

Sine Series Approximation of the Mod Function for Bootstrapping of Approximate HE

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Abstract

While it is well known that the sawtooth function has a point-wise convergent Fourier series, the rate of convergence is not the best possible for the application of approximating the mod function in small intervals around multiples of the modulus. We show a different sine series, such that the sine series of order n has error $O(\epsilon^{2n+1})$ for approximating the mod function in ϵ -sized intervals around multiples of the modulus. Moreover, the resulting polynomial, after Taylor series approximation of the sine function, has small coefficients, and the whole polynomial can be computed at a precision that is only slightly larger than $-(2n + 1) \log \epsilon$, the precision of approximation being sought. This polynomial can then be used to approximate the mod function to almost arbitrary precision, and hence allows practical CKKS-HE bootstrapping with arbitrary precision. We validate our approach by an implementation and obtain 94 bit precision bootstrapping as well as improvements over prior work even at lower precision.

1 Introduction

The work of [8, 7] presented a new homomorphic encryption (HE) scheme for approximate arithmetic (called the CKKS-HE scheme) over real/complex numbers. The CKKS-HE scheme was considerably more efficient than other schemes for evaluating arithmetic circuits and leveraged properties of approximate arithmetic to achieve these efficiency gains. It has found many applications, among them privacy-preserving machine learning and secure genome analysis (see [13, 17, 4, 15, 19, 14] for some examples). However, the initial CKKS-HE scheme lacked a bootstrapping procedure, and, thus, it was not a *fully* homomorphic encryption (FHE) scheme. This was remedied when [6] introduced the first bootstrapping procedure for the CKKS-HE scheme, which followed the general template introduced by Gentry [9] of evaluating the decryption circuit homomorphically. The challenge here is that the decryption procedure for

CKKS requires computing the mod function, which is not easily representable via an arithmetic circuit. In fact, the mod function modulo q on the interval $[-Kq, Kq]$ for some integer K is not even a continuous function. However, [6] made the clever observation that in the CKKS-HE scheme, we have an upper bound m on the size of the message, which can be made much smaller than q . In this situation, we actually only need to be able to compute the mod function on points in $[-Kq, Kq]$ that are a distance at most m from a multiple of q . In this case, the mod function is periodic with period q and is linear on each of the small intervals around a multiple of q . Figure 1 shows the mod function along with the small intervals for approximation.

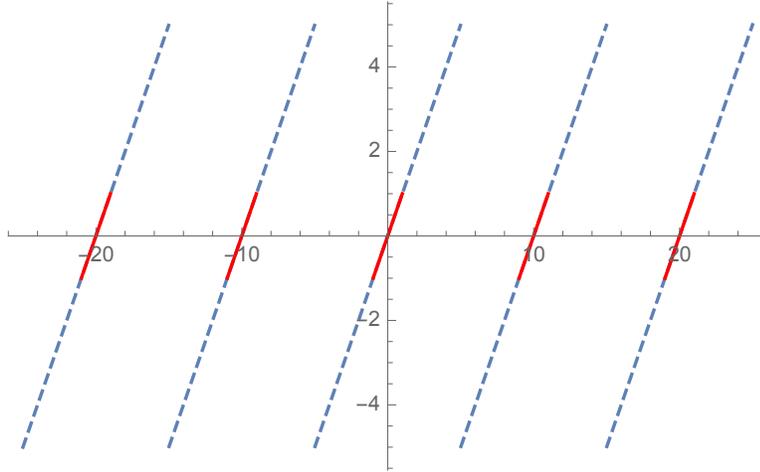


Figure 1: The mod function with modulus $q = 10$. The solid red lines represent the small intervals on which we need to approximate.

The work of [6] observed that the mod function $[t]_q$ on these intervals can be approximated via a scaled sine function $S(t) = \frac{q}{2\pi} \sin\left(\frac{2\pi t}{q}\right)$. This approximation introduces an inherent error that depends on the message upper bound m . Let ϵ denote the ratio $\frac{m}{q}$. Then, it can be shown that

$$|[t]_q - S(t)| \leq \frac{2\pi^2}{3} q \epsilon^3.$$

If ϵ is small enough, then this error can be sufficiently small for use in bootstrapping provided that $S(t)$ can be well-approximated by a low degree polynomial. The work of [6] along with several followup works [5, 11, 16] proceeded to provide methods of approximating this scaled sine function (or scaled cosine function in the case of [11] and scaled sine/cosine and inverse sine in the case of [16]) by a low-degree polynomial, which can then be plugged into the

bootstrapping procedure of [6]. However, due to the inherent error between the mod function $[t]_q$ and the scaled sine function $S(t)$, this approach has a “fundamental error” that will occur regardless of how $S(t)$ is approximated. One of the problems with this is that in order for the error to be $O(1)$ (and, therefore, not destroy the message), m must be $O(q^{2/3})$. This means that we must begin bootstrapping while the size of the encrypted message is considerably smaller than q , which is a source of inefficiency in the bootstrapping procedure, particularly in applications that require high precision. Compounding this problem is the fact that for large q , the size of the coefficients of the approximation of $S(t)$ grow. This requires the basis polynomials to be computed to higher precision and affects the stability of the computation, where the errors introduced by the approximate arithmetic are amplified due to the large coefficients.

The reason obtaining high-precision bootstrapping for CKKS-HE is important is that one of the main applications for CKKS-HE is privacy-preserving machine learning. However, many ML algorithms require high precision computation in order to converge. This may be especially true during the learning phase of neural networks, which involves back propagation and integer division by private integers. Additional nonlinear steps involve pooling functions, threshold functions, etc. Moreover, due to their high depth, computing these ML algorithms homomorphically without bootstrapping is infeasible. Thus, for privacy-preserving ML applications, high-precision bootstrapping is required.

Recently, the work of [12] was able to obtain high-precision bootstrapping by finding direct polynomial approximations of the mod function on small intervals around the modulus via a new technique called modular Lagrange interpolation. This avoided the fundamental error inherent in prior works that occurred due to first approximating the mod function by a scaled sine function. The coefficients of these polynomials were small enough to enable high-precision bootstrapping. However, the coefficients were still large enough that in order to evaluate the polynomial approximations, one would need to operate at a higher precision than the input ciphertext. Ultimately, this fact corresponded to the bootstrapping procedure losing additional levels, since the computations during bootstrapping were operating at a higher precision.

1.1 This Work

In this work, we show how to obtain arbitrary precision bootstrapping via a different method from that of [12] and more in line with the original sine function approach of [6]. Instead of approximating the mod function directly, we first approximate the mod function by a sine series and then approximate the sine function by its Taylor series (more precisely, the Taylor series of e^{ix}). This is then followed by a series of squarings to approximate the other terms in the sine series. We show that the sine series converges to the mod function in small intervals around

the modulus. In particular, our sine series of order n has error $O(\epsilon^{2n+1})$ for approximating the mod function in ϵ -sized intervals around multiples of the modulus.

Thus, we avoid the fundamental error of the scaled sine approach and are able to obtain an approximation with arbitrarily small error in the desired intervals. Furthermore, the coefficients of the sine series are small (in fact, they have norm < 2). This, combined with the fact that the Taylor series expansion of $\sin x$ has small coefficients, leads to a polynomial approximation of the mod function with small coefficients. Due to these small coefficients, the whole polynomial can be computed at a precision only slightly larger than $(-2n + 1) \log \epsilon$, the precision of the approximation being sought. This means that during bootstrapping, we can operate at a precision level that is approximately the same as that of the input ciphertext and do not lose additional levels due to having to operate at a higher precision during bootstrapping. We validate our approach by an implementation and obtain 94 bit precision bootstrapping as well as improvements over prior work even at lower precision.

1.2 Problem Overview

Here, we provide a brief overview of the challenges of approximating the mod function for use in CKKS-HE bootstrapping. We provide a thorough overview of the bootstrapping procedure in Section 3. The goal of CKKS-HE bootstrapping is to take a ciphertext ct at the lowest level and bring it up to the highest level so that homomorphic computation can continue. In other words, we wish to obtain a ciphertext ct' such that

$$\langle \text{ct}, \text{sk} \rangle \bmod q \approx \langle \text{ct}', \text{sk} \rangle \bmod q_\ell,$$

where q is the lowest level modulus and q_ℓ represents a higher level modulus. Since errors accumulated during homomorphic computation are not eliminated by decryption in CKKS-HE, the goal is not to reduce the error in the ciphertext, but, rather, to increase the modulus so that more computations can be performed. If one simply views the ciphertext ct as operating at the highest level q_L , then it follows that $\langle \text{ct}, \text{sk} \rangle \bmod q_L = qI + m$. The magnitude of I can be upper bounded and $m \ll q$ and, thus, the challenge then becomes to compute mod q on small intervals near multiples of q (we defer additional complications such as computing on slots vs. coefficients to Section 3). Since CKKS-HE can compute homomorphic additions and multiplications, we need a polynomial approximation to the mod function. However, there are three crucial criteria that are relevant to the bootstrapping application.

- **Error:** The error of the approximation contributes additional error to the message m , which, if large, will cause a loss in plaintext precision.

- **Degree:** The degree of the polynomial approximation determines the multiplicative depth required to evaluate it. A larger multiplicative depth corresponds to losing more modulus levels and, thus, if too large, the polynomial will not be able to be evaluated homomorphically.
- **Coefficient Magnitude:** The size of the coefficients of the polynomial approximation determine the “evaluation precision” at which one must operate during bootstrapping. Larger coefficients correspond to a larger “evaluation precision” in order to maintain numerical stability, which, in turn, corresponds to losing more modulus bits per level.

Thus, it is critical that we obtain good low-degree polynomial approximations to the mod function in small intervals around multiples of the modulus that additionally have small coefficients.

1.3 Sine Series Intuition

As mentioned previously, prior approaches to CKKS-HE bootstrapping approximated the mod function via a scaled sine function. For simplicity, we will ignore the scaling for the moment and try to obtain a good approximation to the mod 2π function. Thus, prior works used $\sin x$ as the approximation and noted that, for $|x| < \epsilon$, the error of approximation is $O(\epsilon^3)$. It is well-known that the Fourier series of the mod function (or sawtooth function) converges everywhere except the discontinuities. Unfortunately, the rate of convergence is too slow, and the Fourier series does not give a good approximation when the number of terms is small. Instead, we will approximate the mod function by a different sine series such that it converges to the mod function near multiples of the modulus very quickly. As a warmup, suppose we added a $\sin 2x$ term to our approximation of the mod function. If we can determine coefficients β_1 and β_2 such that the Taylor series expansion of $\beta_1 \sin x + \beta_2 \sin 2x$ is $x + x^5 p(x)$ for some polynomial $p(x)$, then for $|x| < \epsilon$, the error of approximation will be $O(\epsilon^5)$, an improvement on $\sin x$. Thus, looking at the x and x^3 terms in the Taylor series expansions of $\sin x$ and $\sin 2x$, we wish to determine β_1, β_2 such that $\beta_1 + 2\beta_2 = 1$ (so that the coefficient of x is 1) and $\beta_1 + 2^3\beta_2 = 0$ (so that the coefficient of x^3 is 0). This can be solved to yield $\beta_1 = 4/3, \beta_2 = -1/6$. This intuition can then be extended to give an n -term sine series with error $O(\epsilon^{2n+1})$. We will show that the β_i 's are small and, thus, the resulting low-degree polynomial approximation has small coefficients. Moreover, we will show that the constants hiding in the big-O notation are reasonable, and the dependence on n is minor.

1.4 Organization

In Section 2, we formalize the above intuition and prove explicit error bounds for the sine series approximation of the mod function. In Section 3, we overview the bootstrapping procedure for CKKS-HE. In Section 4, we explain how to approximate the sine series by a low-degree polynomial for bootstrapping. In Section 5, we implement bootstrapping using our sine series approximation and give performance metrics and comparisons with prior approaches.

2 Sine Series Approximation

In this section, we will show the following theorem and corollaries, giving a sine series approximation to the mod function in small intervals around the modulus that can be used for CKKS-HE bootstrapping.

Theorem 1 *For every $n \geq 1$, there exists a sequence of rational numbers β_1, \dots, β_n such that for every ϵ , $0 < \epsilon < 2/\sqrt{n}$, for every $|x| < \epsilon$,*

$$\left| x - \sum_{k=1}^n \beta_k \sin(kx) \right| < e^2 * (n + 1) * (\epsilon/2)^{2n+1}$$

Using the periodicity of the sine function, we immediately arrive at the following corollary.

Corollary 2 *For every $n \geq 1$, there exists a sequence of rational numbers β_1, \dots, β_n such that for every ϵ , $0 < \epsilon < 2/\sqrt{n}$, for every integer m , for every x such that $|x - 2m\pi| < \epsilon$,*

$$\left| (x \bmod 2\pi) - \sum_{k=1}^n \beta_k \sin(kx) \right| < e^2 * (n + 1) * (\epsilon/2)^{2n+1}$$

A further simple manipulation leads to the following scaled version of the corollary.

Corollary 3 *For every $n \geq 1$, there exists a sequence of rational numbers β_1, \dots, β_n such that for every ϵ , $0 < \epsilon < \frac{1}{\pi\sqrt{n}}$, for every integer $q \geq 1$, for every integer m , for every x such that $|x - m * q| < \epsilon * q$,*

$$\left| (x \bmod q) - \frac{q}{2\pi} * \sum_{k=1}^n \beta_k \sin(2\pi k * x/q) \right| < \frac{e^2 * q}{2\pi} * (n + 1) * (\epsilon\pi)^{2n+1}$$

Determining the β_i 's: To prove Theorem 1, for each n , we will determine the rational numbers $\{\beta_i\}_{i \in [n]}$. In particular, these are *not* the same as the Fourier coefficients of the sawtooth function, as we are focused on x that is potentially much smaller than the period of the sawtooth function. Recall that we wish to determine $\{\beta_i\}_{i \in [n]}$ such that the resulting sine series has a Taylor series expansion of the form $x + x^{2n+1}p(x)$ for some polynomial $p(x)$. In particular, there are no terms of degree $< 2n + 1$ (except for x). These constraints give a system of equations that can be solved to determine the β_i 's.

We begin by formalizing this intuition. For every $n > 0$, for every sequence of n distinct integers $\mathbf{a} = (a_1, \dots, a_n)$, let $V^{(n)}(\mathbf{a})$ denote the Vandermonde matrix of \mathbf{a} , i.e. it is the $n \times n$ matrix with the (i, j) -th element a_i^{j-1} (for $i, j \in [1..n]$). Define $S^{(n)}(\mathbf{a})$ to be the $n \times n$ matrix with the (i, j) -th element a_i^{2j-1} , i.e. each row is the odd powers of the elements of \mathbf{a} . Note that the first column of this matrix is just \mathbf{a} . Also, define a related matrix $\hat{S}^{(n)}(\mathbf{a})$ to be the $n \times n$ matrix which is same as $S^{(n)}(\mathbf{a})$ except that the first column (i.e. \mathbf{a}) is replaced by $(2n + 1)$ -th powers of \mathbf{a} . In other words, the $(i, 1)$ -th element of this matrix is $a_i^{(2n+1)}$.

Let $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ be an n -vector of rational numbers. For the sine series approximation, we would like to determine $\vec{\beta}$ so that the transpose of the matrix $S^{(n)}(\mathbf{a})$ multiplied by $\vec{\beta}$ is a vector with all entries zero except the first, which is one. Since β_i refers to the coefficient of the $\sin(a_i x)$ term in the sine series, the above requirement ensures that when we Taylor expand each sine term in the sine series about the origin (or a multiple of 2π) and sum the terms, the resulting polynomial will be $x + x^{2n+1}p(x)$ for some polynomial $p(x)$. Thus, the $x^3, x^5, \dots, x^{2n-1}$ terms in the Taylor series expansions of the $\sin(ix)$'s cancel out. We note that since our sine series will include $\sin x, \sin 2x, \sin 3x, \dots$ terms, we will later instantiate \mathbf{a} with $(1, 2, \dots, n)$. The required condition is drawn below.

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1^3 & a_2^3 & \dots & a_n^3 \\ & & \vdots & \\ a_1^{2n-1} & a_2^{2n-1} & \dots & a_n^{2n-1} \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (1)$$

Let d_i denote the $(i, 1)$ -th minor of $S^{(n)}(\mathbf{a})$. In other words, the list $\{d_i\}_i$ is the list of minors of the first column of $S^{(n)}(\mathbf{a})$.

Lemma 4

$$\beta_i = (-1)^{i+1} * \frac{d_i}{\det(S^{(n)}(\mathbf{a}))}.$$

Proof: From the above equation, $\vec{\beta}$ is just the first column of the inverse of $(S^{(n)}(\mathbf{a}))^T$. Note that the $(i, 1)$ -th element of the inverse of the transpose of $S^{(n)}(\mathbf{a})$ is $(-1)^{i+1} * d_i$ divided by the determinant of $S^{(n)}(\mathbf{a})$. \square

We now give an explicit formula for the determinant of $S^{(n)}(\mathbf{a})$. We will also give an explicit formula for the determinant of $\hat{S}^{(n)}(\mathbf{a})$, which will be of use later. We will use the well-known fact that the determinant of the Vandermonde matrix is given by the following formula.

$$\det(V^{(n)}(\mathbf{a})) = \prod_{i=1}^n \prod_{1 \leq j < i} (a_i - a_j).$$

Lemma 5 *The determinant of the matrix $S^{(n)}(\mathbf{a})$ is*

$$\left(\prod_{i=1}^n a_i \right) * \prod_{i=1}^n \prod_{1 \leq j < i} (a_i^2 - a_j^2).$$

The determinant of the matrix $\hat{S}^{(n)}(\mathbf{a})$ is

$$(-1)^{n-1} * \det(S^{(n)}(\mathbf{a})) * \prod_{i=1}^n a_i^2.$$

Proof: We will first focus on the matrix $S^{(n)}(\mathbf{a})$. For computing the determinant, for each row i , we get a contribution of a factor a_i towards the determinant, and the remaining matrix is then just a Vandermonde matrix with all powers of a_i^2 . Thus,

$$\det(S^{(n)}(\mathbf{a})) = \left(\prod_{i=1}^n a_i \right) * \det(V^{(n)}(\mathbf{a}')),$$

where $\mathbf{a}' = (a_1^2, \dots, a_n^2)$. The result then follows from the well-known determinant of Vandermonde matrices.

As for the claim for the matrix $\hat{S}^{(n)}(\mathbf{a})$, first consider a modified matrix that is obtained by moving the first column to the last. Since this can be accomplished by $(n-1)$ column exchanges, the determinant of the modified matrix is $(-1)^{n-1}$ times the determinant of $\hat{S}^{(n)}(\mathbf{a})$. Furthermore, the determinant of the modified matrix is easily related to determinant of $S^{(n)}(\mathbf{a})$ by noting that i -th row in the modified matrix is a_i^2 times the i -th row in $S^{(n)}(\mathbf{a})$. \square

We observe from the formula for the determinant of $S^{(n)}(\mathbf{a})$ that if the sequence of integers \mathbf{a} are in increasing order and lower bounded by one, then the determinant of $S^{(n)}(\mathbf{a})$ is positive. We now show the following lemma, characterizing the β_i 's.

Lemma 6 *For the matrix $S^{(n)}(\mathbf{a})$ with \mathbf{a} set to the sequence of integers from one to n ,*

$$\beta_1 = \frac{2n}{n+1} < 2$$

and, for $i \geq 2$

$$|\beta_i| < 1.$$

Moreover, the β_i 's alternate in sign and decrease in magnitude as i increases. That is,

$$|\beta_{i+1}| < |\beta_i|$$

for all $i \in [n]$, $\beta_{2j+1} > 0$, and $\beta_{2j} < 0$.

Proof: We will show this using the formula for β_i from Lemma 4. By definition,

$$d_i = \det \begin{pmatrix} a_1^3 & a_1^5 & \dots & a_1^{2n-1} \\ a_2^3 & a_2^5 & \dots & a_2^{2n-1} \\ & & \vdots & \\ a_{i-1}^3 & a_{i-1}^5 & \dots & a_{i-1}^{2n-1} \\ a_{i+1}^3 & a_{i+1}^5 & \dots & a_{i+1}^{2n-1} \\ & & \vdots & \\ a_n^3 & a_n^5 & \dots & a_n^{2n-1} \end{pmatrix}$$

Thus,

$$d_i = \left(\prod_{j=1, j \neq i}^n a_j^2 \right) * \det(S^{(n-1)}(\mathbf{a}')),$$

where \mathbf{a}' is \mathbf{a} with a_i removed. Thus,

$$\beta_i = (-1)^{i+1} * \frac{\left(\prod_{j=1, j \neq i}^n a_j^2 \right)}{a_i * \left(\prod_{j=1}^{i-1} (a_i^2 - a_j^2) \right) * \left(\prod_{j=i+1}^n (a_j^2 - a_i^2) \right)}.$$

We observe that every term in the above expression is positive except for $(-1)^{i+1}$ and, thus, the β_i 's alternate sign with $\beta_{2j+1} > 0$ and $\beta_{2j} < 0$. It follows that

$$\beta_1 = \frac{2(n!)^2}{(n+1)!(n-1)!} = \frac{2n}{n+1} < 2.$$

Moreover, for $i \geq 2$,

$$|\beta_i| = \frac{1}{i} * \frac{2(n!)^2}{(2n)!} * \binom{2n}{n+i}$$

Observe that $|\beta_{i+1}| < |\beta_i|$. Moreover, since $\binom{2n}{n+i} < \binom{2n}{n}$ for $i \geq 2$, it follows that

$$|\beta_i| < \frac{2}{i} \leq 1$$

for $i \geq 2$. □

Bounding the Error: A First Attempt Having characterized the β_i 's, we now turn our focus to bounding the error between $f(x) = \sum_{k=1}^n \beta_k \sin(kx)$ and x for $|x| < \epsilon$. We note that $f(x)$ is an analytic function since it is the sum of analytic functions and, therefore, its Taylor series converges to $f(x)$. Thus, taking the Taylor series expansion of $f(x)$ around 0,

$$f(x) = x + \sum_{m=2n+1}^{\infty} \frac{f^{(m)}(0)}{m!} x^m.$$

We can bound $|x - f(x)|$ for $|x| < \epsilon$ using the Lagrange remainder term of the $2n$ -th Taylor polynomial of $f(x)$. Thus,

$$|x - f(x)| = \left| \frac{f^{(2n+1)}(\xi)}{(2n+1)!} x^{2n+1} \right|$$

for some real number ξ between 0 and x . We have that

$$f^{(2n+1)}(x) = \pm \sum_{k=1}^n \beta_k k^{2n+1} \cos(kx).$$

Upper bounding $f^{(2n+1)}(\xi)$ gives

$$|x - f(x)| < \sum_{k=1}^n |\beta_k| k^{2n+1} \frac{|x|^{2n+1}}{(2n+1)!}.$$

By Lemma 6, $\beta_k < 2/k$, which gives

$$|x - f(x)| < |x|^{2n+1} * \frac{2}{(2n+1)!} * \sum_{k=1}^n k^{2n}.$$

This then gives an upper bound of $\epsilon^{2n+1} * \frac{2 * n * n^{2n}}{(2n+1)!}$, and no better than $\epsilon^{2n+1} * \frac{2 * n^{2n}}{(2n+1)!}$. However, we will now show that a more sophisticated yet elementary approach that improves upon this bound by a factor of e^{2n} .

A Better Bound via the Alternating Series Test: To obtain a better error bound, we will show that the Taylor series expansion of our sine series satisfies Leibniz's alternating series test. This will enable us to bound the error of the sine series $f(x)$ from the mod function by the $2n + 1$ th term in the Taylor series expansion (the first nonzero term after x). We can write the Taylor series expansion of $f(x)$ as $x + \sum_{m=n+1}^{\infty} (-1)^m * b_m$, where

$$b_m = (-1)^{n-1} * \sum_{i=1}^n \beta_i * \frac{(ix)^{2m-1}}{(2m-1)!}. \quad (2)$$

To bound the error, we will show, for any x in the domain of approximation, that the series $\sum_{m=n+1}^{\infty} (-1)^m * b_m$ satisfies the alternating series test. The alternating series test requires that the b_m satisfy the following three conditions.

1. $\lim_{m \rightarrow \infty} b_m = 0$
2. All b_m are positive (or all b_m are negative)
3. $|b_m| \geq |b_{m+1}|$ for all natural numbers $m \geq n + 1$.

Theorem 7 Alternating Series Test [Leibniz]. *If the series above satisfies the alternating series test then $\sum_{m=n+1}^{\infty} (-1)^m * b_m$ converges. Moreover, for all $k \geq 0$,*

$$\left| \sum_{m=n+1}^{\infty} (-1)^m * b_m - \sum_{m=n+1}^{n+1+k-1} (-1)^m * b_m \right| \leq |b_{n+1+k}|.$$

We will show the following lemma.

Lemma 8 *For every $|x| < 2/\sqrt{n}$, the above series given by b_m satisfies the Leibniz alternating series test.*

A Naive Proof Attempt: We briefly explain why the following naive approach to proving this lemma fails. For simplicity, assume that n is odd, so that β_n is positive and β_{n-1} is negative by Lemma 6. Then, the naive approach would be to prove that

$$\beta_n * \frac{(n * x)^{2m-1}}{(2m-1)!} + \beta_{n-1} * \frac{((n-1) * x)^{2m-1}}{(2m-1)!}$$

(and similarly paired other terms) decreases as m increases, starting from $m = n + 1$. Since powers of $n * x$ are larger than powers of $(n - 1) * x$, this would eventually be true for some

$m > n + 1$. However, since $|\beta_n| < |\beta_{n-1}|$ and β_{n-1} is negative (see Lemma 6), this is not necessarily true at $m = n + 1$. In fact, calculations show that this indeed fails for a few terms beyond $m = n + 1$. Thus, a more advanced approach is required to prove that the Leibniz test holds starting at $m = n + 1$. We will show that the test holds for $|x| < 2/\sqrt{n}$.

Preparing for the Proof: We prove Lemma 8 in the next subsection, but first we show several additional lemmas which will assist us in the proof of Lemma 8.

Define $V^{(n,k)}(\mathbf{a})$ to be an $n \times n$ matrix, which is same as the Vandermonde matrix $V^{(n)}(\mathbf{a})$ except the last column is replaced by the $(n - 1 + k)$ -th powers (instead of the $(n - 1)$ -th powers).

Lemma 9 For $k \geq 1$, the determinant of the matrix $V^{(n,k)}(\mathbf{a})$ is

$$\det(V^{(n)}(\mathbf{a})) * s_k(\mathbf{a}),$$

where $s_k(\mathbf{a})$ is a (symmetric) polynomial in \mathbf{a} given by

$$s_k(\mathbf{a}) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} a_{i_1} * \dots * a_{i_k}.$$

Note that the polynomials $s_k(\mathbf{a})$ differ from the well-known elementary symmetric polynomials $e_k(\mathbf{a})$ since in the latter, the summation is taken over $1 \leq i_1 < \dots < i_k \leq n$.

Proof:

Fix any $k \geq 1$. Consider an $n \times n$ matrix M which is same as $V^{(n,k)}(\mathbf{a})$ except that the last row is powers of an indeterminate x . Let \mathbf{a}' stand for the $(n - 1)$ length truncation of \mathbf{a} . Treating the elements of \mathbf{a}' as scalars, the determinant of the matrix M is a polynomial in x of degree $n - 1 + k$. Call this polynomial $f(x)$. Since the determinant of a matrix with two equal (or even scaled by a constant) rows is zero, the polynomial $f(x)$ has roots \mathbf{a}' . Thus,

$$f(x) = g(x) * \prod_{i=1}^{n-1} (x - a_i), \tag{3}$$

where $g(x)$ is a polynomial (to be determined) of degree k . However, $f(x)$, the degree $n - 1 + k$ polynomial, has zero coefficients for all monomials x^j with j in $[n - 1..n - 1 + k - 1]$. If we introduce a new formal variable $t = 1/x$, then the above equation (3) can be written as

$$\tilde{f}(t) = \tilde{g}(t) * \prod_{i=1}^{n-1} (1 - ta_i). \tag{4}$$

where \tilde{f} (resp. \tilde{g}) is the polynomial f (resp. g) with coefficients reversed. Note, all the zero coefficients of $f(x)$ described above implies that $\tilde{f}(t) = f_{n-1+k} \bmod t^{k+1}$, where f_{n-1+k} denotes the coefficient of x^{n-1+k} in $f(x)$. Thus,

$$f_{n-1+k} * \prod_{i=1}^{n-1} (1 - ta_i)^{-1} = \tilde{g}(t) \bmod t^{k+1}. \quad (5)$$

Hence,

$$f_{n-1+k} * \prod_{i=1}^{n-1} \sum_{j=0}^k (1 + (ta_i)^j) = \tilde{g}(t) \bmod t^{k+1}. \quad (6)$$

Since $g(x)$ is of degree k , $\tilde{g}(t)$ has degree at most k as well. Denote by \tilde{g}_z the coefficient of t^z in \tilde{g}_z , which is same as g_{k-z} . Then, by comparing coefficients of t^z on both sides we get, that for each $z \in [0..k]$,

$$g_{k-z} = \tilde{g}_z = f_{n-1+k} * s_z(\mathbf{a}'),$$

where $s_0(\mathbf{a}')$ is defined to be 1.

Thus, having determined $g(x)$, we also have $f(x)$ by (3). Letting $x = a_n$, then we get

$$\begin{aligned} \det(V^{(n,k)}(\mathbf{a})) &= f(a_n) \\ &= \prod_{i=1}^{n-1} (a_n - a_i) * g(a_n) \\ &= \prod_{i=1}^{n-1} (a_n - a_i) * f_{n-1+k} * \sum_{z=0}^k a_n^{k-z} s_z(\mathbf{a}') \\ &= \prod_{i=1}^{n-1} (a_n - a_i) * f_{n-1+k} * s_k(\mathbf{a}) \\ &= \det(V^{(n)}(\mathbf{a})) * s_k(\mathbf{a}), \end{aligned}$$

where the last equality follows by noting that the top coefficient of $f(x)$, i.e. f_{n-1+k} is the (n, n) -minor of $V^{(n,k)}(\mathbf{a})$, which is same as the (n, n) -minor of Vandermonde matrix $V^{(n)}(\mathbf{a})$, which, in turn, is $(-1)^{n+n} * \det V^{(n-1)}(\mathbf{a}')$. \square

Lemma 10 For $\mathbf{a} = (1^2, 2^2, 3^2, \dots, n^2)$, for all $k \geq 0$,

$$\frac{s_{k+1}(\mathbf{a})}{s_k(\mathbf{a})} \leq n^3.$$

Proof: First note that $s_{k+1}(\mathbf{a}) = \sum_{i=1}^n a_i * s_k(\mathbf{a}_{(i)})$, where $\mathbf{a}_{(i)}$ is \mathbf{a} restricted to first i entries. Since a_i are monotonically increasing, it follows that $s_{k+1}(\mathbf{a}) \leq n * a_n * s_k(\mathbf{a})$, from which the claim follows. \square

Lemma 11 For the matrix $S^{(n)}(\mathbf{a})$ with \mathbf{a} set to the sequence of integers from one to n , let β_i be given by the formula in Lemma 4. Then,

$$\sum_{i=1}^n \beta_i * i^{2n+1} = (-1)^{n-1} * (n!)^2.$$

Proof: $\sum_{i=1}^n \beta_i * i^{2n+1}$ is the inner product of the first column of $\hat{S}^{(n)}(\mathbf{a})$ and $\vec{\beta}$. By Lemma 4, we have

$$\begin{aligned} \vec{\beta}^\top \cdot (\hat{S}^{(n)}(\mathbf{a}))_1 &= ((S^{(n)}(\mathbf{a}))^{-\top})_1^\top \cdot (\hat{S}^{(n)}(\mathbf{a}))_1 \\ &= \frac{1}{\det(S^{(n)}(\mathbf{a}))} * \sum_{i=1}^n (-1)^{i+1} d_i * (\hat{S}^{(n)}(\mathbf{a}))_{i,1} \\ &= \frac{\det(\hat{S}^{(n)}(\mathbf{a}))}{\det(S^{(n)}(\mathbf{a}))} \\ &= (-1)^{n-1} * \prod_{i=1}^n a_i^2 \\ &= (-1)^{n-1} * (n!)^2, \end{aligned}$$

where we have used Lemma 5 in the second-to-last equality. \square

2.1 Alternating Series Test

Having shown Lemmas 9, 10, and 11, we are now ready to show Lemma 8.

Proof: (of Lemma 8) In this proof, we will fix \mathbf{a} to be the sequence of integers from 1 to n . Note, each b_m can be written as $b_m = c_m * \frac{x^{2m-1}}{(2m-1)!}$, where $c_m = (-1)^{n-1} * \sum_{i=1}^n \beta_i * i^{2m-1}$.

1. Since n is fixed and all β_i are bounded by Lemma 6, we just need to show that for every x in the domain of approximation, for every $i \in [n]$, $\frac{(ix)^{2m-1}}{(2m-1)!}$ goes to zero as m goes to infinity. Since the domain of approximation is bounded, $|x|$ itself is bounded. Since, $k! \geq e(k/e)^k$, the above is upper bounded by $e^{-1} * (ie x / (2m-1))^{2m-1}$, which goes to zero as m goes to infinity.

2. To show that all b_m are positive (or all are negative), it suffices to show that all c_m are positive (or all c_m are negative, resp.). We first show that c_{n+1} is positive (i.e. m set to $n+1$). By Lemma 11, this quantity is simply $(-1)^{2(n-1)} * (n!)^2$ and hence is positive.

Let $\hat{S}^{(n,k)}(\mathbf{a})$ be the matrix that is the same as $\hat{S}^{(n)}(\mathbf{a})$ except that the first column is replaced by the $(2n-1+2k)$ powers of \mathbf{a} . Thus, $\hat{S}^{(n,1)}(\mathbf{a})$ is same as $\hat{S}^{(n)}(\mathbf{a})$. As in the proof of Lemma 11,

$$\begin{aligned}
(-1)^{n-1} * c_{n+k} &= \sum_{i=1}^n \beta_i * i^{2n-1+2k} \\
&= \vec{\beta}^\top \cdot (\hat{S}^{(n,k)}(\mathbf{a}))_1 \\
&= ((S^{(n)}(\mathbf{a}))^{-\top})_1^\top \cdot (\hat{S}^{(n,k)}(\mathbf{a}))_1 \\
&= \frac{1}{\det(S^{(n)}(\mathbf{a}))} * \sum_{i=1}^n (-1)^{i+1} d_i * (\hat{S}^{(n,k)}(\mathbf{a}))_{i,1} \\
&= \frac{\det(\hat{S}^{(n,k)}(\mathbf{a}))}{\det(S^{(n)}(\mathbf{a}))}
\end{aligned}$$

Note that $\det(\hat{S}^{(n,k)}(\mathbf{a}))$ is a Schur polynomial in \mathbf{a} . It is an alternating polynomial, i.e. its sign changes if any two entries of \mathbf{a} are exchanged. All alternating polynomials are divisible by the Vandermonde polynomial, i.e. $\det(S^{(n)}(\mathbf{a}))$; this is clear by noting that every $(a_i - a_j)$ is also a factor of $\det(\hat{S}^{(n,k)}(\mathbf{a}))$, as this determinant is zero if $a_i = a_j$. Regardless, we will not pursue this approach here, and directly use Lemma 5. From that lemma, it follows (after some maneuvering) that $\det(\hat{S}^{(n,k)}(\mathbf{a}))$ is

$$(-1)^{n-1} * s_{k-1}(\mathbf{a}^{(2)}) * \prod_{i=1}^n \prod_{1 \leq j < i} (a_i^2 - a_j^2) * \prod_{i=1}^n a_i^3,$$

where $\mathbf{a}^{(2)}$ is the sequence \mathbf{a} , but with each entry squared. Thus, all $c_{n,k}$ are positive, for $k \geq 1$.

3. We now show that $|b_m| \geq |b_{m+1}|$ for all $m \geq n + 1$. We have,

$$\begin{aligned}
\frac{|b_{m+1}|}{|b_m|} &= \frac{(-1)^{n-1} * s_{m+1-(n+1)}(\mathbf{a}^{(2)}) * \prod_{i=1}^n \prod_{1 \leq j < i} (a_i^2 - a_j^2) * \prod_{i=1}^n a_i^3 * \frac{x^{2m+1}}{(2m+1)!}}{(-1)^{n-1} * s_{m-(n+1)}(\mathbf{a}^{(2)}) * \prod_{i=1}^n \prod_{1 \leq j < i} (a_i^2 - a_j^2) * \prod_{i=1}^n a_i^3 * \frac{x^{2m-1}}{(2m-1)!}} \\
&= \frac{s_{m+1-(n+1)}(\mathbf{a}^{(2)}) * \frac{x^{2m+1}}{(2m+1)!}}{s_{m-(n+1)}(\mathbf{a}^{(2)}) * \frac{x^{2m-1}}{(2m-1)!}} \\
&= \frac{s_{m+1-(n+1)}(\mathbf{a}^{(2)})}{s_{m-(n+1)}(\mathbf{a}^{(2)})} * \frac{x^2}{2m(2m+1)} \\
&\leq n^3 * \frac{x^2}{2m(2m+1)} \text{ (by Lemma 10)} \\
&\leq 1 \text{ (for } |x| < 2/\sqrt{n}\text{)}.
\end{aligned}$$

□

We are now ready to show Theorem 1.

Proof: (of Theorem 1) Let β_k , for $k \in [1..n]$, be defined as in equation (1) with \mathbf{a} set to the sequence of numbers from 1 to n . From the Taylor series expansion of the sine series, which converges since the sine series is analytic, it follows that

$$\sum_{k=1}^n \beta_k \sin(kx) = x + \sum_{m=n+1}^{\infty} (-1)^m * b_m,$$

where b_m are defined in equation (2). Thus, by Lemma 8 and Leibniz's alternating series test (Theorem 7), we have for $|x| < 2/\sqrt{n}$,

$$\begin{aligned}
\left| x - \sum_{k=1}^n \beta_k \sin(kx) \right| &\leq \left| \sum_{k=1}^n \beta_k * \frac{(kx)^{2n+1}}{(2n+1)!} \right| \\
&= \left| \sum_{k=1}^n \beta_k * k^{2n+1} * \frac{x^{2n+1}}{(2n+1)!} \right| \\
&= \frac{|x^{2n+1}|}{(2n+1)!} * \left| \sum_{k=1}^n \beta_k * k^{2n+1} \right| \\
&= \frac{(n!)^2}{(2n+1)!} * |x^{2n+1}|,
\end{aligned}$$

where we used Lemma 11 in the last equality.

Restricting $|x| < \epsilon$, Theorem 1 follows from the fact that

$$\begin{aligned}
\frac{(n!)^2}{(2n+1)!} \epsilon^{2n+1} &< \frac{((n+1)/e)^{2n+2} e^2}{((2n+1)/e)^{2n+1}} \epsilon^{2n+1} \\
&= e * (n+1) * \left(\frac{n+1}{2n+1}\right)^{2n+1} * \epsilon^{2n+1} \\
&= e * (n+1) * \left(\frac{2n+2}{2n+1}\right)^{2n+1} * \left(\frac{\epsilon}{2}\right)^{2n+1} \\
&< e^2 * (n+1) * \left(\frac{\epsilon}{2}\right)^{2n+1},
\end{aligned}$$

where we have used the fact that

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+1}{e}\right)^{n+1} e$$

for all $n \geq 1$ and that $(1 + 1/n)^n < e$ for all $n \geq 1$. □

3 Application to Bootstrapping for Approximate HE

In Section 1, we explained that approximating the mod function on small intervals around the modulus is a necessary step in bootstrapping for approximate homomorphic encryption (CKKS). In this section, we will briefly overview the bootstrapping procedure for the CKKS-HE scheme introduced in [6].

Notation and Necessary Preliminaries: Let M be a power of 2 and $\Phi_M(X) = X^N + 1$ be the M th cyclotomic polynomial of degree $N = M/2$. Let $\mathcal{R} = \mathbb{Z}[X]/\Phi_M(X)$. For an integer q , let $\mathcal{R}_q = \mathbb{Z}_q[X]/\Phi_M(X)$. Using the canonical embedding σ , it is possible to map an element $m(X) \in \mathcal{R}$ into \mathbb{C}^N by evaluating $m(X)$ at the M th primitive roots of unity. Using the same canonical embedding, it is also possible to define an isometric ring isomorphism between $\mathcal{S} = \mathbb{R}[X]/\Phi_M(X)$ and $\mathbb{C}^{N/2}$, where for an element $m(X) \in \mathcal{S}$, it has the canonical embedding norm $\|m\|_\infty^{\text{can}} = \|\sigma(m)\|_\infty$.

Overview of the CKKS-HE Scheme: The CKKS-HE scheme [8] is an HE scheme for approximate arithmetic over real/complex numbers. Its security is based on the ring-LWE (RLWE) assumption. The message space of the scheme is polynomials $m(X)$ in \mathcal{R} with $\|m\|_\infty^{\text{can}} < q/2$ for a prime q . Using the canonical embedding and appropriate scaling, one

can map a vector in $\mathbb{C}^{N/2}$ of fixed precision into \mathcal{R} . The fact that canonical embedding induces an isometric ring isomorphism between \mathcal{S} and $\mathbb{C}^{N/2}$ implies that operations on the message space \mathcal{R} map to the same operations performed coordinate-wise on $\mathbb{C}^{N/2}$. Thus, the CKKS-HE scheme supports packing $N/2$ complex numbers into a single plaintext and operating on them in single instruction multiple data (SIMD) manner. Please refer to [8] for more details on this encoding procedure. We will refer to $m(X) \in \mathcal{R}$ as the plaintext/message and the corresponding vector in $\mathbb{C}^{N/2}$ as the plaintext “slots.”

A ciphertext ct encrypting a message $m \in \mathcal{R}$ is an element of $\mathcal{R}_{q_\ell}^2$ for some $\ell \in \{0, \dots, L\}$. ℓ refers to the “level” of the ciphertext. In [8], $q_\ell = p^\ell * q$ for primes p and q . However, q_ℓ can be set in other ways (such as via an RNS basis [7]). The decryption structure is $\langle \text{ct}, \text{sk} \rangle \bmod q_\ell = m + e$ for some small error $e \in \mathcal{R}$. Observe that there is no way to remove e and some of the least significant bits of m are unrecoverable. A fresh ciphertext is generated at the highest level L . Homomorphic operations increase the magnitude of the error and the message and one must apply a rescaling procedure or modular reduction to bring a ciphertext to a lower level to continue homomorphic computation. Eventually, a ciphertext is at the lowest level (an element of \mathcal{R}_q^2), and no further operations can be performed.

Bootstrapping Procedure for CKKS-HE: [6] introduced the first bootstrapping procedure for the CKKS-HE scheme. Subsequent works [5, 10, 11] improved various aspects of bootstrapping, but the overall procedure remains the same. The goal is to take a ciphertext at the lowest level and bring it up to a higher level so that homomorphic computation can continue. Thus, given a ciphertext ct at the lowest level, we want to obtain another ciphertext ct' such that

$$\langle \text{ct}, \text{sk} \rangle \bmod q \approx \langle \text{ct}', \text{sk} \rangle \bmod q_\ell$$

for some $\ell > 1$. For simplicity in the following, we will include the starting decryption error in the message m . That is, we will assume that $\langle \text{ct}, \text{sk} \rangle \bmod q = m$.

Bootstrapping is done via the following sequence of steps:

1. **Modulus Raising:** By simply considering ct as a ciphertext at the highest level, it follows that $\langle \text{ct}, \text{sk} \rangle \bmod q_L = qI + m$ for some $I \in \mathcal{R}$.
2. **Coefficients to Slots:** We need to perform the modular reduction on the polynomial coefficients of $t = qI + m$. However, recall that homomorphic computations evaluate coordinate-wise on the plaintext “slots,” not the polynomial coefficients. Thus, we need to transform our ciphertext so that the polynomial coefficients are in the “slots.” This can be done by evaluating a linear transformation homomorphically.

3. **Compute the Mod Function:** We need a procedure to compute/approximate the mod function homomorphically. This is a significant challenge since we can only compute arithmetic operations homomorphically.
4. **Slots to Coefficients:** Finally, we need to undo the coefficients to slots step. This can be done by homomorphically evaluating the inverse of the previous linear transform.

Observe that if we can approximate the mod function, then the above procedure will give us a ct' at some higher level ℓ that decrypts to $m + e$ for some small error e . Since we are dealing with approximate arithmetic, this error from bootstrapping can be absorbed into the other errors that occur during approximate arithmetic and homomorphic evaluation. We can upper bound $|I| < K$ for some integer K so that we only need to approximate the mod function on the interval $[-Kq - m, Kq + m]$, where we have overloaded notation to make m an upper bound on the size of the message.

4 Evaluating the Sine Series Approximation of the Mod Function

In order to use the sine series approximation of the mod function given by Corollary 3 for bootstrapping, we must approximate the sine series by a low-degree polynomial, since the CKKS-HE scheme cannot compute sine directly. In this section, using our sine series approximation of the mod function and the well-known Taylor series expansion of the sine function, we will give explicit low-degree polynomial approximations of the mod function on small intervals around multiples of the modulus to (almost) arbitrary precision. The resulting polynomials have small coefficients, as the Taylor series of the sine function has small coefficients, and the sine series itself has small coefficients by Lemma 6. Recall that small coefficients are beneficial in contrast to large coefficients, as in the latter case one is forced to compute the different power monomials to much higher precision in order to obtain an accurate polynomial evaluation. This, in turn, causes the computational precision that we must operate at during bootstrapping to be higher, which causes each “level” to consume more bits of the modulus. We next explain how we evaluate the sine series and then determine the degree and evaluation precision required for the Taylor series approximation of sine.

Evaluating the Sine Series: To evaluate the sine series, we first compute a Taylor series approximation of e^{ix} (recall that CKKS-HE allows us to compute over complex numbers).

We can obtain an approximation to $\sin x$ by extracting the imaginary part. The other higher order $\sin kx$ terms can be obtained conveniently by computing e^{ikx} from e^{ix} and extracting the imaginary part. As for computing the Taylor series approximation of the sine function, note that the domain of approximation is small intervals around ℓq , where $\ell \in [-K..K]$ and q is the modulus. The bound K comes from the bound on the Hamming-weight of the secret key and is typically 8 to 24. If our input is $X = x + \ell q$ for some small offset x and $\ell \in [-K..K]$, our goal is to compute $e^{i(2\pi(x+\ell q)/q)}$. This then requires a Taylor series that has powers of $2\pi(x + \ell q)/q$, which can be more than one. Earlier works noted that one can instead first compute $e^{i(2\pi(x+\ell q)/(q2^r))}$ using a Taylor series expansion (for some $r > 0$) and then compute $e^{i(2\pi(x+\ell q)/q)}$ using r squarings. For instance, by setting $r = \lceil \log((K + 1)2\pi) \rceil$, we can ensure that $2\pi i(x + \ell q)/(q2^r)$ has norm less than one, and thus, the higher powers in the Taylor series rapidly approach zero.

Determining the Degree of the Taylor Series Approximation: Next, we must determine the degree to which we compute the Taylor series expansion of $e^{2\pi i(x+\ell q)/(q2^r)}$. The Taylor series expansion is

$$\sum_{m=0}^{\infty} (2\pi i(x + \ell q)/(q2^r))^m / m!.$$

We now determine for which range of values of $(x + \ell q)$ the above restricted to the sine terms, i.e. the imaginary terms or odd powers of x , satisfies the alternating series test (so that the partial series error can be bound by the absolute value of the next missing term). Thus, we need to determine the conditions under which

$$\begin{aligned} 1 &> \frac{(2\pi|(x + \ell q)/(q2^r)|)^{(2m+1)} / (2m + 1)!}{(2\pi|(x + \ell q)/(q2^r)|)^{(2m-1)} / (2m - 1)!} \\ &= \frac{(2\pi|(x + \ell q)/(q2^r)|)^2}{(2m + 1)2m} \end{aligned}$$

Assuming $x \ll q$ and $2^r \geq (K + 1)$, the above reduces to $(2m + 1)2m > 2\pi$, or $m > \sqrt{\pi/2}$. Thus, if the Taylor series is computed partially up to any degree $2m - 1$, $m > \sqrt{\pi/2}$, then the error in the approximation of sine is at most

$$(2\pi)^{2m+1} / (2m + 1)! < (2\pi e / (2m + 1))^{2m+1},$$

which is at most 2^{-2m+1} if we further require that $m > 2\pi e (> \sqrt{\pi/2})$.

Thus, having computed $\sin(2\pi(x + \ell q)/(q2^r))$ partially up to m terms, we now investigate the error for the higher order terms in the sine series, i.e. $\sin(2\pi k(x + \ell q)/q)$ for $k \geq 1$. If

the error in the approximation of the original term is small, say $\delta \ll 1$, then the error for this k -th term is approximately $k2^r * \delta$. Thus, the total error in the sine series due to the Taylor series approximation of $\sum_{k=1}^n \beta_k \sin(2\pi k(x + \ell q)/q)$ is upper bounded in absolute value by $\sum_{k=1}^n |\beta_k| * k2^r |\delta|$, which is at most $n^2(K+1)\delta$, which, in turn, is at most $n^2(K+1)2^{-2m+1}$.

Finally, using Corollary 3, the total error in the mod function approximation, for an input $X = x + \ell q$ with $\ell \in [-K..K]$ and $|x| < \epsilon * q$ for any $\epsilon < 1/\pi\sqrt{n}$ is

$$(q/2\pi) * n^2(K+1)2^{-2m+1} + \frac{e^2 * q}{2\pi} * (n+1) * (\epsilon * \pi)^{2n+1}.$$

Thus, it makes sense to have m about $-\log_2(\epsilon/2) * n$ (which is typically greater than $2\pi e$ for $n > 1$; if this value is less than $2\pi e$, then one must use an r such that $2^r/(K+1)$ is correspondingly larger than one).

Determining the Evaluation Precision: We must also determine the precision to which to evaluate the polynomials. Setting $Y = 2\pi(x + \ell q)/(q2^r)$, we observe that the degree m Taylor expansion of $e^{2\pi i(x + \ell q)/(q2^r)}$ is simply the polynomial

$$\sum_{j=0}^m (iY)^j / j!.$$

Recall that we have chosen r so that $|Y| < 1$. Moreover, setting $c_j = i^j/j!$, the polynomial becomes $\sum_{j=0}^m c_j Y^j$, where $|c_j| \leq 1$. We need to determine the precision to which we evaluate the powers Y^j (we will first evaluate the Y^{2^j} 's by repeated squaring and then use these powers to evaluate all intermediate powers). Let Y^j denote the exact values and let \tilde{Y}^j denote the approximated values (to some precision to be determined). Suppose we evaluate the powers Y^j up to w bits (and simply chop off the additional bits). Then, $|\tilde{Y} - Y| < 2^{-w}$. Computing \tilde{Y}^2 by squaring \tilde{Y} and rounding, we have that \tilde{Y}^2 differs from Y^2 by at most $\approx 2 * 2^{-w}$. To see this, note that $\tilde{Y} = Y \pm \delta$, where $\delta < 2^{-w}$. Then, $\tilde{Y}^2 = Y^2 \pm 2Y\delta + \delta^2 < Y^2 \pm 2\delta + \delta^2 \approx Y^2 \pm 2 * 2^{-w}$. By an analogous argument, it follows that \tilde{Y}^j differs from Y^j by at most approximately $j * 2^{-w}$. Thus, the error of $\sum_{j=0}^m c_j \tilde{Y}^j$ is bounded by

$$\sum_{j=0}^m j * 2^{-w} * \frac{1}{j!} = \sum_{j=1}^m \frac{2^{-w}}{(j-1)!} < e * 2^{-w}.$$

Thus, to obtain error 2^{-d} , it suffices to compute the powers \tilde{Y}^j to precision w for $w > d + \log_2 e$, only slightly higher than the minimum precision d required to obtain this approximation.

In the above, we saw that having small coefficients c_j (and coefficients that decrease in magnitude as j increases) enabled the approximation of the polynomial $\sum_{j=0}^m c_j Y^j$ by evaluating the powers of Y to precision only a couple bits larger than the minimum precision required for the desired error. This is crucial during bootstrapping as a higher evaluation precision directly corresponds to losing more bits of the modulus during the polynomial evaluation. In contrast, suppose that the c_j 's were large and bounded in magnitude $|c_j| < 2^k$ for some k . Then, if the powers of Y are evaluated to precision w , the error of the polynomial evaluation is bounded by

$$\sum_{j=0}^m j * 2^{-w} * 2^k < \frac{m(m+1)}{2} * 2^{k-w}.$$

Thus, to obtain error 2^{-d} , the powers of Y would need to be evaluated to precision $w > d + k + 2 \log m - 1$. Note the additional dependence on both k and the number of terms m .

5 Implementation

To demonstrate the applicability of our polynomial approximation to high precision bootstrapping for approximate homomorphic encryption, we updated the bootstrapping procedure of the HEAAN library [1] to utilize our sine series during the ‘‘Compute the Mod Function’’ step (see Section 3). Additionally, we updated HEAAN to use the `quadmath` library, since we wanted to achieve bootstrapping error smaller than the precision of a `double`. We ran our implementation using a PC with an Intel Xeon 2.0 GHz 12-Core CPU.

Table 1 gives our bootstrapping results for sine series of various orders. As before, ϵ represents the ratio p/q , where p is an upper bound on the size of the message (including any errors associated from the approximate arithmetic and prior homomorphic operations) and q is the size of the modulus prior to bootstrapping. In Table 1, ϵ is set to 2^{-10} . The key-sparsity is set to $h = 256$, so that on average K is about $\sqrt{h} = 16$. However, our implementation can handle K as large as 31. q_L denotes the modulus of the largest level, which is the modulus of a fresh ciphertext prior to any homomorphic operations. N denotes the ring dimension, which we increase as q_L increases to maintain 128-bit security [2, 3]. Results in this table were obtained using 8 slots, and the dependence on a larger number of slots is reported below. $q_{\ell'}$ denotes the modulus of the ciphertext after bootstrapping. The reported error is the decryption error after performing bootstrapping. In other words, if the decryption before bootstrapping would have resulted in message slot value M , then the decryption after bootstrapping would result in a message slot value M' such that $|M' - M| \leq \beta_{\text{bs}}|M|$. As can be seen from Table 1, for $\log_2 p = 80$, the bootstrapping error is essentially zero. This is because the bootstrapping procedure is performed at a precision that is ten bits more than the number of bits required

Table 1: High-Precision Bootstrapping Results for $\epsilon = 2^{-10}$ and degree 63 Taylor series sine approx. The secret-key sparsity is set to $h = 256$. The errors reported are for K up to 31.

Input Precision [†] $\log_2 p$	Sine Series Order	Modulus (Fresh) $\log_2 q_L$	Ring Dim. N	Boot. prec.	Modulus (After) $\log_2 q_{\ell'}$	Error (Boot.) $\beta_{\mathbf{bs}} = \text{err}/p$	Runtime ^{††} (secs)
30	2	1200	2^{15}	55	235	2^{-25}	16
50	3	1600	2^{15}	75	220	2^{-45}	19
60	4	2400	2^{16}	85	839	2^{-54}	72
80	5	2400	2^{16}	105	374	$< 2^{-80}$	73
100	6	3000	2^{16}	125	594	2^{-95}	100
120	7	3200	2^{16}	145	414	2^{-95}	111

[†] The modulus q_{ℓ} of the ciphertext prior to bootstrapping is p/ϵ . The number of bits of q_{ℓ} is $p - \log \epsilon = p + 10$, and bootstrapping (computational) precision is set to $(p - \log \epsilon + \log_2 K) + 10$.

^{††} Includes runtime of ‘‘Coefficients to Slots’’ and ‘‘Slots to Coefficients’’ steps. Number of slots fixed to be 8 so that the ‘‘Compute the Mod Function’’ step dominates runtime. Results reported are from a 2.0 GHz 12-core Intel Xeon CPU using quadmath, NTL and GMP software libraries.

Table 2: Timing and Error Dependence on Number of Slots. In this table $\epsilon = 2^{-10}$, $\log_2 p = 80$, and the sine series order is fixed to $n = 5$.

Num Slots	Input Precision $\log_2 p$	Sine Series Order	Modulus (Fresh) $\log_2 q_L$	Ring Dim. N	Boot. prec.	Modulus (After) $\log_2 q_e$	Error (Boot.) $\beta_{\text{bs}} = \text{err}/p$	Runtime ^{††} (secs)
8	80	5	2400	2^{16}	105	374	$< 2^{-80}$	73
16	80	5	2400	2^{16}	105	374	2^{-80}	84
32	80	5	2400	2^{16}	105	374	2^{-78}	107
64	80	5	2400	2^{16}	105	374	2^{-75}	110
128	80	5	2400	2^{16}	105	374	2^{-69}	152
256	80	5	2400	2^{16}	105	374	2^{-68}	228

^{††} Includes runtime of “Coefficients to Slots” and “Slots to Coefficients” steps. For all rows, the mod function evaluation time is almost the same at 52 secs.

to represent $M + Kq$ (i.e. the value which needs to be reduced mod q).

Recall that the sine series approach begins by approximating e^{ix} using a Taylor series approximation, since CKKS-HE allows computation on complex numbers. In this particular implementation e^{ix} was approximated to degree 63 using the Paterson-Stockmeyer polynomial evaluation optimization [18]. Below, we report results for other variants for approximating e^{ix} .

We see that our methodology is capable of achieving high precision bootstrapping, with the resulting message precision as large as 94 bits. Prior to our work, the highest precision bootstrapping of CKKS was the recent work of [12] which could achieve a resulting message precision of up to 67 bits. However, that result was only for $K = 12$ and key sparsity $h = 64$, whereas our 94 bit precision bootstrapping is for $h = 256$ and can handle K up to 31. Thus, we view our result as a substantial improvement for bootstrapping in settings where high precision is required, such as the inference step of a convolution neural network or even the learning stage of the neural network. As mentioned earlier, since CKKS is for approximate arithmetic, it is only possible to have unlimited computation for stable computations that do not lose precision. However, even such stable computations lose precision in early stages prior to convergence. Thus, it is important to begin such computations with high precision and, later, one can switch to smaller precision during the stable regime.

5.1 Time and Error Dependence on the Number of Slots

As the number of slots is increased, the time of the mod function evaluation step during bootstrapping remains the same. However, the linear transforms that send the coefficients

to slots and vice versa take a substantial hit. Since this linear transform also involves more rotations, key-switchings, multiplications by constants, and additions, for every doubling of the number of slots, the bootstrapping error also increases proportionately. However, since our error is so low, the error for a high number of slots still remains low enough to be termed high-precision. This dependence of runtime and bootstrapping error is reported in Table 2 for one particular parameter, where the sine series is of order five.

5.2 Comparison with Basic Sine and Other Variants

While the implementation results reported in Table 1 used a Taylor series approximation of degree 63 of e^{ix} , the implementation in [1] instead used a degree 7 approximation of $e^{ix/2^r}$ (for $r = 4$) followed by r squarings. We investigated if we could use a similar approach for the sine series, as the different order sine terms are obtained by squarings of e^{ix} anyway. We found that for small precision, i.e. $\log_2 p \leq 40$, this approach can lead to a faster implementation while yielding effectively the same error. However, for $\log_2 p \geq 50$, this approach led to substantially worse error. For example, at $\log_2 p = 50$, the error increased from 2^{-45} to 2^{-30} . But, as mentioned, for smaller $\log_2 p$ we get the following improvements. First of all, the basic sine approach (i.e. $n = 1$) with $r = 4$ and degree 7 Taylor series yields an error of 2^{-19} for $\log_2 p = 30$. If the fresh modulus used is 1600 bits, then the modulus after bootstrapping has 741 bits. The time taken is 7.6 secs. Interestingly, with sine series of order two, i.e. $n = 2$, using the same approach we get an error of 2^{-26} , with modulus after bootstrapping having 631 bits. Moreover, the time taken is 7.8 secs. Yet another implementation, with a degree 31 Taylor series approximation, and $r = 0$, also yields error 2^{-25} , but takes time 12 secs. However, the modulus after bootstrapping has more bits at 690 bits. Regardless, it seems that the sine series of order two with a degree 7 Taylor series and $r = 4$ seems to be beneficial at low precision.

We also experimented with different values of ϵ , in particular ϵ set to 2^{-5} , 2^{-10} , 2^{-15} , 2^{-20} . The errors at each input precision were not much different, and, in fact, $\epsilon = 2^{-10}$ seems to be the best option.

5.3 Comparison with Other Earlier Works

The work [5] followed an interesting approach of obtaining Chebyshev interpolants of the scaled sine function. In particular, using the Fourier series of $\sin(2\pi K \cos x)$, they obtained approximations of $\sin(2\pi Kx)$ in terms of Chebyshev polynomials. Furthermore, this approach also leads to an almost optimal minmax polynomial approximation, as well as yielding small coefficients. Since the scaling K is already incorporated in the function, it removes the $\log K$

Table 3: Comparison with Modular Lagrange Interpolation [12]. Note, [12] cites results for $K = 12$, whereas our results are for K up to 31.

Input Precision	[12]			This Work		
	Key Sparsity (h)	Ciphertext Bits Lost	Error (Boot.)	Key Sparsity (h)	Ciphertext Bits Lost	Error (Boot.)
30	64	935	2^{-24}	256	965	2^{-25}
50	64	1725	2^{-46}	256	1380	2^{-45}
60	64	1800	2^{-54}	256	1561	2^{-54}
80	64	2150	2^{-63}	256	2026	2^{-80}
100	N/A	N/A	N/A	256	2406	2^{-95}
120	N/A	N/A	N/A	256	2786	2^{-95}

squarings required in [6] and in this work. However, Chebyshev interpolants do not readily submit to the Paterson-Stockmeyer evaluation optimization and while [5] did show a variant of this method, it leads to coefficients increasing in size. Thus, as explained in Section 3, this then requires a larger computational precision that leads to loss of many more (ciphertext modulus) bits per multiplication depth in the bootstrapping circuit. For a direct comparison of our approach to [5], we take data from Tables 2-4 from that work, and note that the best approximation they obtain has error 2^{-21} for data set IV*. A look at our Table 1 shows that the worst error we obtain is 2^{-25} for $\log_2 p = 30$. The number of ciphertext (modulus) bits lost for that error is $1200 - 235 = 965$, whereas [5] loses $1240 - 43 * 6 = 982$ bits. Moreover, our implementation can handle K up to 31, whereas [5] numbers are cited for $K = 12$. Thus, our approach is clearly better at even this low precision.

In [11], the authors obtain better approximation error than [5] by leveraging the fact the approximation is only needed in small intervals around multiples of the modulus. However, their approach also uses a baby-step giant-step, or alternately the Paterson-Stockmeyer variant applied to Chebyshev polynomials that can lead to a blowup in the size of coefficients. The authors do not give details on the number of ciphertext (modulus) bits lost in the bootstrapping procedure. In [16], the authors report high-precision bootstrapping using a composition of sine/cosine and the inverse sine function. The authors report a practical implementation of up to 40-bits precision bootstrapping.

The work [12] gives a direct approximation of the mod function, i.e. without going through the sine function, and hence bypasses the fundamental error of the sine function approach. Thus, they can get arbitrarily high precision, and they also show that the coefficients of their polynomial approximation are not too large. Nevertheless, the coefficients are large enough

that our approach beats [12]. Moreover, they only give implementation numbers for $K = 12$, and for $K = 24$, the number of ciphertext modulus bits lost during bootstrapping would be higher. In Table 3, we compare their results with ours for $\epsilon = 2^{-10}$ and various plaintext precisions.

References

- [1] Heaan, <https://github.com/snucrypto/HEAAN> 5, 5.2
- [2] Albrecht, M.R.: On dual lattice attacks against small-secret lwe and parameter choices in helib and seal. In: Coron, J.S., Nielsen, J.B. (eds.) *Advances in Cryptology – EUROCRYPT 2017*. pp. 103–129. Springer International Publishing, Cham (2017) 5
- [3] Albrecht, M.R., Player, R., Scott, S.: On the concrete hardness of learning with errors. *J. Math. Cryptol.* 9(3), 169–203 (2015), <http://www.degruyter.com/view/j/jmc.2015.9.issue-3/jmc-2015-0016/jmc-2015-0016.xml> 5
- [4] Bergamaschi, F., Halevi, S., Halevi, T.T., Hunt, H.: Homomorphic training of 30,000 logistic regression models. In: Deng, R.H., Gauthier-Umaña, V., Ochoa, M., Yung, M. (eds.) *Applied Cryptography and Network Security*. pp. 592–611. Springer International Publishing, Cham (2019) 1
- [5] Chen, H., Chillotti, I., Song, Y.: Improved bootstrapping for approximate homomorphic encryption. In: *EUROCRYPT*. pp. 34–54 (2019) 1, 3, 5.3
- [6] Cheon, J., Han, K., Kim, A., Kim, M., Song, Y.: Bootstrapping for approximate homomorphic encryption. In: *EUROCRYPT*. pp. 360–384 (01 2018) 1, 1, 1.1, 3, 3, 5.3
- [7] Cheon, J.H., Han, K., Kim, A., Kim, M., Song, Y.: A full rns variant of approximate homomorphic encryption. In: *Selected Areas in Cryptography – SAC 2018* (2018) 1, 3
- [8] Cheon, J.H., Kim, A., Kim, M., Song, Y.: Homomorphic encryption for arithmetic of approximate numbers. In: *ASIACRYPT* (2017) 1, 3
- [9] Gentry, C.: Fully homomorphic encryption using ideal lattices. In: *STOC*. pp. 169–178 (2009) 1
- [10] Han, K., Hhan, M., Cheon, J.H.: Improved homomorphic discrete fourier transforms and the bootstrapping. *IEEE Access* 7, 57361–57370 (2019) 3

- [11] Han, K., Ki, D.: Better bootstrapping for approximate homomorphic encryption. In: Jarecki, S. (ed.) *Topics in Cryptology – CT-RSA 2020*. pp. 364–390. Springer International Publishing, Cham (2020) 1, 3, 5.3
- [12] Jutla, C.S., Manohar, N.: Modular lagrange interpolation of the mod function for bootstrapping of approximate he. *Cryptology ePrint Archive*, Report 2020/1355 (2020), <https://eprint.iacr.org/2020/1355> 1, 1.1, 5, 3, 5.2, 5.3
- [13] Kim, A., Song, Y., Kim, M., Lee, K., Cheon, J.H.: Logistic regression model training based on the approximate homomorphic encryption. *BMC Medical Genomics* 11(4), 83 (2018), <https://doi.org/10.1186/s12920-018-0401-7> 1
- [14] Kim, M., Harmanci, A., Bossuat, J.P., Carpov, S., Cheon, J., Chilotti, I., Cho, W., Froelicher, D., Gama, N., Georgieva, M., Hong, S., Hubaux, J.P., Kim, D., Lauter, K., Ma, Y., Ohno-Machado, L., Sofia, H., Son, Y., Song, Y., Jiang, X.: Ultra-fast homomorphic encryption models enable secure outsourcing of genotype imputation. *bioRxiv* (2020) 1
- [15] Kim, M., Song, Y., Wang, S., Xia, Y., Jiang, X.: Secure logistic regression based on homomorphic encryption: Design and evaluation. *JMIR Med Inform* 6(2), e19 (Apr 2018), <http://www.ncbi.nlm.nih.gov/pubmed/29666041> 1
- [16] Lee, J., Lee, E., Lee, Y., Kim, Y., No, J.: High-precision bootstrapping of rns-ckks homomorphic encryption using optimal minimax polynomial approximation and inverse sine function. *IACR Cryptol. ePrint Arch. 2020*, 552 (2020) 1, 5.3
- [17] Masters, O., Hunt, H., Steffinlongo, E., Crawford, J., Bergamaschi, F., Rosa, M.E.D., Quini, C.C., Alves, C.T., de Souza, F., Ferreira, D.G.: Towards a homomorphic machine learning big data pipeline for the financial services sector. In: *RWC* (2020) 1
- [18] Paterson, M.S., Stockmeyer, L.J.: On the number of nonscalar multiplications necessary to evaluate polynomials. *SIAM J. Comput.* 2, pp. 60–66 (1973) 5
- [19] Sav, S., Pyrgelis, A., Troncoso-Pastoriza, J.R., Froelicher, D., Bossuat, J.P., Sousa, J.S., Hubaux, J.P.: Poseidon: Privacy-preserving federated neural network learning (2020) 1