

T₅: Hashing Five Inputs with Three Compression Calls

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Abstract. Given $2n$ -to- n compression functions h_1, h_2, h_3 , we build a new $5n$ -to- n compression function T_5 , using only 3 compression calls:

$$T_5(m_1, m_2, m_3, m_4, m_5) := h_3(h_1(m_1, m_2) \oplus m_5, h_2(m_3, m_4) \oplus m_5) \oplus m_5$$

We prove that this construction matches Stam’s bound, by providing $\tilde{O}(q^2/2^n)$ collision security and $O(q^3/2^{2n} + nq/2^n)$ preimage security (the latter term dominates in the region of interest, when $q < 2^{n/2}$). In particular, it provides birthday security for hashing 5 inputs using three $2n$ -to- n compression calls, instead of only 4 inputs in prior constructions.

Thus, we get a sequential variant of the Merkle-Damgård (MD) hashing, where t message blocks are hashed using only $3t/4$ calls to the $2n$ -to- n compression functions; a 25% *saving* over traditional hash function constructions. This time reduces to $t/4$ (resp. $t/2$) sequential calls using 3 (resp. 2) parallel execution units; saving a factor of 4 (resp. 2) over the traditional MD-hashing, where parallelism does not help to process one message.

We also get a novel variant of a Merkle tree, where t message blocks can be processed using $0.75(t - 1)$ compression function calls and depth $0.86 \log_2 t$, thereby *saving 25% in the number of calls and 14% in the update time* over Merkle trees. We provide two modes for a local opening of a particular message block: conservative and aggressive. The former retains the birthday security, but provides longer proofs and local verification time than the traditional Merkle tree.

For the aggressive variant, we reduce the proof length to a 29% overhead compared to Merkle trees ($1.29 \log_2 t$ vs $\log_2 t$), but the verification time is now 14% *faster* ($0.86 \log_2 t$ vs $\log_2 t$). However, birthday security is only shown under a plausible conjecture related to the 3-XOR problem, and only for the (common, but not universal) setting where the root of the Merkle tree is known to correspond to a valid t -block message.

1 Introduction

A fundamental problem in cryptography is the construction of a hash function using idealized building blocks. A natural way to approach this problem is

to use λn -to- n -bit compression functions. Two well-known and widely-deployed constructions follow this approach:

- the *Merkle-Damgård construction* [10, 18], a sequential construction that is used in hash functions such as MD5, SHA-1 and SHA-2, and
- the *Merkle tree* [17], a parallel construction used in hash-based signatures (of interest due to their post-quantum security), version control systems such as git, and cryptocurrencies such as Ethereum.

The collision resistance of the Merkle-Damgård construction and the Merkle tree can be proven, based on the collision-resistance of the compression functions. The number of compression function calls is (essentially) the same for both constructions. For example, setting $\lambda = 2$, which is the focus of this work,⁵ they both process t message blocks using t and $(t - 1)$ compression function calls, respectively.

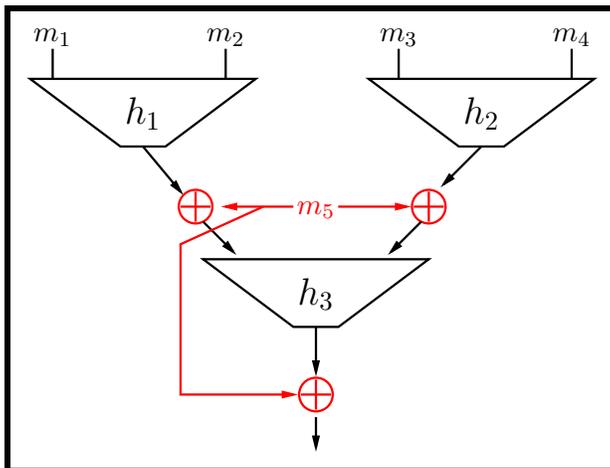


Fig. 1. The T_5 construction with five message blocks m_1, m_2, m_3, m_4, m_5 and three compression function calls.

NEW COMPRESSION FUNCTION T_5 . In this paper, we introduce the T_5 construction (see Fig. 1) that processes five message blocks using three $2n$ -to- n -bit compression function calls, thereby improving over the state-of-the-art of Merkle-Damgård (with IV counted as message block) and Merkle trees by processing an additional message block with the same number of compression function calls and essentially the same level of collision security.

Although T_5 is of independent theoretical interest to the construction of a compression function, we will also investigate Merkle-Damgård and Merkle trees when instantiated with T_5 .

⁵ Without loss of generality, all our results easily generalize to any $\lambda \geq 2$.

T₅ WITH MERKLE-DAMGÅRD. Our variant of the Merkle-Damgård construction, depicted in Figure 4, processes t message blocks using $3t/4$ calls to the $2n$ -to- n compression functions. If the chaining value is provided as m_5 in T_5 , then h_1 and h_2 can not only be computed in parallel, but independently of the chaining value. This allows a fully parallel implementation of h_1 , h_2 , and h_3 (with h_3 “one-round behind”), which is four times faster than MD which requires four sequential compression function calls to process a single input of the same length.

T₅ WITH MERKLE TREES. For our variant of Merkle trees, depicted in Figure 5, we will consider how they are often used in practice: for proof-of-inclusion of data in a larger set.

A *full opening* of a Merkle tree corresponds to the list of all message blocks, which can be used to verify that the message corresponds to a given hash value. An advantage of Merkle trees is it is possible to provide a *local opening*: to verify that one message block belongs to the tree, it suffices to provide a list of compression function outputs that is proportional to the depth of the tree.

These two types of openings give rise to three different notions of collision resistance for trees:

- *full-full* collision resistance, the “traditional” notion that requires finding two distinct messages that result in the same hash value,
- *local-local* collision resistance, where the goal is to find two local openings with the same hash value,
- *full-local* collision resistance, the setting of finding a collision between a full tree and a local opening.

The full-local setting is relevant in the common scenario where a hash value is honestly computed, but the proof is composed by an untrusted party. This happens, for example, when a user sends a message to a cloud server after hashing it, and then later wants to retrieve some message block from the server. Another natural application is the Merkle accumulator, where a protocol accumulates message blocks using a Merkle tree, and later parties provide proofs that a message block is in the tree.

Standard Merkle trees provide the same level of security under all three notions of collision resistance, and the same holds for our Merkle tree variant using T_5 if all four siblings are opened. In this case, our variant of the Merkle tree will process t message blocks using $0.75(t - 1)$ instead of $t - 1$ compression function calls, and depth $0.86 \log_2 t$ instead of $\log_2 t$. Due to the need to open four siblings, the opening proof will increase from $\log_2 t$ to $1.72 \log_2 t$, and the verification time increases as well from $\log_2 t$ to $1.29 \log_2 t$.

However, we also propose aggressive variant of our construction that only opens three siblings. See Figure 3. When this saving of one element for T_5 is translated to the whole Merkle tree, one gets a smaller local opening proof of $1.29 \log_2 t$, and a shorter verification time of $0.86 \log_2 t$. We prove that this more aggressive variant nevertheless provides the same full-local collision resistance, under a conjecture related to the 3-XOR problem. For local-local collision resistance, we prove security up to $2^{n/3}$ under a conjecture related to the 4-XOR

problem. The k -XOR problem is the subject of the well-studied generalized birthday problem by Wagner [31], and used in proof-of work algorithms such as Equihash. A full comparison of these three constructions will be given in Table 1 of Section 7.2.

OUR RESULTS FOR T_5 . We prove the following security results for T_5 in terms of adversarial advantage after q queries to the inner compression functions, and provide matching attacks in Section 8:⁶

- $q^2/2^n$ collision resistance, i.e., full-full collision resistance (CR) security (Theorem 1);
- $q^3/2^{2n} + q/2^n$ preimage resistance (Theorem 2).
- $q^2/2^n$ full-local CR security under a conjecture related to the 3-XOR problem (Proposition 1);
- $q^3/2^n$ full-local CR security unconditionally, i.e., 128-bit security for $n = 384$ (Theorem 3);
- $q^3/2^n$ local-local CR security under a conjecture related to the 4-XOR problem (Proposition 2);
- $q^4/2^n$ local-local CR security unconditionally, i.e., 128-bit security for $n = 512$ (Theorem 3).

These results almost immediately imply corresponding security claims for our Merkle-Damgård variant (see Section 6) and our Merkle tree variant (see Section 7 and Table 1).

2 Related Work

The design of a hash function is usually based on one or more *primitives* with fixed-length inputs and outputs. Historically, the most common choice for these primitives were block ciphers. This gives rise to the following question: how can we construct a hash function with the minimum number of block cipher calls?

This question motivated a significant research effort into efficient block-cipher-based hash function constructions. Block ciphers such as Triple-DES and AES have a block size of 64 and 128 bits respectively, which may not provide sufficient collision security when used in the Merkle-Damgård construction. Therefore, one line of work focuses on combining smaller primitives to produce a wider hash function. Results of interest include the Knudsen-Preneel construction based on linear error correcting codes [15], and a double-length construction by Nandi et al. [20] that was generalized by Peyrin et al. [24], and by Seurin and Peyrin [26], which interestingly also links the security to a conjecture related to the 3-sum problem.

A related line of research attempted to improve upon the Merkle-Damgård construction to process additional message blocks. A brief overview of some constructions and an impossibility result was given by Black et al. [6, 7]. More

⁶ For simplicity of reading, we only list dominant terms, and ignore constant and even small $\text{poly}(n)$ factors; i.e., omit \tilde{O} notation below.

specifically, they considered hash functions that make one block cipher call (under a small set of keys) for each message block to be hashed, and showed that all such constructions are vulnerable to a simple attack.

This work was later generalized by Rogaway and Steinberger [25], and refined in subsequent papers by Stam [27], and by Steinberger et al. [28,29]. This result, commonly known as “Stam’s bound,” puts a limit on the efficiency (in terms of primitive calls) of any secure hash function construction.

Stam’s bound states that there always exists a collision attack and a preimage attack with at most $2^{n(\lambda-(t-0.5)/r)}$ and $2^{n(\lambda-(t-1)/r)}$ queries respectively on a tn -to- n -bit hash function making r calls to λn -to- n -bit compression functions. We have $t = 5$, $\lambda = 2$, and $r = 3$ in the case of T_5 , thereby showing that we cannot hope to do better than $2^{n/2}$ collision and $2^{2n/3}$ preimage security.

As explained by Stam [27], the bound applies to hash functions that satisfy the *uniformity assumption*, and applies to cryptographic permutations ($\lambda = 1$) as well as compressing primitives ($\lambda \geq 1$). However, the main focus of this line of work had been on combining smaller non-compressing primitives (see e.g., Mennink and Preneel [16]).

Lastly, we recall that a series of papers have investigated optimal trade-offs between time and space for Merkle tree traversal, e.g., Jakobsson et al. [14], Szydło [30], and Berman et al. [4]. Given that we propose T_5 inside a standard Merkle tree, these trade-offs can also be directly applied to the constructions in this paper. We would also like to mention Haitner et al. [12]’s construction which only has depth one, at the cost of making significantly more calls than the standard Merkle tree.

3 Preliminaries

3.1 Notation

If S is a set, $x \xleftarrow{\$} S$ denotes the uniformly random selection of an element from S . We let $y \leftarrow A(x)$ and $y \xleftarrow{\$} A(x)$ be the assignment to y of the output of a deterministic and randomized algorithm $A(x)$, respectively.

For positive integers m, n , we let $\text{Func}(m, n)$ denote the set of all functions mapping $\{0, 1\}^m$ into $\{0, 1\}^n$. We write $h \xleftarrow{\$} \text{Func}(m, n)$ to denote random sampling from the set $\text{Func}(m, n)$ and assignment to h , and say that h is modeled as an ideal hash function. For fixed m and n , such modeling is attempting to approximate the security of real-world, keyless, fixed-input-size compression functions, such as the compression function of SHA-2.

3.2 Security Definitions of Hash Functions

An *adversary* A is a probabilistic algorithm, possibly with access to oracles $\mathcal{O}_1, \dots, \mathcal{O}_\ell$ denoted by $A^{\mathcal{O}_1, \dots, \mathcal{O}_\ell}$. Our definitions of collision (Coll), and preimage (Pre) security are given for any general fixed-input length hash function H built upon the compression functions h_i for $i = 1, \dots, \ell$ where h_i are modeled as

ideal functions. Namely, for a fixed adversary A and for all $i = 1$ to ℓ with $h_i \xleftarrow{\$} \text{Func}(2n, n)$, we define the following advantage functions:

$$\mathbf{Adv}_H^{\text{Coll}}(A) = \Pr \left[H^{h_1, \dots, h_\ell}(M) = H^{h_1, \dots, h_\ell}(M') \text{ and } M \neq M' \right. \\ \left. \mid (M, M') \xleftarrow{\$} A^{h_1, \dots, h_\ell}(\cdot) \right]$$

and

$$\mathbf{Adv}_H^{\text{Pre}}(A) = \Pr \left[H^{h_1, \dots, h_\ell}(M) = H^{h_1, \dots, h_\ell}(M') \right. \\ \left. \mid M \xleftarrow{\$} \mathcal{M}_H, M' \xleftarrow{\$} A^{h_1, \dots, h_\ell}(H^{h_1, \dots, h_\ell}(M)) \right]$$

We define the $\mathbf{Adv}_H^{\text{atk}}(q)$ against the $\text{atk} = \{\text{Coll}, \text{Pre}\}$ -security of H as the maximum advantage over all adversaries making at most q total queries to its oracles.

3.3 Local Opening Security

We define local opening security of a hash function output (viewed as a commitment of a message). Given a function H built upon compression functions h_1, h_2, \dots where h_i are all modeled as ideal functions, a local opening $\text{Open}^{h_1, \dots}(\cdot, \cdot)$ for $H^{h_1, \dots}$ maps a pair (M, i) to π (called proof) where $M = (m_1, m_2, \dots, m_c)$ is a message (a tuple of blocks) and $1 \leq i \leq c$ is an index.

CORRECTNESS OF LOCAL OPENING. There is an efficient function $\text{Ver}^{h_1, \dots}$ such that for all message M , all index i ,

$$\text{Ver}^{h_1, \dots}(i, m_i, \text{Open}^{h_1, \dots}(M, i), H(M)) = 1.$$

SECURITY OF LOCAL OPENING. We provide two notions of local opening security. For the stronger variant, which we call “local-local,” the adversary wins if it produces an output f corresponding to two contradicting local openings for some position i . For the weaker variant, which we call “full-local,” the adversary wins if it produces an output f corresponding to a local opening contradicting a full opening.

Definition 1 (local-local and full-local opening advantage). Let H be a hash function and Open is a correct local opening for H with Ver is the verification function. For any adversary A , we define the local-local opening advantage as

$$\mathbf{Adv}_H^{\text{local-local}}(A) = \Pr \left[\text{Ver}(i, m, \pi, f) = \text{Ver}(i, m', \pi', f) = 1, m \neq m' \right. \\ \left. \mid (i, m, m', \pi, \pi', f) \xleftarrow{\$} A^{h_1, \dots} \right]$$

We define full-local opening advantage (a weaker variant of the above) of \mathbf{A} as

$$\mathbf{Adv}_H^{\text{full-local}}(\mathbf{A}) = \Pr \left[\mathbf{Ver}(i, m', \pi', H(M)) = 1, m' \neq m_i \mid (i, M, m', \pi') \stackrel{\$}{\leftarrow} \mathbf{A}^{h_1, \dots} \right]$$

Finally, for a function H with local opening algorithms $(\text{Open}, \text{Ver})$ and attack $z \in \{\text{local-local}, \text{full-local}\}$, we define $\mathbf{Adv}_H^z(q) = \max_{\mathbf{A}} \mathbf{Adv}_H^z(\mathbf{A})$ as the maximum advantage over all adversaries making at most q total queries to its oracles.

Intuitively, the weaker definition protects against situations where the initial commitment f was honestly produced, using some long message M . Thus, a contradictory local opening will result in finding a collision between a local opening and a full opening. In contrast, the (traditional) stronger definition is directly concerned with somebody producing two contradictory local openings.

BY-PASS HASH COMPUTATION. We say that H has a *by-pass computation* H_i corresponding to a local opening Open for a fixed index $1 \leq i \leq c$, if for all M ,

$$H_i^{h_1, \dots}(m_i, \text{Open}^{h_1, \dots}(M, i)) = H^{h_1, \dots}(M).$$

In other words, given a proof (output of the Open) and the message block for the index (for which the proof is produced), we can compute the hash output of the message (without knowing the other blocks of the message).

The presence of by-pass computations $\mathcal{H} = \{H_i\}$ for all indices lead to a natural verification algorithm as follows: $\mathbf{Ver}(i, m, \pi, f) = 1$ whenever $H_i^{h_1, \dots}(m, \pi) = f$. For a fixed index i , we define the *cross-collision* advantage between the hash function H and a by-pass computation H_i as

$$\mathbf{Adv}_{H, H_i}^{\text{Coll}}(\mathbf{A}) = \Pr \left[H(M) = H_i(m', \pi) \text{ and } M_i \neq m' \mid (M, m', \pi) \stackrel{\$}{\leftarrow} \mathbf{A}^{h_1, \dots} \right]$$

Similarly, we define the *inter-collision* advantage for by-pass computation H_i as

$$\mathbf{Adv}_{H_i}^{\text{Coll}}(\mathbf{A}) = \Pr \left[H_i(m, \pi) = H_i(m', \pi') \text{ and } m \neq m' \mid (m, \pi, m', \pi) \stackrel{\$}{\leftarrow} \mathbf{A}^{h_1, \dots} \right]$$

Let $\mathcal{H} = \{H_i\}$ be the family of by-pass computations for all indices i . We define

$$\mathbf{Adv}_{H, \mathcal{H}}^{\text{Coll}}(q) = \max_{\mathbf{A}} \max_i \mathbf{Adv}_{H, H_i}^{\text{Coll}}(\mathbf{A}) \quad \text{and} \quad \mathbf{Adv}_{\mathcal{H}}^{\text{Coll}}(q) = \max_{\mathbf{A}} \max_i \mathbf{Adv}_{H_i}^{\text{Coll}}(\mathbf{A})$$

as the maximum advantage over all by-pass computations for all adversaries making at most q total queries to its oracles.

Now we make a simple observation when by-pass computations $\mathcal{H} = \{H_i\}$ for all indices exist for a hash function H , the induced verification procedure Ver satisfies

$$\mathbf{Adv}_H^{\text{full-local}}(q) \leq \mathbf{Adv}_{H, \mathcal{H}}^{\text{Coll}}(q) \quad \text{and} \quad \mathbf{Adv}_H^{\text{local-local}}(q) \leq \mathbf{Adv}_{\mathcal{H}}^{\text{Coll}}(q) \quad (1)$$

The above observation helps us to reduce the local opening security to cross-collision or inter-collision security problem for the family \mathcal{H} . As all the function H_i in this family are often symmetric, a proof for a fixed function H_i implies the one for the entire family \mathcal{H} .

4 Construction T_5

We define $T_5 : \{0, 1\}^{5n} \rightarrow \{0, 1\}^n$ based on the $2n$ -to- n -bit compression functions h_1, h_2, h_3 as follows:

$$T_5(m_1, m_2, m_3, m_4, m_5) := h_3(h_1(m_1, m_2) \oplus m_5, h_2(m_3, m_4) \oplus m_5) \oplus m_5$$

For the purposes of all our proof we will assume that the compression functions h_i for $i = 1$ to 3 are ideal functions.

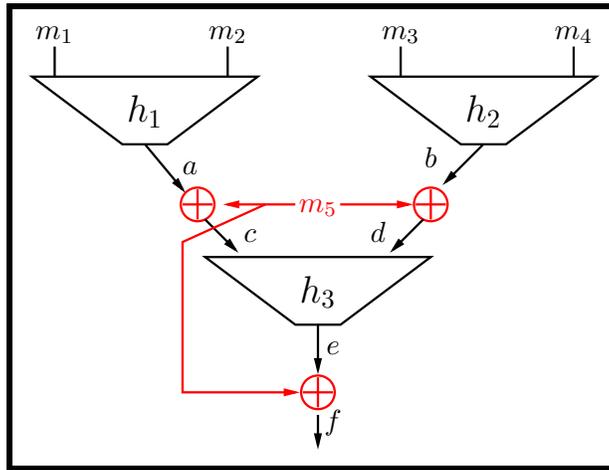


Fig. 2. Modified 2-level Merkle tree $T_5(m_1, m_2, m_3, m_4, m_5)$ with an extra input m_5 for the same 3 hash calls.

Notation. As shown in Figure 2, we use the variables

- m_1 and m_2 (resp. m_3 and m_4) to denote the left and right halves of various inputs to h_1 (resp. h_2);
- a (resp. b) to denote various outputs of h_1 (resp. h_2);
- c and d to denote the left and right halves of various inputs to h_3 ;
- e to denote various outputs of h_3 ;
- $M = (m_1, m_2, m_3, m_4, m_5)$ to denote various inputs to T_5 ;
- f to denote various outputs of T_5 .

Hence, a valid computation of $T_5(M) = T_5(m_1, \dots, m_5)$ proceeds as follows:

1. Set $a = h_1(m_1, m_2)$, $b = h_2(m_3, m_4)$.
2. Set $c = a \oplus m_5$, $d = b \oplus m_5$.
3. Set $e = h_3(c, d)$, and output $f = e \oplus m_5$.

We say that a triple of queries $((m_1, m_2), a)$ to h_1 , $((m_3, m_4), b)$ to h_2 , and $((c, d), e)$ to h_3 is *consistent* if

$$a \oplus b = c \oplus d, \tag{2}$$

in which case we define $m_5 = a \oplus c = b \oplus d$, and say that this consistent triple of queries (uniquely) defines a valid T_5 evaluation (M, f) , where $M = (m_1, m_2, m_3, m_4, m_5)$ and $f = e \oplus a \oplus c = e \oplus b \oplus d$.

Main results of the section. The main result of this paper is to provide the collision and preimage security of the T_5 hash function. The following theorem shows that T_5 achieves nearly birthday collision security, despite hashing one more input than the traditional Merkle-Damgård function of depth 2.

Theorem 1. *The T_5 construction achieves nearly birthday-bound collision security:*

$$\mathbf{Adv}_{T_5}^{\text{Coll}}(q) \leq \frac{(n^2 + 10)q^2}{2^n} \tag{3}$$

The full formal proof of this result is somewhat subtle, and will be given in the Appendix (see App. A). But an informal proof intuition for a representative special case will be given in Section 4.1.

As our second main result, we also show that T_5 maintains nearly optimal preimage security $\tilde{O}(q/2^n)$ for $q < 2^{n/2}$, which means it offers optimal preimage *and* collision-resistance security in the common range of $q < 2^{n/2}$.

However, even when q grows above $2^{n/2}$, T_5 still offers non-trivial security $O(q^3/2^{2n})$ for values of $q < 2^{2n/3-1}$, which is likely sufficient for most applications. As we show in Section 8.2, T_5 is indeed not preimage resistant when $q > 2^{2n/3}$, so our result is tight.

Theorem 2. *Assuming $q \leq 2^{2n/3-1}$, the T_5 construction achieves the following preimage security:*

$$\mathbf{Adv}_{T_5}^{\text{Pre}}(q) \leq \frac{2q^3}{2^{2n}} + O\left(\frac{qn}{2^n}\right) \tag{4}$$

We give a formal proof in the Appendix (see App. B), but present some proof intuition (for an important special case) in Section 4.2, similar to what was done in Section 4.1.

4.1 Proof Intuition for Collision Resistance of T_5

Below we give the proof intuition for the simple, but natural special case where the adversary A makes all of its queries to h_1 and h_2 before any query to h_3 is made. As we explain in the formal proof in the Appendix (see App. A), this assumption is with a *significant* loss of generality, and several special arguments are needed to cover the fully general case. However, this simplified case will already demonstrate some of the main arguments of our analysis.

The proof will roughly consist in arguing that A , making q total hash queries, is unlikely — up to birthday advantage — to succeed in the following four tasks. (These tasks are formalized later in Propositions 3,4,5,6.)

1. **Task 1:** finding any simple collision in h_1 or h_2 .⁷
2. **Task 2:** finding some value z for which more than n distinct pairs of queries $((m_1, m_2), (m_3, m_4))$ to h_1 and h_2 result in⁸

$$h_1(m_1, m_2) \oplus h_2(m_3, m_4) = z$$

3. **Task 3:** generate more than nq valid evaluations of T_5 .
4. **Task 4:** generate a non-trivial collision of T_5 .

The argument of each subsequent task will inductively assume that the adversary indeed failed in the previous task.

For Task 1, this is the trivial birthday bound on h_1 or h_2 .

For Task 2, we will formally define the set $C_{12}(z)$ to consist of pairs of queries $((m_1, m_2), a)$ to h_1 and $((m_3, m_4), b)$ satisfying $a \oplus b = z$. Using the fact that no simple collisions in h_1 and h_2 are found, it is easy to see that each of the n queries to h_1 and h_2 inside $C_{12}(z)$ must be distinct. Moreover, the latter of the two queries $((*, a), (*, b)) \in C_{12}(z)$ must collide with a fixed value z plus the former of the two queries. E.g., if the query $(*, a)$ to h_1 was made before $(*, b)$ to h_2 , this tuple will fall inside $C_{12}(z)$ only if $b = z \oplus a$, which happens with probability 2^{-n} . Taking the union bound over all values z and all possible choices of $2n$ out of q queries to be included inside $C_{12}(z)$, we see that

$$\Pr[\exists z \text{ s.t. } |C_{12}(z)| \geq n] \leq 2^n \cdot q^{2n} \cdot \left(\frac{1}{2^n}\right)^n \leq \left(\frac{2q^2}{2^n}\right)^n \ll O\left(\frac{q^2}{2^n}\right)$$

For Task 3, we will use our simplifying assumption that A makes all of its queries to h_1 and h_2 before any query to h_3 is made. In this case, all consistent triples of queries to h_1 , h_2 and h_3 defining a valid input-output (M, f) to T_5 get created by making a call to h_3 . For each of at most q such queries to h_3 on some input (c, d) , we claim that this query will “match up” with a pair of earlier queries $(*, a)$ to h_1 and $(*, b)$ to h_2 only if $m_5 = a \oplus c = b \oplus d$, which is equivalent to $a \oplus b = c \oplus d$, which means that

$$((*, a), (*, b)) \in C_{12}(c \oplus d)$$

But we already assumed that $|C_{12}(z)| \leq n$ for all z , meaning that each query (c, d) can form a consistent tuple with at most n pairs of queries to h_1 and h_2 . Summing over all (up to) q queries to h_3 , the total number of evaluations will be at most nq .⁹

Finally, for Task 4 that we care about, we will once again use our simplifying assumption. In particular, under our assumption, such a collision can only be

⁷ For the general case, we will also need no collisions in a slightly modified variant of h_3 .

⁸ For the general case, we will also need similar guarantees for the combinations of $h_1 + h_3$ and $h_2 + h_3$.

⁹ As we will see, in the general case a very different proof strategy will be needed to extend this argument to valid evaluations of T_5 completed by calls to h_1 and h_2 .

caused by a call to h_3 on some input (c, d) . From the previous argument, we already know that this query will “match up” with a pair of earlier queries $(*, a)$ to h_1 and $(*, b)$ to h_2 only if $((*, a), (*b)) \in C_{12}(c \oplus d)$, meaning there are at most n new evaluations of T_5 caused by this query. Also, any two of these n new evaluations cannot collide among themselves, as they have two different values $a \neq a'$ (remember, no collisions in h_1), so the final outputs $f = h_3(c, d) \oplus c \oplus a \neq h_3(c, d) \oplus c \oplus a' = f'$. Thus, the only chance the adversary has is if one of these n new evaluations of T_5 (call the output f) collides with one of at most nq already defined previous evaluations f' of T_5 .

But each of the n new output values f will be *individually random*, as it is equal to random $e = h_3(c, d)$ plus $m_5 = a \oplus c$. Hence, this individually random f can collide with the previously defined output f' of T_5 with probability at most $nq/2^n$, because from failing Task 3 we know there are at most nq previous evaluations of T_5 completed so far. Taking the union bound over n values of f , and q queries to h_3 , the final bound $O(n^2q^2/2^n)$ follows.

General case. We will not fully detail the general case (see full argument in the Appendix (see App. A)), but briefly demonstrate that making queries to h_1 or h_2 after some queries to h_3 could be potentially helpful to the adversary, and will require adjustments to our proof strategy above.

For example, our proof sketch above showed that, modulo very rare events, the number of valid evaluations of T_5 can increase by at most n for each new query to h_3 . However, imagine that A first makes a query to h_2 with output b . Then A can make $\Omega(q)$ queries (c_i, d_i) all satisfying $c_i \oplus d_i = z$ (for some z). Now, a query to h_1 (made after these $\Omega(q)$ queries to h_3) has a chance to simultaneously match with $\Omega(q) \gg n$ tuples $((*, b), ((c_i, d_i), *))$. Of course, in order for this to happen, the random answer a must match $b \oplus z$, which happens with tiny probability. In fact, we could apply Markov’s equality to argue that the probability a query to h_1 will produce more than n new evaluation points is at most (what turns out to be by an easy calculation) $O(q^2/2^n)$. By itself, this is good enough, but it will not “survive” a union bound over up to q potential queries to h_1 . Instead, we will use linearity of expectation to make a global, “stochastic” argument that *all* such (up to) q queries to h_1 and h_2 will define more than nq new evaluations with at most “birthday” probability. See Proposition 5 for the details.

Overall, the full proof in the general case will be noticeably more subtle than the proof intuition given above, but will still follow the same high-level structure.

4.2 Proof Intuition for Preimage Resistance of T_5

As in Section 4.1, we will only consider the special case when the adversary A makes all of its queries to h_1 and h_2 before any query to h_3 is made, as it will contain most of the ideas needed in the general proof. Also, we will assume

that $q = \Omega(\sqrt{n} \cdot 2^{n/2})$, as this is the case where the “unexpected” term $q^3/2^{2n}$ appears.¹⁰

In this setting, there are several differences from the case of collision-resistance we considered so far. First, A is given a specific target f to invert. In particular, we will not care about local collisions in functions h_1 , h_2 , and the c - or d -“shifted” versions of h_3 , as such collisions will happen, but will not help the adversary invert f . Instead, we will care that in each new call to $h_3(c, d)$, the number of valid new evaluations of T_5 will be not much higher than what we expect. Recall, in our special case of only h_3 queries causing all new evaluations of T_5 , this number of new evaluations is bounded by $|C_{12}(c \oplus d)|$, where $C_{12}(z)$ to consist of pairs of queries $((m_1, m_2), a)$ to h_1 and $((m_3, m_4), b)$ satisfying $a \oplus b = z$. And since each such evaluation defines an individually random output value $f' = h_3(c, d) \oplus c \oplus a$, the probability this f' matches f is 2^{-n} , meaning A’s overall chance to invert f in this query is $|C_{12}(c \oplus d)|/2^n$.

Of course, the adversary can select *any* value of $z = c \oplus d$ to make sure it chooses the largest set $C_{12}(z)$. Thus, if we want to upper bound A probability of success, we must argue that $K = \max_z |C_{12}(z)|$ is not much higher than its expectation with high probability. In the collision resistance proof, we manage to upper bound $K \leq n$ with “good enough” probability $(q^2/2^n)^n$. Indeed, this was enough to withstand the union bound over z to give final probability $(2q^2/2^n)^n \leq 2q^2/2^n$ that $K \geq n$ in that setting.

We will do a similar union bound in our case as well, except that we have $q \gg 2^{n/2}$, so even in the best case scenario we expect $|C_{12}(z)| \geq q^2/2^n \gg n$ for any given z , let alone the z chosen by the adversary. Moreover, the final birthday bound will not be good enough for us in this setting as well. But first let us optimistically assume that for any fixed z , we managed to get the following very strong concentration bound:¹¹

$$\Pr \left[|C_{12}(z)| \geq \frac{2q^2}{2^n} \right] \leq \frac{1}{2^{2n}} \quad (5)$$

Then we will be done, because we can take the union bound over z to conclude that $\Pr [K \geq 2q^2/2^n] \leq 2^{-n}$. And, finally, there will be at most $2q^2/2^n$ new evaluations f' per each query to h_3 . Thus, taking the union bound over at most q such queries, A’s overall inversion probability (ignoring 2^{-n} failure event above) is upper bounded by:

$$q \cdot \frac{2q^2}{2^n} \cdot \frac{1}{2^n} = \frac{2q^3}{2^{2n}}$$

High concentration bound via tabulation hashing. So it remains to argue the high concentration bound in (5). This turns out to be much harder than in

¹⁰ Our general proof will not treat this case separately, but the calculations are slightly easier to write for the “beyond-birthday” case.

¹¹ This bound, as stated, is only true if $q = \Omega(\sqrt{n} \cdot 2^{n/2})$. In the general case the term becomes $|C_{12}(z)| \geq \Omega(n) + 2q^2/2^n$, as we expect $\Omega(n)$ multi-collisions already when $q \approx 2^{n/2}$.

the collision-resistance case, where the bound we needed was the much weaker $O((q^2/2^n)^n)$, which is meaningless when $q > 2^{n/2}$.

The next naive attempt is to write $|C_{12}(z)|$ as a sum of q^2 indicator variables X_{ij} , equal to 1 if the i -th output a_i of h_1 and the j -th output b_j of h_2 satisfy $a_i \oplus b_j = z$. And then try to use Chernoff bound to argue that the probability sum of these indicator variables is twice as large as its expectation is exponentially low. Unfortunately, the random variables X_{ij} are not even 4-wise independent; e.g., $(a_1 \oplus b_1) \oplus (a_1 \oplus b_2) \oplus (a_2 \oplus b_1) \oplus (a_2 \oplus b_2) = 0$. So we cannot apply the Chernoff bound, and the Chebyshev inequality for pairwise independent random variables is not strong enough. Indeed, the question of getting our concentration bound turned out to be quite deep.

Fortunately, the setting we need turns out to be equivalent to the classical hashing problem, called *simple tabulation hashing*, introduced in the seminal paper of Carter and Wegman [9]. Applied to our setting, given two random “hash tables” T_1 and T_2 with range of size $N = 2^n$, tabulation hashing would map a “ball” $y = (u, v)$ into a “bin” $z = T_1[u] \oplus T_2[v]$. The classical question studied by tabulation hashing is to upper bound occupancy of any such bin z after some Q balls are thrown using tabulation hashing. We defer the details to the formal proof in Section 4.2, but point out that in our setting the tables are implemented using hash functions h_1 and h_2 , and the number of balls $Q \leq q^2$ corresponds to all pairs of queries to h_1 and h_2 .

Designing strong enough concentration bound for tabulation hashing (which is exactly what we need!) was an open problem for many years, until the breakthrough result of Pătraşcu and Thorup [22] showed a Chernoff-type concentration bound which enables us to show (5), and thus complete the proof.

5 Aggressive Opening for T_5

In this section we describe a non-trivial opening for T_5 . We first note that a straightforward way to open a block m_i in T_5 is to provide all four siblings $m_{j \neq i}$. For a single T_5 the full-local and the local-local security (Section 3.3) definitions are the same and correspond to the collision security of T_5 which we have already studied. The performance of this method in a full tree is somewhat less attractive and is given in detail in Section 7.2. Thus we call it *conservative*.

Now we provide another point on the security-performance tradeoff for T_5 , which we call *aggressive*. We see that even though the provable security bounds decrease, the heuristic security (as the complexity of best attacks) remains the same under plausible conjectures. Both openings are depicted in Figure 3.

Our (aggressive) local opening **Open** for T_5 is defined as follows. Let $m = (m_1, m_2, m_3, m_4, m_5)$.

1. $\text{Open}^{h_1, h_2, h_3}(1, m) = (m_2, m_5, h_2(m_3, m_4) \oplus m_5)$.
2. $\text{Open}^{h_1, h_2, h_3}(2, m) = (m_1, m_5, h_2(m_3, m_4) \oplus m_5)$.
3. $\text{Open}^{h_1, h_2, h_3}(3, m) = (m_4, m_5, h_1(m_1, m_2) \oplus m_5)$.
4. $\text{Open}^{h_1, h_2, h_3}(4, m) = (m_3, m_5, h_1(m_1, m_2) \oplus m_5)$.
5. $\text{Open}^{h_1, h_2, h_3}(5, m) = (m_1, m_2, h_2(m_3, m_4) \oplus m_5)$.

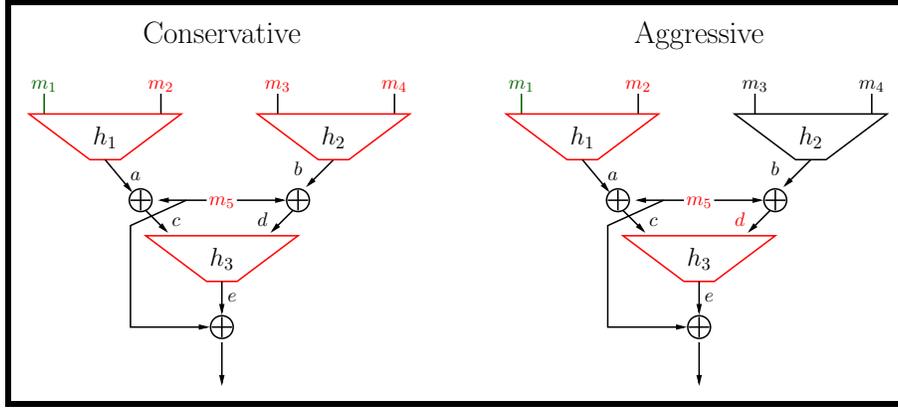


Fig. 3. Conservative and aggressive openings for T_5 . Green m_1 is opened. Red are opening elements (4 vs 3) and recomputed compression functions (3 vs 2).

We first show that the above defined opening has a by-pass computation T'_5 for index 1 (one can similarly show for the other indices) and hence it satisfies the correctness condition of an opening. For the sake of simplicity, we skip the hash oracles notation h_1, h_2, h_3 . We define $T'_5 : \{0, 1\}^{4n} \rightarrow \{0, 1\}^n$ based on the $2n$ -to- n -bit compression functions h_1, h_3 as follows:

$$T'_5(m'_1, m'_2, m'_3, m'_4) := h_3(h_1(m'_1, m'_2) \oplus m'_3, m'_4) \oplus m'_3$$

A straightforward calculation shows that $T'_5(m_1, \text{Open}(1, m)) = T_5(m)$. Hence T'_5 is a by-pass computation of T_5 for the opening function defined above.

Theorem 3.

$$\mathbf{Adv}_{T_5}^{\text{full-local}}(q) \leq \mathbf{Adv}_{T_5, T'_5}^{\text{Coll}}(q) \leq \frac{nq^3 + 9q^2}{2^n} \quad (6)$$

and

$$\mathbf{Adv}_{T_5}^{\text{local-local}}(q) \leq \mathbf{Adv}_{T'_5}^{\text{Coll}}(q) \leq \frac{q^4}{2^n} \quad (7)$$

We note that the relation between the collision advantage and the opening advantage is already described in (1). So it only remains to bound the cross-collision probability and the collision probability for a family of by-pass hash computations. The full formal proof of the cross-collision bound is somewhat similar to the collision security analysis of T_5 , and is provided in the Appendix (see App. C). But an informal proof intuition for a representative special case will be given in Section 5.1 below. However, the bound for collision advantage of the by-pass family is more or less straightforward and described afterwards.

5.1 Proof Intuition for Theorem 3 (Cross-Collision Bound)

Here, we give sketch proof of the cross-collision advantage between T_5 and T'_5 . We first note that in the case of T'_5 , all queries are consistent. Hence q queries

each to h_1 and h_3 can generate a maximum of $q^2 T'_5$ evaluations. And, as we have already seen before, q oracle queries can generate at most nq evaluations of T_5 if bad events B_1 and B_2 don't happen. Now, we approach as in the earlier theorem. We break the event that a collision between T_5 and T'_5 has occurred into three parts, depending upon which oracle was queried by the i -th query and give an upper bound to the collision probability. Then, we apply a union bound to over the q queries to give the intended result. Let X_i be the event that none of bad events B_1 , B_2 or B_3 happened. The three cases are as follows:

- The i -th query is $((m_1, m_2), a)$ to h_1 . As in the earlier proof, a collision can happen in two ways, either this output a induces a T_5 tuple and a T'_5 tuple which collide, or there is some *previous* value $(*, f) \in \mathbf{Eval}^{i-1}$ such that the random answer $h_1(m_1, m_2) = a$ caused the collision with this f . The former case implies only the trivial collision, which we have ruled out. In the latter case, for a collision to happen, there are two subcases depending on whether $(*, f)$ comes from a previous evaluation of T_5 or T'_5 evaluation. If $(*, f)$ comes from a T_5 evaluation, the random answer a can combine with any of the queries $((c, d), e)$ of h_3 such that $f = a \oplus c \oplus e$. We note that there are a maximum of nq prior T_5 evaluations, and q h_3 queries. The probability that $f = a \oplus c \oplus e$ is $1/2^n$ for each such query. Therefore,

$$\Pr[X_i] \leq nq \cdot q \cdot \frac{1}{2^n} = \frac{nq^2}{2^n}$$

If $(*, f)$ comes from a T'_5 evaluation, we note that there are at most q^2 such evaluations. There exist tuples $((*, b), ((c, d), e)) \in C_{23}(f)$, which can be at most n in number, and the random answer $a = h_1(m_1, m_2)$ should be equal to $b \oplus c \oplus d$. This again gives

$$\Pr[X_i] \leq q^2 \cdot n \cdot \frac{1}{2^n} = \frac{nq^2}{2^n}$$

- The i -th query is $((m_3, m_4), b)$ to h_2 . The only way this query can cause a collision is that there exists some previous value $(*, f) \in \mathbf{Eval}_{T'_5}^{i-1}$ such that the random answer creates a T_5 output equal to f . This case is exactly similar to the first subcase of Case 1.
- The i -th query is $((c, d), e)$ to h_3 . The case generated by this query is the same as that in the first case. If this query generates a T_5 tuple and a T'_5 tuple that collide, we again get only the trivial collision. If there is some previous T_5 evaluation with which this query collides, then again, q such queries can combine with n evaluations of T_5 , and the collision probability for each combination is $1/2^n$, resulting in the same probability as in the first case. If this query collides with some previous T'_5 evaluation, which are nq in number, we note that there exist tuples $((*, a), (*, b)) \in C_{12}(c \oplus d)$ which can be at most n in number. Again, we get the same probability.

Taking union bound over the q queries, we find that

$$\Pr [B_4 \cap \overline{B_1 \cap B_2 \cap B_3}] \leq q \cdot \max_i \Pr[X_i] \leq \frac{nq^3}{2^n}.$$

5.2 Proof of (7) (Collision Bound of Family of By-Pass Hash)

Now we prove the second part of the result (collision of by-pass hash family). Here we show the collision probability for T'_5 (i.e., for the index 1). The proof for the other indices will be very similar and will have same bound. As we take the maximum collision probability for all indices, the result will follow. Following a similar notation, let $h_1(m_1, m_2) = b$, $h_1(m'_1, m'_2) = b'$, $c = b \oplus m_5$ and $c' = b' \oplus m'_5$. So, the hash outputs are $T'_5(m_1, m_2, m_5, d) = f = h_3(c, d) \oplus m_5$ and $T'_5(m'_1, m'_2, m'_5, d') = f' = h_3(c', d) \oplus m'_5$. If $f = f'$ with $(m'_1, m'_2, m'_5, d') \neq (m_1, m_2, m_5, d)$ (i.e., collision happens) then we have

$$h_1(m'_1, m'_2) \oplus h_1(m_1, m_2) \oplus (c \oplus h_3(c, d)) \oplus (c' \oplus h_3(c', d')) = 0 \quad (8)$$

Let us write $h_3(x, y) \oplus x$ as $h'_3(x, y)$. It is obvious that h'_3 behaves exactly like a random function (independent with h_1, h_2). Thus, we have shown that collision problem of T'_5 is reduced to finding $((m_1, m_2), (c, d)) \neq ((m'_1, m'_2), (c', d'))$ such that

$$h_1(m_1, m_2) \oplus h_1(m'_1, m'_2) \oplus h'_3(c, d) \oplus h'_3(c', d') = 0 \quad (9)$$

We call this problem 4-XOR' (a variant of 4-XOR problem described in the following section). It is also not difficult to construct a collision pair from four pairs satisfying a 4-XOR' relation. In other words, 4-XOR' is equivalent to finding collision of T'_5 . Now, by applying union bound, the collision probability can be simply bounded above by $q^4/2^n$.

5.3 Reduction of Local Opening Security to 3-XOR/4-XOR Problem

k-XOR problem. Let F_1, F_2, \dots, F_k be k oracles that output n -bit strings. Find x_1, x_2, \dots, x_k such that

$$F_1(x_1) \oplus F_2(x_2) \oplus \dots \oplus F_k(x_k) = 0.$$

Reduction for full-local. When $k = 3$, the k -XOR problem has an information-theoretical security of $n/3$ -bits, however the best algorithm requires more than $2^{n/2}/\sqrt{n}$ time (ignoring a $\log n$ factor) and queries [8, 21]. Bridging this gap would imply a solution to the long-standing 3-XOR problem, which would be a substantial breakthrough.

Conjecture 1 (3-XOR hardness) *For any algorithm \mathcal{A} that makes less than $q < 2^{n/2}$ queries to F_1, F_2, F_3 and runs for time q , the probability that for randomly chosen F_1, F_2, F_3 it solves the 3-XOR problem is at most $n^2 q^2 / 2^n$.*

Note that we have kept some margin on the power of n in our conjecture. Till now we have attack with advantage about $nq^2/2^n$. Now we provide a heuristic reduction of full-local security to the 3-XOR problem. Let us

look at the full-local security definition. To break it, we are required to find $(m_1, m_2, m_3, m_4, m_5, m'_1, m'_2, m'_5, d')$ such that

$$T_5(m_1, m_2, m_3, m_4, m_5) = h_3(h_1(m'_1, m'_2) \oplus m'_5, d') \oplus m'_5$$

or equivalently

$$T_5(m_1, m_2, m_3, m_4, m_5) = h_3(c', d') \oplus h_1(m'_1, m'_2) \oplus c'. \quad (10)$$

In our reduction we consider only algorithms which we call *valid-aware*. Those (1) make all queries to h_1, h_2 before h_3 and (2) for any query (c, d) they make to h_3 are aware of all valid T_5 executions that are created this way. Concretely, with any query (c, d) they attach (a potentially empty) list of all pairs h_1, h_2 that are valid with c, d . This list cannot be long as we had argued for Theorem 1 where there cannot be more than n of them. We stress that this is a natural assumption as every valid execution ever considered by an algorithm must be discovered in some way, and if all h_3 queries are made last, it only makes sense to make queries that yield valid executions. The speculative nature of this argument makes us claim that the reduction is only heuristic.

Lemma 1. *Given any valid-aware algorithm A , which makes at most q queries to oracles h_1, h_2, h_3 , which solves (10) with probability ϵ , there is an algorithm A' making at most $q < 2^{n/2}$ queries to oracles F_1, F_2, F_3 (with runtime almost the same as A) that solves 3-XOR problem with probability at least $\epsilon/(4n)$.*

Proof. First we note that a solution to (10) must have $(c, d) \neq (c', d')$, i.e., h_3 gets a non-zero difference. Indeed, otherwise the output e of h_3 and thus m_5 must both have a zero difference, which in turn implies that a and b have a zero difference, i.e., we have found a collision in h_1 or h_2 which we are ruling out (or we can extend our reduction with this outcome).

Now, A works as follows:

- It calls A to find a collision between T_5 and T'_5 .
- When A queries $h_1(m_1, m_2)$ it is given $F_1(0||m_1||m_2)$.
- When A queries $h_2(m_3, m_4)$ it is given $F_1(1||m_3||m_4)$.
- When A queries $h_3(c, d)$ with list L of valid tuples (m_1, m_2, m_3, m_4) such that $h_1 \oplus h_2 = c \oplus d$. If the list is empty, A' returns $F_2(c||d) \oplus c$ to A . Otherwise A' flips a coin:
 - For tails A' selects a random entry of L and returns $F_3(c||d) \oplus c \oplus F_1(0||m_1||m_2)$ to A .
 - For heads A' returns $F_2(c||d) \oplus c$ to A .

Now suppose A finds a solution to (10). This implies that

$$h_1(m_1, m_2) \oplus h_3(c, d) \oplus c = h_1(m'_1, m'_2) \oplus h_3(c', d') \oplus c'.$$

Recall that $(c, d) \neq (c', d')$. Now note that:

- (c, d) yields a valid execution of T_5 . Note that the validity was known at the time of query.

- With probability at least $1/(2n)$ the value for $h_3(c, d)$ is selected as $F_3(c||d) \oplus c \oplus F_1(0||m_1||m_2)$ (i.e., we had selected the same m_1, m_2 as in h_1 of the solution) since we have at most n pairs (h_1, h_2) with given difference $c \oplus d$.
- With probability $1/2$ the value for $h_3(c', d')$ is selected as $F_3(c' || d') \oplus c'$.
- The value for $h_1(m_1, m_2)$ is $F_1(0||m_1||m_2)$ and should cancel out due to our outcome for $h_3(c, d)$.

Therefore with total probability at least $1/(4n)$ the solution found by **A** is translated into

$$F_3(c||d) = F_1(0||m'_1||m'_2) \oplus F_2(c' || d')$$

which yields a solution for the 3-XOR problem.

Together with the hardness conjecture, we obtain the following proposition.

Proposition 1 *Assuming the 3-XOR hardness and that the best algorithm that breaks full-local aggressive collision security is valid-aware, the adversary that runs in time q finds a full-local collision with probability at most $4n^3 q^2 / 2^n$.*

Reduction for local-local. The best known algorithm for solving 4-XOR problem runs in $O(n2^{n/3})$ time and $O(2^{n/3})$ queries [31]. So, we pose a similar conjecture for the 4-XOR problem.

Conjecture 2 (4-XOR hardness) *For any algorithm \mathcal{A} that makes $q < 2^{n/3}$ queries to random oracles F_1, F_2, F_3, F_4 , runs for time q , the probability that it solves the 4-XOR problem is at most $q^3 / 2^n$.*

Now we consider a simple variant of 4-XOR problem, called 4-XOR', which uses two lists. Let F, F' be oracles that output random n -bit strings. Find x, y, z, w with $(x, y) \neq (z, w)$ such that

$$F(x) \oplus F(y) \oplus F'(z) \oplus F'(w) = 0.$$

Lemma 2. *Given any algorithm \mathcal{A}' making at most q queries to all its oracles which solves the 4-XOR' problem with probability ϵ there is an algorithm **A** making at most q queries to all its oracles (with run time almost same as \mathcal{A}') which solves 4-XOR problem with probability at least $\epsilon/4$.*

Proof. The reduction from \mathcal{A}' to **A** works as follows. We run \mathcal{A}' and it makes two types of queries, namely F and F' . For each query we choose b randomly from $\{1, 2\}$. If it is an $F(x)$ query, then **A** returns $F_b(x)$. Similarly, if it is an $F'(z)$ query, **A** returns $F_{2+b}(z)$ to \mathcal{A}' . Finally, \mathcal{A}' returns (x, y, z, w) such that $F(x) \oplus F(y) \oplus F'(z) \oplus F'(w) = 0$. Now, **A** succeeds if $F(x) = F_b(x)$, $F(y) = F_{3-b}(y)$ and $F'(z) = F_{2+b'}(z)$, $F'(w) = F_{5-b'}(w)$. We note that the b values are chosen randomly. As F 's are independent random functions, the output to \mathcal{A}' is independent with b . In other words, the view of \mathcal{A}' remains independent with the b values chosen by **A**. Thus, **A** succeeds with probability $1/4$ given that \mathcal{A}' succeeds. \square

We have already seen that given a collision adversary B of T'_5 we can construct an algorithm A' for solving the 4-XOR' problem and hence we can construct an algorithm A solving the 4-XOR problem. Moreover the success probability of solving the 4-XOR is at least $\frac{1}{4} \cdot \text{Adv}_{T'_5}^{\text{Coll}}(B)$. This leads us to conclude with the following claim.

Proposition 2 *Assuming the 4-XOR hardness and that the best algorithm that breaks full-local aggressive collision security is valid-aware, the adversary that runs in time q finds a full-local collision with probability at most $4q^3/2^n$.*

6 Merkle-Damgård Variant

We can plug in our 5-to-1 compression function to the standard Merkle-Damgård (MD) mode to get a sequential hash t -to-1 function making only $3t/4$ calls to the underlying $2n$ -to- n compression functions h_1, h_2 and h_3 , and inheriting the birthday security of T_5 . This function is depicted in Figure 4.

PARALLEL IMPLEMENTATION. In addition to providing a 25% speedup when compared to the traditional MD mode applied to a $2n$ -to- n -compression function h , another advantage of our new variant is that it is easily parallelizable for architectures that support parallel execution.

For example, if we have three execution units P_1, P_2, P_3 , the unit P_i can be responsible for all h_i computations. This allows to compute a hash of $t = 4k$ message blocks using only $(k + 2) = (t/4 + 2)$ parallel rounds of hashing, saving a factor (almost) 4 over the traditional MD mode. The execution unit P_3 will be “one-round behind” units P_1 and P_2 (so that in the first round only P_1 and P_2 hash (m_1, m_2) and (m_3, m_4) , and in the last round only P_3 produces the final hash). Namely, when P_3 just completed the computation of the previous initial value IV_j , P_1 completed $a_j = h_1(m_{4j+1}, m_{4j+2})$, and P_2 completed $b_j = h_2(m_{4j+3}, m_{4j+4})$, in the next round P_3 will set the next initial value

$$IV_{j+1} = h_3(IV_j \oplus a, IV_j \oplus b) \oplus IV_j,$$

while P_1 and P_2 respectively compute $a_{j+1} = h_1(m_{4j+5}, m_{4j+6})$ and $b_{j+1} = h_2(m_{4j+7}, m_{4j+8})$.

Similarly, two execution units P'_1, P'_2 can perform the same task in only $2k = t/2$ parallel rounds, saving a factor 2 over the traditional MD mode.

LARGER COMPRESSION. Although our results are stated for $2n$ -to- n compression functions, we can also easily extend them to the λn -to- n case, by simply adding $(\lambda - 2)$ “dummy” inputs to each of h_1, h_2, h_3 . For example, for $\lambda = 3$, this gives us an $8n$ -to- n analog of T_5 , using three calls to some $3n$ -to- n hash function. Which gives a new MD mode for $t = 7k$ block messages, using only $3k = 3t/7$ compression calls. In contrast, a naive MD mode will use $t/2$ calls, saving a factor $(1 - (3t/7)/(t/2)) = 1/7 \approx 14.2\%$. Similar calculations can be done for larger λ .

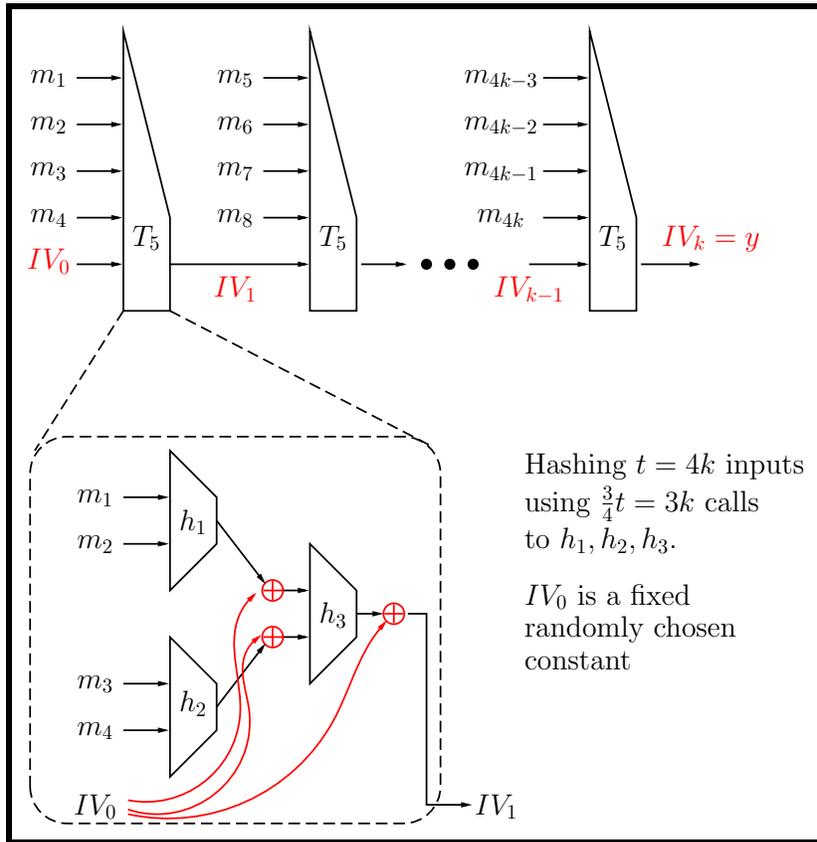


Fig. 4. Merkle-Damgård Hash based on T_5 .

7 Merkle Tree Variant

Our 2-level construction extends straightforwardly to a full-blown t -to-1 tree H . In Section 7.1 we briefly recall how to build Merkle Trees (MT) from smaller compression functions (such as T_5). We then present our faster and shallower MT variant and its properties in Section 7.2.

7.1 General Merkle Trees and Their Security

The Merkle tree is a data structure to store long lists of t n -bit elements, so that insertion, deletion, update, or proof of membership for an element need only $O(\log t)$ time and space. It is built on a λn -to- n compression function H_λ (typically $\lambda = 2$ but other values are used too) and retains all its security properties regarding collision and preimage resistance from the compression function. The

tree is defined recursively:

$$\begin{aligned} \text{MT}_\lambda(\underbrace{m_1, \dots, m_\lambda}_{m_{[1:\lambda]}}) &= H_\lambda(m_1, \dots, m_\lambda); \\ \text{MT}_{\lambda t}(m_{[1:\lambda t]}) &= H_\lambda(\text{MT}_\lambda(m_{[1:\lambda]}), \dots, \text{MT}_\lambda(m_{[\lambda t - t + 1:\lambda t]})) \end{aligned}$$

where $t = \lambda^k \geq 2$, with the last hash called a root and m_i called leafs.

As for the compression function, we define the terms full opening and local opening for the entire MT, with the former being all t elements and the latter for a leaf in a tree is a sequence of $\log_\lambda t$ local openings of an element in all compression functions on the path from the leaf to the root. In the simplest case $\lambda = 2$ an opening is one element per tree layer, whereas for wider compression functions it is $\lambda - 1$ or fewer (in case H_λ has “aggressive” local opening). The full-local security for the tree is defined analogously to the compression function and corresponds to the case of a public tree to which an adversary makes a forged membership proof. The local-local security matches the case when the adversary provides two valid openings for an alleged tree root but the full tree is unknown. Both full-local and local-local security for the tree follows from their compression function counterparts. So overall we have the following parameters:

- Efficiency $E(t)$ as the number of compression function calls is $(t-1)/(\lambda-1)$.
- Depth $D(t)$ of the tree is $\log_\lambda t$.
- Update/insert/delete complexity as the number of compression function recomputations equals $D(t)$.
- The total length of *conservative* opening $L(t)$ is $(\lambda-1) \log_\lambda t$.
- In case H_λ might have a more compact (i.e., “aggressive”) local opening of length $\ell \leq \lambda-1$, then the total length of the resulting “aggressive” local opening for MT_λ becomes $L(t) = \ell \log_\lambda t$.
- Number of calls to F needed to verify the opening: $V(t) = \log_\lambda t$.
- Collision, full-local, and local-local opening security the same as that of H_λ (ideally $2^{n/2}$, if one uses a $2n$ -to- n hash function as last building block).
- Preimage security the same as that of H_λ (ideally 2^n , if one uses $2n$ -to- n hash function as last building block).

7.2 Faster and Shallower Merkle Tree based on T_5

Our construction extends straightforwardly to a full-blown t -to-1 tree H of any depth k , as depicted in Figure 5, with optimal $t = 5^k$. Notice, unlike our compression function T_5 for H , which needed domain separation for functions h_1, h_2, h_3 (see Section 8.4), the final tree H can reuse the same h_1, h_2, h_3 across all invocations, which follows from general security properties of standard Merkle trees.

The overall trade-offs of our construction are summarized in Table 1, but we expand on it below.

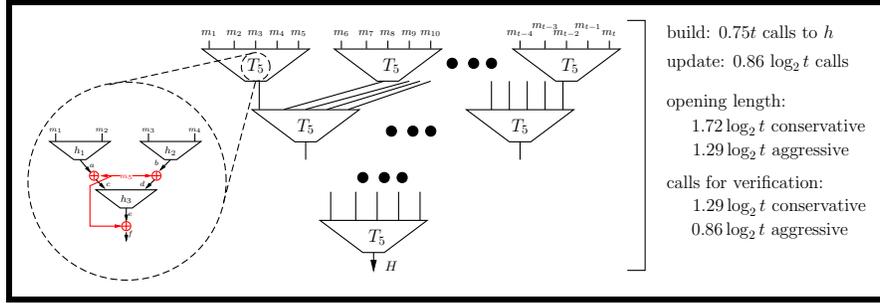


Fig. 5. Full-blown tree based on T_5 . Security is $2^{n/2}$ (tight) for conservative openings, $2^{n/3}$ provable and $2^{n/2}$ heuristic for full-local aggressive opening, $2^{n/4}$ provable and $2^{n/3}$ heuristic for local-local aggressive opening.

Efficiency. Let us summarize the performance of our tree construction:

- **Build:** In H we compress every 5 inputs using our $5n$ -to- n compression function T_5 , therefore using $(t-1)/4$ calls to T_5 , which is equivalent to

$$E(t) = 0.75(t-1)$$

calls to the h_j 's, thus giving us 25% improvement over the regular Merkle tree.

- **Depth/update:** The depth $D(t)$ of the tree, measured in the calls to the h_j 's, reduces from $\log_2 t$ to

$$D(t) = 2 \log_5 t \approx 0.86 \log_2 t$$

which is a 14% saving compared to standard Merkle trees.

- **Opening length (conservative):** In the conservative opening we open all 4 siblings, thus $L(t)$ increases from $\log_2 t$ of standard Merkle trees to $4 \log_5 t \approx 1.72 \log_2 t$ (loss of 72%). It is still useful though when bandwidth is not critical which is sometimes the case.
- **Opening length (aggressive):** In the aggressive opening we open 3 elements, thus $L(t)$ increases to only $3 \log_5 t \approx 1.29 \log_2 t$ (loss of 29%, which may be tolerated for some applications).
- **Verification time (conservative):** the number of h_i calls needed to verify the proof increases to $V(t) = 3 \log_5 t \approx 1.29 \log_2 t$.
- **Verification time (aggressive):** the number of h_i calls needed to verify the proof decreases to $V(t) = 2 \log_5 t \approx 0.86 \log_2 t$. This is very handy in applications when opening verification time is crucial (such as zero-knowledge membership proofs where proving time linearly depend on the circuit size needed for the opening verification). However we pay for this with security (see below).

Table 1. We compare standard Merkle trees to two variants of Merkle trees with T_5 . The conservative variant requires opening four siblings in a local opening proof, compared to three siblings in the aggressive variant. The boxed formulas are conjectures based on the 3-XOR and 4-XOR problems. We have $2/\log_2 5 \approx 0.86$, $3/\log_2 5 \approx 1.29$, $4/\log_2 5 \approx 1.72$. Note that full-full collision resistance (CR) security is listed for completeness, this is the “traditional” collision resistance involving two distinct messages (and the entire corresponding Merkle trees).

	Standard Merkle	Merkle with T_5 (conservative)	Merkle with T_5 (aggressive)
build calls/ t	1	0.75	0.75
update/ $\log_2 t$	1	0.86	0.86
verify/ $\log_2 t$	1	1.29	0.86
opening/ $\log_2 t$	1	1.72	1.29
full-full CR security	$n/2$	$n/2$	$n/2$
full-local CR security	$n/2$	$n/2$	$n/3 \rightarrow \boxed{n/2}$
local-local CR security	$n/2$	$n/2$	$n/4 \rightarrow \boxed{n/3}$

Security. Since our construction applies the standard Merkle paradigm to the 5-to-1 compression function T_5 , the resulting hash function H inherits the same global (or local opening) collision and preimage security as the compression function T_5 :

- **Full-local (aggressive) security:** provable security up to (taking $poly(n)$ factors aside) $2^{n/3}$ queries (Theorem 3), heuristic security up to $2^{n/2}$ running time (Proposition 1).
- **Local-local (aggressive) security:** provable security up to $2^{n/4}$ queries (Theorem 3), heuristic security up to $2^{n/3}$ running time (Proposition 2).
- **Both full-local and local-local (conservative) security:** provable security up to $2^{n/2}$ queries (Theorem 1).

Additionally, non-trivial preimage security holds up to $2^{2n/3}$ queries (and at optimal level $O(q/2^n)$, in the region of interest where $q \leq 2^{n/2}$).

Applications. Merkle trees are used in a number of protocols, but their exhaustive list is beyond the scope of this paper. To name just a few: anonymous cryptocurrencies and mixers [13, 23], interactive oracle proof (IOP) compilers [2, 3], and post-quantum hash-based signatures [5]. Among them, cryptocurrencies and mixers are the examples of publicly controlled Merkle trees, i.e., where the full-local opening makes sense and where the aggressive opening with its improved verification time is appealing.

From all this diverse range of applications, the ones who benefit also from the conservative opening strategy are those for which the performance and depth are more important than the local opening size. Of course, basic hashing by

itself is a very important example of such an application. The next examples are zero-knowledge proof systems where circuit depth is a major performance factor [32, 33]. Finally, a more speculative area where our construction could help is multiparty computation protocols applied to functionalities involving hashing, whose complexity depends on the circuit depth (e.g., variants of the original BGW [1] and GMW [11] protocols).

Summary. Our construction has clear advantage over the regular Merkle tree in build and update efficiency, but loses in the opening length. For the verification time and security we have a tradeoff: an aggressive opening needs fewer h_i calls but only heuristic security argument of $2^{n/2}$ with provable security reaching $2^{n/3}$ only, whereas the conservative opening has the same security properties as in the regular tree but requires 30% more verification calls. Thus whether or not our construction outperforms the regular Merkle tree depends on the setting.

8 Matching Attacks and Warnings

In this section we discuss the tightness of our results (by providing matching attacks), as well as show that certain natural improvement or simplification ideas do not work.

8.1 Matching Collision Attack on T_5

A simple birthday collision search with complexity $2^{n/2}$ applies to T_5 in the full-full and full-local collision settings, matching Theorem 1 and Proposition 1.

8.2 Matching Preimage Attack on T_5

We provide a brief description of a preimage attack with query complexity $2^{2n/3}$, which naturally follows from a bad event we aimed to exclude in the preimage security proof.

Preimage attack. Given challenge f , let us make $2^{2n/3}$ queries to both h_1 and h_2 . There will be about $2^{4n/3-n} = 2^{n/3}$ collisions between h_1 and h_2 . Let \mathcal{A} denote the $2^{n/3}$ outputs of these collisions, so that $|\mathcal{A}| = 2^{n/3}$ and we know a valid preimage (m_1, m_2, m_3, m_4) for the pair (a, a) under h_1 and h_2 , for each $a \in \mathcal{A}$.

Now we make $2^{2n/3}$ queries to h_3 in form (c, c) , and for each output e check if there exists $a \in \mathcal{A}$ s.t.

$$a \oplus c = e \oplus f$$

Each such equation holds with probability 2^{-n} , but we try $2^{n/3} \cdot 2^{2n/3} = 2^n$ different choices of a and c , so expect one such equation to hold with constant probability. When the match happens, we can set $m_5 = a \oplus c = e \oplus f$, and the tuple $(m_1, m_2, m_3, m_4, m_5)$ will be a valid preimage of f , where (m_1, m_2, m_3, m_4) is the colliding preimage of (a, a) under h_1 and h_2 .

Analysis. This attack requires $2^{2n/3}$ time and memory but can be computed memoryless using unbalanced meet-in-the-middle attack [19] with time complexity about $2^{3n/4}$.

8.3 Matching Attack on Aggressive Opening Mode

Here we demonstrate a matching attack on the aggressive opening, which uses 3 (vs. 4) siblings to open a T_5 commitment.

Concretely we construct a collision for T_5^1 . Consider the equation

$$h_3(c, d) \oplus c \oplus h_1(m_1, m_2) = h_3(c', d') \oplus c' \oplus h_1(m'_1, m'_2)$$

By making $2^{n/4}$ queries to h_1 and h_3 , we expect such colliding tuples will exist with constant probability, even if we insist on having $m_1 \neq m'_1$. Given such a tuple, let

$$h_3(c, d) \oplus c \oplus h_1(m_1, m_2) = f = h_3(c', d') \oplus c' \oplus h_1(m'_1, m'_2),$$

$m_5 = c \oplus h_1(m_1, m_2)$, and $m'_5 = c' \oplus h_1(m'_1, m'_2)$. Then it is easy to see that (m_2, m_5, d) and (m'_2, m'_5, d') are valid preimages of f to $m_1 \neq m'_1$, respectively.

Similar attack works for any other permutation involving opening some m_i using only 3 other values. Note however, that even though theoretically $2^{n/4}$ queries suffice, the best complexity of this attack is $2^{n/3}$ rather than $2^{n/4}$ (cf. the 4-XOR problem in Section 5).

8.4 Attack on T_5 with Identical Compression Functions

Now we show that our requirement that all h_i are distinct is crucial: without it, a subtle attack is possible. Let $T_5^1 : \{0, 1\}^{5n} \rightarrow \{0, 1\}^n$ based on the *identical* $2n$ to n -bit compression functions $h_1, h_2, h_3 = h$ be defined as follows:

$$T_5^1(m_1, m_2, m_3, m_4, m_5) := h(h(m_1, m_2) \oplus m_5, h(m_3, m_4) \oplus m_5) \oplus m_5$$

Collision attack on T_5^1 . Note that the only difference here with T_5 is the fact that the internal compression functions in T_5^1 are identical. In the $2^{n/4}$ query attack below, we will only use $h_1 = h_2$, though, so we could keep the index h_3 intact. However, similar attack will hold if we only use $h_1 = h_3$ or $h_2 = h_3$, meaning that domain separation is needed on all three compression functions.

We make $2^{n/4}$ queries (m_1, m_2) to h , and let \mathcal{A} be the set of $((m_1, m_2), a)$ input-output pairs obtained. Also, make $2^{n/4}$ queries (c, c) to $h_3 = h$, and let \mathcal{C} be the set of $((c, c), e)$ input-output pairs obtained. Solve the equation

$$a \oplus (e \oplus c) = a' \oplus (e' + c') \tag{11}$$

for $a \neq a'$, where $((m_1, m_2), a), ((m'_1, m'_2), a') \in \mathcal{A}, ((c, c), e), ((c', c'), e') \in \mathcal{C}$. We expect to find a solution with constant probability. Define

$$m_3 = m_1, m_4 = m_2, m_5 = a \oplus c, m'_3 = m'_1, m'_4 = m'_2, m'_5 = a' \oplus c'$$

Then, using (11),

$$\begin{aligned}
T_5^1(m_1, m_2, m_3, m_4, m_5) &= T_5^1(m_1, m_2, m_1, m_2, a \oplus c) \\
&= h(a \oplus (a \oplus c), a \oplus (a \oplus c)) \oplus (a \oplus c) \\
&= e \oplus a \oplus c = e' \oplus a' \oplus c' \\
&= h(a' \oplus (a' \oplus c'), a' \oplus (a' \oplus c')) \oplus (a' \oplus c') \\
&= T_5^1(m'_1, m'_2, m'_1, m'_2, a' \oplus c') \\
&= T_5^1(m'_1, m'_2, m'_3, m'_4, m'_5).
\end{aligned}$$

8.5 Attack on T_5 Without the Final XOR

The next attack is on the modified T_5 construction where the final XOR into the output of h_3 from the T_5 construction is omitted; this attack provides more insight on the rationale behind the main T_5 construction, by showing that the XOR of m_5 into the output of h_3 is essential.

Let $T_5^2 : \{0, 1\}^{5n} \rightarrow \{0, 1\}^n$ based on the $2n$ to n -bit compression functions h_1, h_2, h_3 be defined as follows:

$$T_5^2(m_1, m_2, m_3, m_4, m_5) := h_3(h_1(m_1, m_2) \oplus m_5, h_2(m_3, m_4) \oplus m_5)$$

Collision attack on T_5^2 . The attack follows from the generalized birthday attack which finds a quartet of distinct inputs $((m_1, m_2), (m_3, m_4), (m'_1, m'_2), (m'_3, m'_4))$ such that

$$h_1(m_1, m_2) \oplus h_1(m'_1, m'_2) \oplus h_2(m_3, m_4) \oplus h_2(m'_3, m'_4) = 0.$$

This can be done after $2^{n/4}$ queries to each of the hash function with constant probability. Let $h_1(m_1, m_2) \oplus h_1(m'_1, m'_2) = h_2(m_3, m_4) \oplus h_2(m'_3, m'_4) = \Delta$, then by setting $m'_5 = m_5 \oplus \Delta$ we derive

$$\begin{aligned}
T_5^2(m_1, m_2, m_3, m_4, m_5) &= h_3(h_1(m_1, m_2) \oplus m_5, h_2(m_3, m_4) \oplus m_5) \\
&= h_3(h_1(m'_1, m'_2) \oplus \Delta \oplus m_5, h_2(m'_3, m'_4) \oplus \Delta \oplus m_5) \\
&= h_3(h_1(m'_1, m'_2) \oplus m'_5, h_2(m'_3, m'_4) \oplus m'_5) \\
&= T_5^2(m'_1, m'_2, m'_3, m'_4, m'_5)
\end{aligned}$$

9 Generalizations and Future Work

Our construction is likely to be extended to wider compression functions. For example, a natural generalization of our construction can hash $T = 3 \cdot 2^k - 1$ inputs using only $E = 2 \cdot 2^k - 1 = (2T - 1)/3$ evaluations, where standard $2n$ -to- n hash h corresponds to $k = 0$, and our T_5 corresponds to $k = 1$. As k increases, $E(k)/T(k) \rightarrow 2/3$, which matches Stam's bound for building Tn -to- n hash functions from $2n$ -to- n compression functions. Unfortunately, as k grows, the local opening size also grows, so it is unclear if this overall hash saving is

worthwhile for applications. We leave the security analysis of this, and other optimized hashing constructions to future work.

Finally, it remains to prove the reduction of the full-local aggressive security to the 3-XOR problem unconditionally, i.e., without restrictions on the adversary.

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A Formal Proof of Collision Resistance of T_5

Consider any collision finding adversary A making a total of q queries to h_1, h_2, h_3 . Without loss of generality, we assume that A does not duplicate any of its queries, and is deterministic. Thus, all the probabilities are computed over the randomness of the ideal hash functions h_1, h_2, h_3 .

For any index $i = 0 \dots q$, the run of A defines the following random variables:

- Q_1^i, Q_2^i, Q_3^i : set of input-output pairs of h_1, h_2, h_3 known to A after its i -th query. In particular, $((m_1, m_2), a) \in Q_1^i$ if and only if one of the first i queries of A was a query (m_1, m_2) to h_1 , and a was the answer to this query. Same for h_2 and h_3 .
- \mathbf{Eval}^i : the set of input-output pairs for the entire construction T_5 known to A after its i -th query. Formally, we look at all triples of *consistent* queries $((m_1, m_2), a) \in Q_1^i, ((m_3, m_4), b) \in Q_2^i$ and $((c, d), e) \in Q_3^i$, and for each such triple we add the resulting input-output pair (M, f) defined by this triple to the set \mathbf{Eval}^i .
- For any $z \in \{0, 1\}^n$,

$$C_{12}^i(z) = \{ (((m_1, m_2), a) \in Q_1^i, ((m_3, m_4), b) \in Q_2^i) \mid a \oplus b = z \} \quad (12)$$

Jumping ahead, these sets will determine how many fresh consistent triples of queries can be obtained by A , by making a new query (c, d) to h_3 .

- For any $f \in \{0, 1\}^n$,

$$C_{13}^i(f) = \{ (((m_1, m_2), a) \in Q_1^i, ((c, d), e) \in Q_3^i) \mid a \oplus e = f \oplus c \}$$

$$C_{23}^i(f) = \{ (((m_3, m_4), b) \in Q_2^i, ((c, d), e) \in Q_3^i) \mid b \oplus e = f \oplus d \}$$

Jumping ahead, for any output value f of T_5 (presumably obtained by some consistent triple of queries), the set $C_{13}^i(f)$ (resp. $C_{23}^i(f)$) will determine how many tuples of queries to h_1 (resp. h_2) and h_3 have a chance to cause a collision with f , before a new query to h_2 (resp. h_1) is made.

We let $Q_1 = Q_1^q, Q_2 = Q_2^q, Q_3 = Q_3^q, C_{12}(z) = C_{12}^q(z), C_{13}(f) = C_{13}^q(f), C_{23}(f) = C_{23}^q(f)$ and $\mathbf{Eval} = \mathbf{Eval}^q$ be the final values of the corresponding sets at the end of execution. We can now define the following “bad events” at the end of A ’s execution:

- B_1 : collision in either Q_1, Q_2 , or “shifted” Q_3 . Formally:
 - $\exists ((m_1, m_2), a), ((m'_1, m'_2), a) \in Q_1$ with $(m_1, m_2) \neq (m'_1, m'_2)$, or

- $\exists((m_3, m_4), b), ((m'_3, m'_4), b) \in Q_2$ with $(m_3, m_4) \neq (m'_3, m'_4)$, or
- $\exists((c, d), e), ((c', d'), e') \in Q_3$ with $(c, d) \neq (c', d')$ and either $e \oplus c = e' \oplus c'$ or $e \oplus d = e' \oplus d'$.
- B_2 : $\max_z |C_{12}(z)| \geq n$, or $\max_f |C_{13}(f)| \geq n$, or $\max_f |C_{23}(f)| \geq n$.
- B_3 : $|\mathbf{Eval}| \geq qn$; i.e., number of valid evaluations of T_5 is “too large”.
- B_4 : collision on T_5 found; i.e., $(M, f), (M', f) \in \mathbf{Eval}$ with $M \neq M'$.

Our goal is to upper bound the probability of B_4 . For that, we split this probability into four disjoint events, as follows:

$$\Pr[B_4] \leq \Pr[B_1] + \Pr[B_2 \cap \overline{B_1}] + \Pr[B_3 \cap \overline{B_1 \cap B_2}] + \Pr[B_4 \cap \overline{B_1 \cap B_2 \cap B_3}]$$

In the following propositions (proven in subsequent subsections), we bound each of these 4 terms separately.

Proposition 3 $\Pr[B_1] \leq \frac{2q^2}{2^n}$

Proposition 4 $\Pr[B_2 \cap \overline{B_1}] \leq 3 \cdot \left(\frac{2q^2}{2^n}\right)^n$

Proposition 5 $\Pr[B_3 \cap \overline{B_1 \cap B_2}] \leq \frac{2q^2}{n2^n}$

Proposition 6 $\Pr[B_4 \cap \overline{B_1 \cap B_2 \cap B_3}] \leq \frac{n^2 q^2}{2^n}$

Combining the results of Propositions 3,4,5,6, we get our final bound in (3):

$$\Pr[B_4] \leq \frac{2q^2}{2^n} + 3 \cdot \left(\frac{2q^2}{2^n}\right)^n + \frac{2q^2}{n2^n} + \frac{n^2 q^2}{2^n} \leq \frac{(n^2 + 10)q^2}{2^n},$$

assuming that $\frac{2q^2}{2^n} \leq 1$. Hence, it suffices to prove Propositions 3,4,5,6.

Bounding Simple Collisions (Proof of Proposition 3) For completeness, we repeat the claimed bound:

$$\Pr[B_1] \leq \frac{2q^2}{2^n}$$

Each of the 4 events in B_1 (collision in h_1 , h_2 , and two types of collisions in h_3) correspond to standard birthday bounds in ideal hash functions $h_1(m_1, m_2)$, $h_2(m_3, m_4)$, $h_3(c, d) \oplus c$ and $h_3(c, d) \oplus d$. Each such bound is $q^2/2^{n+1}$, so the final bound is this birthday bound times 4.

Bounding Multi-Collisions (Proof of Proposition 4) For completeness, we repeat the claimed bound:

$$\Pr[B_2 \cap \overline{B_1}] \leq 3 \cdot \left(\frac{2q^2}{2^n}\right)^n$$

Intuitively, each of the sets $C_{jk}(\cdot)$ corresponds to the fact that “many” pairs of queries to h_j and h_k produced outputs whose XOR is equal to a fixed value. Clearly, this probability gets lower and lower as we increase the value of “many.” The calculation below shows that setting it to n drives this probability below the “birthday” bound we are seeking. We use the standard union bound several times to upper bound the desired probability as follows:

$$\begin{aligned}
\Pr [B_2 \cap \overline{B_1}] &\leq \Pr \left[(\max_z |C_{12}(z)| \geq n) \cap \overline{B_1} \right] + \\
&\quad \Pr \left[(\max_f |C_{13}(f)| \geq n) \cap \overline{B_1} \right] + \\
&\quad \Pr \left[(\max_f |C_{23}(f)| \geq n) \cap \overline{B_1} \right] \\
&\leq 2^n \cdot \max_z \Pr [(|C_{12}(z)| \geq n) \cap \overline{B_1}] + \\
&\quad 2^n \cdot \max_f \Pr [(|C_{13}(f)| \geq n) \cap \overline{B_1}] + \\
&\quad 2^n \cdot \max_f \Pr [(|C_{23}(f)| \geq n) \cap \overline{B_1}]
\end{aligned}$$

We upper bound each of the three probabilities below (see (15), (16), and (17)) by $(q^2/2^n)^n$, which completes the proof, since then

$$\Pr [B_2 \cap \overline{B_1}] \leq 3 \cdot 2^n \cdot \left(\frac{q^2}{2^n} \right)^n = 3 \cdot \left(\frac{2q^2}{2^n} \right)^n$$

Let us start with $\Pr [(|C_{12}(z)| \geq n) \cap \overline{B_1}]$ for any fixed z . First, recall that $C_{12}(z) = \{ ((*, a) \in Q_1, (*, b) \in Q_2) \mid a \oplus b = z \}$. Define $\Delta_{12}^i(z) = |C_{12}^i(z) \setminus C_{12}^{i-1}(z)|$, which is the number of tuples added to $C_{12}^i(z)$ because a new evaluation in query i was made. Then $|C_{12}(z)| = \sum_{i=1}^q \Delta_{12}^i(z)$. Now, in order for the sum of q non-negative integers to be at least n , either one of the terms is strictly greater than 1, or at least n of the terms are non-zero:

$$\begin{aligned}
\Pr [(|C_{12}(z)| \geq n) \cap \overline{B_1}] &= \Pr \left[\left(\sum_{i=1}^q \Delta_{12}^i(z) \geq n \right) \cap \overline{B_1} \right] \\
&\leq \Pr [(\exists i \text{ s.t. } \Delta_{12}^i(z) > 1) \cap \overline{B_1}] + \\
&\quad \Pr [\exists \geq n \text{ indices } i \text{ s.t. } \Delta_{12}^i(z) > 0] \\
&\leq q \cdot \max_i (\Pr [(\Delta_{12}^i(z) > 1) \cap \overline{B_1}]) + \tag{13}
\end{aligned}$$

$$q^n \cdot \max_{i_1 \dots i_n} (\Pr [\forall i \in \{i_1 \dots i_n\}, \Delta_{12}^{i_i}(z) > 0]) \tag{14}$$

Now, let us look at the i -th query of A . Clearly, a query to h_3 does not increase $|C_{12}^i(z)|$, so by symmetry let us assume that query i was a query (m_1, m_2) to h_1 .

Assume the random answer is $h_1(m_1, m_2) = a$. This new query $((m_1, m_2), a)$ will “match up” with an old query $((m_3, m_4), b) \in Q_2^i$ only if $a = b \oplus z$. Moreover, the only way it will also match up with another such query $((m'_3, m'_4), b') \in Q_2^i$ is if $a = b \oplus z = b' \oplus z$. But this means that $b = b'$ is a collision in Q_2 , triggering the bad event B_1 . Hence, the probability in (13) is 0:

$$\Pr [(\Delta_{12}^i(z) > 1) \cap \overline{B_1}] = 0$$

Moreover, since $|Q_2^i| \leq q$, there are at most q values $\{b \oplus z \mid (*, b) \in Q_2^i\}$. And since the answer a to $h_1(m_1, m_2)$ is truly random, the chance it will be equal one of the values in this set is at most $q/2^n$, which means

$$\Pr [\Delta_{12}^i(z) > 0] \leq \frac{q}{2^n}$$

Coming back to (14), we see that the n events corresponding to distinct indices $i_1 \dots i_n$ are independent, meaning

$$\Pr [\forall i \in \{i_1 \dots i_n\}, \Delta_{12}^i(z) > 0] \leq \left(\frac{q}{2^n}\right)^n$$

Hence, for any z ,

$$\Pr [(|C_{12}(z)| \geq n) \cap \overline{B_1}] \leq q \cdot 0 + q^n \cdot \left(\frac{q}{2^n}\right)^n \leq \left(\frac{q^2}{2^n}\right)^n \quad (15)$$

Completely analogously, we can argue that for any f ,

$$\Pr [(|C_{13}(f)| \geq n) \cap \overline{B_1}] \leq \left(\frac{q^2}{2^n}\right)^n \quad (16)$$

$$\Pr [(|C_{23}(f)| \geq n) \cap \overline{B_1}] \leq \left(\frac{q^2}{2^n}\right)^n \quad (17)$$

For example, the only difference between the argument for $C_{13}(f)$ and $C_{12}(z)$ is that the key equation becomes $a \oplus e = f \oplus c$ (instead of $a = b \oplus z$). When the i -th query is to $h_3(c, d)$, the answer e matching up with two values a and a' will imply a collision in Q_1 , as before (since $a = a' = e \oplus f \oplus c$). However, when the i -th query is to $h_1(m_1, m_2)$, the answer a matching up with two values e and e' will imply a collision in “shifted” Q_3 , because now we have $a \oplus f = e \oplus c = e' \oplus c'$. And this is why we defined this as the “ h_3 -collision” when defining the bad event B_1 . Similarly, the argument for $C_{23}(f)$ will result in $b \oplus f = e \oplus d = e' \oplus d'$.

This completes the proof.

Bounding Evaluation Set of T_5 (Proof of Proposition 5) For completeness, we repeat the claimed bound:

$$\Pr [B_3 \cap \overline{B_1} \cap \overline{B_2}] \leq \frac{2q^2}{n2^n}$$

Recall, B_3 states that $|\mathbf{Eval}| \geq qn$, meaning A completed at least qn evaluations of T_5 in its q queries. Let us denote by q_j , for $j = 1, 2, 3$, the (random variable equal to the) number of calls to h_j . And by \mathbf{Eval}_j the number of new evaluations of T_5 resulting from the q_j calls to h_j ; namely, it was the call to h_j that finalized the consistent triplet which defined this evaluation of T_5 . Clearly, $q_1 + q_2 + q_3 = q$ and $|\mathbf{Eval}| = |\mathbf{Eval}_1| + |\mathbf{Eval}_2| + |\mathbf{Eval}_3|$, which means that

$$\Pr [(|\mathbf{Eval}| \geq qn) \cap \overline{B_1 \cap B_2}] \leq \sum_{j=1}^3 \Pr [(|\mathbf{Eval}_j| \geq q_j n) \cap \overline{B_1 \cap B_2}] \quad (18)$$

Namely, to complete more than qn total evaluations, at least one of the functions h_1, h_2, h_3 should have been “responsible” for at least $q_j n$ such evaluations. To complete the proof of the proposition, it thus suffices to show that

$$\Pr [(|\mathbf{Eval}_1| \geq q_1 n) \cap \overline{B_1 \cap B_2}] \leq \Pr [|\mathbf{Eval}_1| \geq q_1 n] \leq \frac{q^2}{n2^n} \quad (19)$$

$$\Pr [(|\mathbf{Eval}_2| \geq q_2 n) \cap \overline{B_1 \cap B_2}] \leq \Pr [|\mathbf{Eval}_2| \geq q_2 n] \leq \frac{q^2}{n2^n} \quad (20)$$

$$\Pr [(|\mathbf{Eval}_3| \geq q_3 n) \cap \overline{B_1 \cap B_2}] = 0 \quad (21)$$

We start with the easier (21), showing that queries to h_3 simply cannot cause so many new evaluations without triggering the bad event B_2 . Recall, the set \mathbf{Eval}_3 can grow from 0 to its final value in q_3 queries to h_3 (somehow dispersed among A 's q total queries). And to achieve the final value $q_3 n$, at least one such query to h_3 must define at least n new evaluations of T_5 . Namely, for some query $1 \leq i \leq q$, this query (c, d) was made to h_3 , and $|\mathbf{Eval}^i \setminus \mathbf{Eval}^{i-1}| \geq n$. But the only way this query (c, d) could “match up” to two prior queries $(*, a) \in Q_1^i$ and $(*, b) \in Q_2^i$ is if $m_5 = a \oplus c = b \oplus d$. This is equivalent to $a \oplus b = c \oplus d$, which is in turn equivalent to saying that the tuple of queries $((*, a), (*, b)) \in C_{12}(c \oplus d)$. So in order for such query (c, d) to match up with at least n tuples $((*, a), (*, b)) \in Q_1^i \times Q_2^i$, it must be the case that $|C_{12}(c \oplus d)| \geq n$. But this immediately triggers the bad event B_2 , meaning that the probability in question is 0, completing the proof of (21).

Thus, it remains to show (19), as (20) is symmetric. Assume $i_1, \dots, i_{q_1} \in \{1 \dots q\}$ are the indices of the q_1 queries to h_1 , and let $\Delta^k = |\mathbf{Eval}^{i_k} \setminus \mathbf{Eval}^{(i_k-1)}|$ be the number of new evaluations of T_5 during the k -th query to h_1 , where $1 \leq k \leq q_1$. Then $|\mathbf{Eval}_1| = \Delta_1 + \dots + \Delta_{q_1}$. Moreover, we claim that the expected value of Δ_k is at most $q^2/2^n$: $\mathbb{E}[\Delta_k] \leq q^2/2^n$. To see this, $|Q_2|, |Q_3| \leq q$, and each pair $((*, b), ((c, d), *)) \in Q_2 \times Q_3$ would only match the value a returned by h_1 if $a = b \oplus c \oplus d$. The probability of this event is $1/2^n$, and the maximum number of such pairs is at most $|Q_2 \times Q_3| \leq q^2$, giving $\mathbb{E}[\Delta_k] \leq q^2/2^n$. But this means

$$\mathbb{E}[|\mathbf{Eval}_1|] = \sum_{k=1}^{q_1} \mathbb{E}[\Delta_k] \leq \frac{q_1 q^2}{2^n}$$

By Markov's inequality,

$$\Pr[|\mathbf{Eval}_1| \geq q_1 n] \leq \Pr\left[|\mathbf{Eval}_1| \geq \mathbb{E}[|\mathbf{Eval}_1|] \cdot \frac{n2^n}{q^2}\right] \leq \frac{q^2}{n2^n}$$

establishing (19), and completing the proof of proposition.

Final Collision Probability (Proof of Proposition 6) For completeness, we repeat the claimed bound:

$$\Pr[B_4 \cap \overline{B_1 \cap B_2 \cap B_3}] \leq \frac{n^2 q^2}{2^n}$$

Let X_i be the event that none of bad events B_1 , B_2 or B_3 happened, and that the i -th query of A generates a fresh collision to T_5 , where $1 \leq i \leq q$. We upper bound $\Pr[X_i]$ separately depending on whether this was the call to h_1 , h_2 or h_3 . In each of these case we show $\Pr[X_i] \leq n^2 q / 2^n$, meaning that this bound is correct irrespective of the identity of the i -th query.

Cases 1-2: query to h_1 or h_2 . Since these cases are symmetric, assume the i -th query is the query (m_1, m_2) to h_1 . In order for this query to cause a collision to T_5 , one of two things must happen. First, it could be that this very query (m_1, m_2) defined a value a which induced two distinct consistent triples

$$\begin{aligned} & ((m_1, m_2), a), ((m_3, m_4), b), ((c, d), e) \\ & \neq ((m_1, m_2), a), ((m'_3, m'_4), b'), ((c', d'), e') \end{aligned}$$

which collided. But this means that $f = a \oplus (e \oplus c) = a \oplus (e' \oplus c')$, or $e \oplus c = e' \oplus c'$. But since we assumed B_1 failed, there are no collisions in “ c -shifted” h_3 , which means $(c, d) = (c', d')$. But then, since both tuples are consistent,

$$b = a \oplus c \oplus d = a \oplus c' \oplus d' = b'$$

which means that there is a collision on h_2 , which again violates B_1 .

Hence, in order for the query (m_1, m_2) to cause a collision in T_5 , there must be some *previous* value $(*, f) \in \mathbf{Eval}^{i-1}$ s.t. the random answer $h_1(m_1, m_2) = a$ caused the collision with this f . In turn, this means that there are two queries $(*, b) \in Q_2^{i-1}$ and $((c, d), e) \in Q_3^{i-1}$ such that the values $m_5 = b \oplus d = e \oplus f$ match, which means that $b \oplus e = f \oplus d$, and, hence the tuple $((*, b), ((c, d), e) \in C_{23}(f)$. Moreover, for the value a to actually match this tuple, we must have $a = b \oplus c \oplus d$, which happens with probability $1/2^n$.

To summarize, in order for the i -th query to h_1 to cause the collision, the following events must happen:

- There exists a tuple $(*, f) \in \mathbf{Eval}^{i-1}$.
- There exists a tuple $((*, b), ((c, d), e)) \in C_{23}(f)$.
- The random answer $a = h_1(m_1, m_2)$ should be equal to $b \oplus c \oplus d$.

Moreover, since we are only interested in the subcase when neither of the bad events B_1, B_2, B_3 happen, we can assume that $|\mathbf{Eval}^{i-1}| \leq |\mathbf{Eval}| < nq$ and $|C_{23}(f)| \leq n$ (as otherwise X_i does not happen anyway). This means that

$$\Pr[X_i] \leq nq \cdot n \cdot \frac{1}{2^n} = \frac{n^2q}{2^n}$$

Case 3: query to h_3 . Assume the i -th query is the query (c, d) to h_3 . In order for this query to cause a collision to T_5 , one of two things must happen. First, it could be that this very query (c, d) defined a value e which induced two distinct consistent triples

$$\begin{aligned} & ((m_1, m_2), a), ((m_3, m_4), b), ((c, d), e) \\ & \neq ((m'_1, m'_2), a'), ((m'_3, m'_4), b'), ((c, d), e) \end{aligned}$$

which collided. But this means that $f = a \oplus (e \oplus c) = a' \oplus (e \oplus c)$, or $a = a'$, and, similarly, $f = b \oplus (e \oplus d) = b' \oplus (e \oplus d)$, or $b = b'$. But since we assumed there no collisions on h_1 and h_2 , this means that $(m_1, m_2) = (m'_1, m'_2)$, $(m_3, m_4) = (m'_3, m'_4)$, and the two consistent triples above are identical.

Hence, in order for the query (c, d) to cause a collision in T_5 , there must be some *previous* value $(*, f) \in \mathbf{Eval}^{i-1}$ s.t. the random answer $h_3(c, d) = e$ caused the collision with this f . In turn, this means that there are two queries $(*, a) \in Q_1^{i-1}$ and $(*, b) \in Q_2^{i-1}$ such that the values $m_5 = a \oplus c = b \oplus d$ match, which means that $a \oplus b = c \oplus d$, and, hence the tuple $((*, a), (*, b)) \in C_{12}(c \oplus d)$. Moreover, for the value e to actually match this tuple, we must have $e = f \oplus (a \oplus c)$ (which is then the same as $f \oplus (b \oplus d)$), which happens with probability $1/2^n$.

To summarize, in order for the i -th query to h_3 to cause the collision, the following events must happen:

- There exists a tuple $(*, f) \in \mathbf{Eval}^{i-1}$.
- There exists a tuple $((*, a), (*, b)) \in C_{12}(c \oplus d)$.
- The random answer $e = h_3(c, d)$ should be equal to $f \oplus a \oplus c$.

Moreover, since we are only interested in the subcase when neither of the bad events B_1, B_2, B_3 happen, we can assume that $|\mathbf{Eval}^{i-1}| \leq |\mathbf{Eval}| < nq$ and $|C_{12}(c \oplus d)| \leq n$ (as otherwise X_i does not happen anyway). This means that

$$\Pr[X_i] \leq nq \cdot n \cdot \frac{1}{2^n} = \frac{n^2q}{2^n}$$

Having established that $\Pr[X_i] \leq n^2q/2^n$ for all i (irrespective of what the i -th query was), we simply take the union bound over $i \in \{1 \dots q\}$ to obtain

$$\Pr[B_4 \cap \overline{B_1} \cap \overline{B_2} \cap \overline{B_3}] \leq q \cdot \max_i \Pr[X_i] \leq \frac{n^2q^2}{2^n}$$

B Formal Proof of Preimage Resistance of T_5

Consider any (w.l.o.g., deterministic and not repeating queries) collision finding adversary A making a total of q queries to h_1, h_2, h_3 . Our proof will follow the same rough strategy as the collision security proof in App. A, but with some differences.

First, we are given a specific value $f = T_5(M)$ for a random (unknown) M , and now need to invert this f . Since each evaluation of T_5 happened on a truly random input, and the value of f is truly random, it is statistically unlikely that A will make any of the 3 queries used to evaluate $T_5(M)$: the probability of this event is at most $O(q/2^n)$. Hence, we will assume that f is truly random, and A starts with fresh (unqueried) hash functions h_1, h_2, h_3 .

Next, we will define the same sets $Q_1^i, Q_2^i, Q_3^i, \mathbf{Eval}^i, C_{12}^i(z), C_{13}^i(f), C_{23}^i(f)$ as in the proof of App. A, and will omit superscripts at the end of the execution (i.e., when $i = q$). Note that we will only care about the sets $C_{13}(f), C_{23}(f)$ for our “challenge” value f rather than any f . More significantly, our “bad events” at the end of A ’s execution will be defined differently. Intuitively, we will no longer care about individual collisions in various hash functions (old event B_1), as such collisions will not help the adversary to find an inversion. Instead, we will only care about certain sets becoming too large, as this could help A invert f quicker than expected:

- B_5 : for a particular constant $\alpha > 0$ defined later,¹²

$$\max(\max_z |C_{12}(z)|, |C_{13}(f)|, |C_{23}(f)|) > \frac{2n}{\alpha} + \frac{2q^2}{2^n}$$

- B_6 : $(*, f) \in \mathbf{Eval}$; i.e., the adversary found a preimage of f .

Notice, event B_5 is similar to the “old” event B_2 , except we increased the “multi-collision threshold” from n to $K = O(n) + 2q^2/2^n$, and the event B_6 is the one we care about. We show the following:

Proposition 7 For some constant α , $\Pr[B_5] \leq \frac{3}{2^n}$

Proposition 8 $\Pr[B_6 \cap \overline{B_5}] \leq \frac{2q^3}{2^{2n}} + \frac{2qn}{\alpha 2^n}$

Combining the results of Propositions 7,8 (proven below), we get our final bound in (4):

$$\Pr[B_6] \leq \Pr[B_5] + \Pr[B_6 \cap \overline{B_5}] \leq \frac{3}{2^n} + \frac{2q^3}{2^{2n}} + \frac{2qn}{\alpha 2^n} = \frac{2q^3}{2^{2n}} + O\left(\frac{qn}{2^n}\right)$$

¹² We did not “dig out” this constant from the complex calculations of [22], whose result we use, since it will not matter asymptotically for our bound. We assume $\alpha > 0.01$, for example.

Bounding Multi-Collisions (Proof of Proposition 7) Consider $C_{12}(z)$ for some fixed value z . For simplicity, we will allow the adversary to make exactly q queries to each function h_1 and h_2 (rather than at most q total). The random process defining $|C_{12}(z)|$ is equivalent to the following.

- Two random tables T_1 and T_2 are selected, each consisting of q random n -bit strings $T_1[i] = a_i$ and $T_2[j] = b_j$, $1 \leq i, j \leq q$.
- All $Q = q^2$ “balls” $y = (i, j)$ are mapped to “bins” $z(y) = T_1[i] \oplus T_2[j]$.
- $C_{12}(z)$ is set to be the number of balls thrown into bin z .

This is a special case of the general process of hashing Q balls into $N = 2^n$ bins using simple 2-tabulated hashing. For this case, we will use the following result of [22] (Theorem 1, Equation (2) of their paper), who proved very strong Chernoff-type bounds for simple tabulation hashing.

Theorem 4 ([22], Theorem 1, Equation (2)). *Let T_1, T_2 be random tables as above. Consider hashing Q balls into $N \geq Q^{3/4}$ bins by simple 2-tabulation, and let $X[z]$ be the number of balls in bin z . Then for any $\gamma > 0$ there exists a constant $\alpha = \alpha(\gamma) > 0$, such that any fixed z and any $\delta \geq 1$:*

$$\Pr \left[X[z] > (1 + \delta) \cdot \frac{Q}{N} \right] \leq (1 + \delta)^{-\alpha(1+\delta)\frac{Q}{N}} + \frac{1}{N^\gamma} \quad (22)$$

To map this to our setting, we let

$$\gamma = 2, \quad \alpha = \alpha(2), \quad Q = q^2, \quad N = 2^n, \quad X[z] = C_{12}(z)$$

The constraint $N \geq Q^{3/4}$ becomes $2^n \geq q^{3/2}$, or $q \leq 2^{2n/3}$, which is the precondition of Theorem 2. We also write $K = (1 + \delta)Q/N = (1 + \delta)q^2/2^n$, and note that we can choose any $K \geq 2q^2/2^n$. Then, since $1 + \delta \geq 2$, we get that for any $K \geq 2q^2/2^n$, (22) implies:

$$\Pr [C_{12}(z) > K] \leq \frac{1}{2^{\alpha K}} + \frac{1}{2^{2n}}$$

Finally, we set $K = 2n/\alpha + 2q^2/2^n > 2q^2/2^n$. We get $\alpha K > 2n$, and thus

$$\Pr \left[C_{12}(z) > \frac{2n}{\alpha} + \frac{2q^2}{2^n} \right] \leq \frac{2}{2^{2n}}$$

Taking the union bound over all z implies

$$\Pr \left[\max_z (C_{12}(z)) > \frac{2n}{\alpha} + \frac{2q^2}{2^n} \right] \leq \frac{2}{2^n}$$

The same argument as for $C_{12}(z)$ applies to $C_{23}(f)$ and $C_{13}(f)$, except we don't even need to take the union bound. This is because the question can be

also reduced to tabulation hashing where the second table T_2 is using a “shifted” functions $h_3(c, d) \oplus c$ and $h_3(c, d) \oplus d$. Thus,

$$\Pr \left[C_{13}(f) > \frac{2n}{\alpha} + \frac{2q^2}{2^n} \right] \leq \frac{2}{2^{2n}}$$

$$\Pr \left[C_{23}(f) > \frac{2n}{\alpha} + \frac{2q^2}{2^n} \right] \leq \frac{2}{2^{2n}}$$

And overall $\Pr[B_5] \leq 2/2^n + 4/2^{2n} \leq 3/2^n$, as claimed.

Bounding Inversion Probability (Proof of Proposition 8) Let us set $K = \frac{2n}{\alpha} + \frac{2q^2}{2^n}$, and assume B_5 did not happen (else $B_6 \cap \overline{B_5}$ does happen anyway), meaning that $\max_z |C_{12}(z)|, |C_{13}(f)|, |C_{23}(f)| \leq K$.

It suffices to show that in this case $\Pr[(*, f) \in \mathbf{Eval}] \leq qK/2^n$, as this is the bound claimed in Proposition 8. For that it suffices to argue that for every query i made by A , $\Pr[(*, f) \in \mathbf{Eval}^i \setminus \mathbf{Eval}^{i-1}] \leq K/2^n$. Consider query i made by A , and split the argument into two cases.

Case 1-2: query to h_1 or h_2 . Assume the i -th query is the query (m_1, m_2) to h_1 (same proof for h_2). The same argument as we made in Proposition 6 implies that the answer b defines a consistent triple producing an evaluation with output f only if:

- There exists a tuple $((*, b), ((c, d), e)) \in C_{23}(f)$.
- The random answer $a = h_1(m_1, m_2)$ should be equal to $b \oplus c \oplus d$.

Since $|C_{23}(f)| \leq K$ and a is random, the probability of this happening is at most $K/2^n$.

Case 3: query to h_3 . Assume the i -th query is the query (c, d) to h_3 . The same argument as we made in Proposition 6 implies that the answer e defines a consistent triple producing an evaluation with output f only if:

- There exists a tuple $((*, a), (*, b)) \in C_{12}(c \oplus d)$.
- The random answer $e = h_3(c, d)$ should be equal to $f \oplus a \oplus c$.

Since $|C_{12}(c \oplus d)| \leq \max_z |C_{12}(z)| \leq K$ and e is random, the probability of this happening is at most $K/2^n$.

This completes the proof.

C Proof of Theorem 3 (cross-collision bound)

Proof. As in the case of T_5 , for T'_5 , we use the variables

- m'_1 and m'_2 to denote the left and right halves of various inputs to h_1 ;
- a' to denote various outputs of h_1 ;

- c' and m'_4 to denote the left and right halves of various inputs to h_3 . We might use d in place of m'_4 while talking about h_3 queries in general;
- e' to denote various outputs of h_3 ;
- $M = (m'_1, m'_2, m'_3, m'_4)$ to denote various inputs to T'_5 ;
- f' to denote various outputs of T'_5 .

Hence, a valid computation of $T'_5(M) = T'_5(m'_1, \dots, m'_4)$ proceeds as follows:

1. Set $a' = h_1(m'_1, m'_2)$.
2. Set $c' = a' \oplus m'_3$.
3. Set $e' = h_3(c', m'_4)$, and output $f' = e' \oplus m'_3$.

However, in case of T'_5 , all the queries are consistent. For any two queries $((m'_1, m'_2), a')$ and $((c', m'_4), e')$, we define $m'_3 = a' \oplus c'$, and say that this pair of queries (uniquely) defines a valid T'_5 evaluation (M', f') , where $M' = (m'_1, m'_2, m'_3, m'_4)$ and $f' = e' \oplus a' \oplus c'$.

Most of the assumptions and definitions are the same as in the T_5 collision proof. We only note the points of difference here.

- \mathbf{Eval}^i now denotes the set of input-output pairs for the entire constructions T_5 and T'_5 known to \mathbf{A} after its i -th query. We denote the set of T_5 outputs by $\mathbf{Eval}_{T_5}^i$ and similarly the set of T'_5 outputs by $\mathbf{Eval}_{T'_5}^i$. Thus, $|\mathbf{Eval}^i| = |\mathbf{Eval}_{T_5}^i| + |\mathbf{Eval}_{T'_5}^i|$.
- Bad event B_3 : $|\mathbf{Eval}| \geq qn + q^2$.
- Bad event B_4 : a non-trivial collision between T_5 and T'_5 is found; i.e., $(M, f), (M', f) \in \mathbf{Eval}$ with $M \in \{0, 1\}^{5n}$, $M' \in \{0, 1\}^{4n}$ and $g(M) \neq M'$, where $g(m_1, m_2, m_3, m_4, m_5) = (m_1, m_2, m_5, h_2(m_3, m_4) \oplus m_5)$.

Our goal remains the same, to upper bound the probability of B_4 . We also note that the four bad events are still disjoint. Therefore, we have

$$\Pr[B_4] \leq \Pr[B_1] + \Pr[B_2 \cap \overline{B_1}] + \Pr[B_3 \cap \overline{B_1} \cap \overline{B_2}] + \Pr[B_4 \cap \overline{B_1} \cap \overline{B_2} \cap \overline{B_3}]$$

The first two propositions and their proof are the same as above. However, for completion, we restate the propositions.

Proposition 9 $\Pr[B_1] \leq \frac{2q^2}{2^n}$

Proposition 10 $\Pr[B_2 \cap \overline{B_1}] \leq 3 \cdot \left(\frac{2q^2}{2^n}\right)^n$

Proposition 11 $\Pr[B_3 \cap \overline{B_1} \cap \overline{B_2}] \leq \frac{2q^2}{n2^n}$

Proposition 12 $\Pr[B_4 \cap \overline{B_1} \cap \overline{B_2} \cap \overline{B_3}] \leq \frac{nq^3}{2^n}$

Combining the results of Propositions 3,4,11,12, we get our final bound in (6):

$$\Pr[B_4] \leq \frac{2q^2}{2^n} + 3 \cdot \left(\frac{2q^2}{2^n}\right)^n + \frac{2q^2}{n2^n} + \frac{nq^3}{2^n} \leq \frac{nq^3 + 9q^2}{2^n}$$

Hence, it suffices to prove Propositions 11,12.

Bounding Evaluation Set of T_5 and T'_5 (Proof of Proposition 11) We break this into two parts. We note that since $|\mathbf{Eval}^i| = |\mathbf{Eval}_{T_5}^i| + |\mathbf{Eval}_{T'_5}^i|$, we must have $|\mathbf{Eval}| = |\mathbf{Eval}_{T_5}| + |\mathbf{Eval}_{T'_5}|$.

First, we prove that A cannot compute more than q^2 evaluations of T'_5 in its q queries.

Let us denote by q_j , for $j = 1, 2, 3$, the (random variable equal to the) number of calls to h_j . T'_5 evaluations only depend on q_1 calls to h_1 and q_3 calls to h_3 . Also, there is no computability condition for T'_5 . Hence, total number of T'_5 evaluations is equal to $q_1 q_3$, which is less than q^2 . Therefore, $|\mathbf{Eval}_{T'_5}| \leq q^2$.

Now, we can follow the exact proof of 5 in the previous section and the proof is completed.

Final Collision Probability (Proof of Proposition 12) For completeness, we repeat the claimed bound:

$$\Pr [B_4 \cap \overline{B_1 \cap B_2 \cap B_3}] \leq \frac{nq^3}{2^n}$$

Let X_i be the event that none of bad events B_1, B_2 or B_3 happened, and that the i -th query of A generates a fresh collision between T_5 and T'_5 , where $1 \leq i \leq q$. We upper bound $\Pr[X_i]$ separately depending on whether this was the call to h_1, h_2 or h_3 . In each of these case we show $\Pr[X_i] \leq nq^2/2^n$, meaning that this bound is correct irrespective of the identity of the i -th query.

Case 1: query to h_1 . Assume the i -th query is the query (m_1, m_2) to h_1 . In order for this query to cause a collision, one of two things must happen. First, it could be that this very query (m_1, m_2) defined a value a which induced two distinct consistent tuples

$$((m_1, m_2), a), ((m_3, m_4), b), ((c, d), e) \neq ((m_1, m_2), a), ((c', d'), e')$$

which collided. Here the first tuple corresponds to a valid T_5 evaluation and the second tuple corresponds to a valid T'_5 evaluation. But this means that $f = a \oplus (e \oplus c) = a \oplus (e' \oplus c')$, or $e \oplus c = e' \oplus c'$. But since we assumed B_1 failed, there are no collisions in “ c -shifted” h_3 , which means $(c, d) = (c', d')$. But this will imply the trivial collision which we have ruled out.

Hence, in order for the query (m_1, m_2) to cause a collision in T_5 , there must be some *previous* value $(*, f) \in \mathbf{Eval}^{i-1}$ s.t. the random answer $h_1(m_1, m_2) = a$ caused the collision with this f . Now again, there are two subcases:

a) $(, f)$ comes from an evaluation of T'_5 .* In turn, this means that there is a query $(*, b) \in Q_2^{i-1}$ and $((c, d), e) \in Q_3^{i-1}$ such that the tuple $((*, b), ((c, d), e) \in C_{23}(f)$. Moreover, for the value a to actually match this tuple, we must have $a = b \oplus c \oplus d$, which happens with probability $1/2^n$.

To summarize, in order for the i -th query to h_1 to cause the collision, the following events must happen:

- There exists a tuple $(*, f) \in \mathbf{Eval}_{T_5'}^{i-1}$.
- There exists a tuple $((*, b), ((c, d), e)) \in C_{23}(f)$.
- The random answer $a = h_1(m_1, m_2)$ should be equal to $b \oplus c \oplus d$.

Moreover, since we are only interested in the subcase when neither of the bad events B_1, B_2, B_3 happen, we can assume that $|\mathbf{Eval}_{T_5'}^{i-1}| \leq |\mathbf{Eval}_{T_5}| < q^2$ and $|C_{23}(f)| \leq n$ (as otherwise X_i does not happen anyway). This means that

$$\Pr[X_i] \leq q^2 \cdot n \cdot \frac{1}{2^n} = \frac{nq^2}{2^n}$$

b) $(*, f)$ comes from an evaluation of T_5 . Then, the query (m_1, m_2) to h_1 must result in an evaluation of T_5' which causes a collision. For this to happen, the following conditions must be satisfied:

- There exists a tuple $(*, f) \in \mathbf{Eval}_{T_5}^{i-1}$.
- The random answer a can combine with any of the queries $((c, d), e)$ of h_3 such that $f = a \oplus c \oplus e$.

Since we are only interested in the subcase when neither of the bad events B_1, B_2, B_3 happen, we can assume that $|\mathbf{Eval}_{T_5}^{i-1}| \leq |\mathbf{Eval}_{T_5}| < nq$. Also, there are at most q queries to h_3 . The probability that $f = a \oplus c \oplus e$ is $1/2^n$ for each such query. Therefore,

$$\Pr[X_i] \leq nq \cdot q \cdot \frac{1}{2^n} = \frac{nq^2}{2^n}$$

Case 2: query to h_2 . Assume the i -th query is the query (m_3, m_4) to h_2 . Now, this query itself cannot output two values that collide since this query does not give any new T_5' outputs. The only way this query can cause a collision is that there exists some previous value $(*, f) \in \mathbf{Eval}_{T_5'}^{i-1}$ such that the random answer creates a T_5 output equal to f . This case is exactly similar to the first subcase of Case 1.

Case 3: query to h_3 . Assume the i -th query is the query (c, d) to h_3 .

In order for this query to cause a collision, one of two things must happen. First, it could be that this very query (c, d) defined a value e which induced two distinct consistent tuples

$$((m_1, m_2), a), ((m_3, m_4), b), ((c, d), e) \neq ((m_1', m_2'), a'), ((c, d), e)$$

which collided. Here the first tuple corresponds to a valid T_5 evaluation and the second tuple corresponds to a valid T_5' evaluation. But this means that $f = a \oplus (e \oplus c) = a' \oplus (e \oplus c)$, or $a = a'$. But since we assumed there are no collisions on h_1 , this means that $(m_1, m_2) = (m_1', m_2')$. But this will imply only the trivial collision, which we have ruled out.

Hence, in order for the query (c, d) to cause a collision, there must be some previous value $(*, f) \in \mathbf{Eval}^{i-1}$ s.t. the random answer $h_3(c, d) = e$ caused the collision with this f . Again, we have two subcases:

a) $(*,f)$ comes from an evaluation of T'_5 . This means that there is a query $(*,a) \in Q_1^{i-1}$ and $(*,b) \in Q_2^{i-1}$ such that the values $m_5 = a \oplus c = b \oplus d$ match, which means that $a \oplus b = c \oplus d$, and, hence the tuple $((*,a), (*,b)) \in C_{12}(c \oplus d)$. Moreover, for the value e to actually match this tuple, we must have $e = f \oplus (a \oplus c)$ (which is then the same as $f \oplus (b \oplus d)$), which happens with probability $1/2^n$.

To summarize, in order for the i -th query to h_3 to cause the collision, the following events must happen:

- There exists a tuple $(*,f) \in \mathbf{Eval}_{T'_5}^{i-1}$.
- There exists a tuple $((*,a), (*,b)) \in C_{12}(c \oplus d)$.
- The random answer $e = h_3(c, d)$ should be equal to $f \oplus a \oplus c$.

Moreover, since we are only interested in the subcase when neither of the bad events B_1, B_2, B_3 happen, we can assume that $|\mathbf{Eval}_{T'_5}^{i-1}| \leq |\mathbf{Eval}_{T'_5}| < q^2$ and $|C_{12}(c \oplus d)| \leq n$ (as otherwise X_i does not happen anyway). This means that

$$\Pr[X_i] \leq q^2 \cdot n \cdot \frac{1}{2^n} = \frac{nq^2}{2^n}$$

b) $(*,f)$ comes from an evaluation of T_5 . Then, the query (c, d) to h_3 must result in an evaluation of T'_5 which causes a collision. For this to happen, the following conditions must be satisfied:

- There exists a tuple $(*,f) \in \mathbf{Eval}_{T_5}^{i-1}$.
- The random answer e can combine with any of the queries $((m_1, m_2), a)$ of h_1 such that $f = a \oplus c \oplus e$.

Since we are only interested in the subcase when neither of the bad events B_1, B_2, B_3 happen, we can assume that $|\mathbf{Eval}_{T_5}^{i-1}| \leq |\mathbf{Eval}_{T_5}| < nq$. Also, there are at most q queries to h_1 . The probability that $f = a \oplus c \oplus e$ is $1/2^n$ for each such query. Therefore,

$$\Pr[X_i] \leq nq \cdot q \cdot \frac{1}{2^n} = \frac{nq^2}{2^n}$$

Having established that $\Pr[X_i] \leq nq^2/2^n$ for all i (irrespective of what the i -th query was), we simply take the union bound over $i \in \{1 \dots q\}$ to obtain

$$\Pr [B_4 \cap \overline{B_1 \cap B_2 \cap B_3}] \leq q \cdot \max_i \Pr[X_i] \leq \frac{nq^3}{2^n}.$$