, y_1), \dots, (x_{j+1}, y_{j+1}))</math>  44 Continue running <math>\mathcal{A}</math> with quantum access to <math>P_{T_{j+1},k}</math> and classical access to <math>E_k[P]</math></p>

**Fig. 2.** Syntactic rewritings of  $\text{Expt}'_j$ .