

# Quantum Key-length Extension

Joseph Jaeger<sup>1</sup>, Fang Song<sup>2</sup>, and Stefano Tessaro<sup>1</sup>

<sup>1</sup> Paul G. Allen School of Computer Science & Engineering  
University of Washington, Seattle, US  
{jsjaeger,tessaro}@cs.washington.edu

<sup>2</sup> Portland State University, Portland, Oregon, US  
fang.song@pdx.edu

**Abstract.** Should quantum computers become available, they will reduce the effective key length of basic secret-key primitives, such as blockciphers. To address this we will either need to use blockciphers which inherently have longer keys or use key-length extension techniques which employ a blockcipher to construct a more secure blockcipher that uses longer keys.

We consider the latter approach – in particular, analyzing the security of the FX and double encryption constructions. Classically, these constructions were considered as key-length extension techniques for DES. FX was proven to be a secure key-length extension technique, while double encryption was shown to be no more secure than single encryption due to a meet-in-the-middle attack. In this work we provide positive results, with concrete and tight bounds, for the security of *both* of these constructions against quantum attackers in ideal models.

For FX, we consider security in the so-called “Q1 model,” a natural model in which the attacker has quantum access to the ideal primitive, but only classic access to FX. We provide two partial results for FX in this model. The first establishes the security of FX against non-adaptive attackers. The second establishes security against fully adaptive attackers when considering a variant of FX using a random oracle in place of an ideal cipher. This result relies on the techniques of Zhandry (CRYPTO ’19) for lazily sampling a quantum random oracle and are thus hard to extend to the true FX construction because it is currently unknown if a quantum random permutation can be lazily sampled. To the best of our knowledge, this result also is the first to introduce techniques to handle Q1 security in ideal models without analyzing the classical and quantum oracles separately, which may be of broader interest.

For double encryption we apply a technique of Tessaro and Thiruvengadam (TCC ’18) to establish that security reduces to the difficulty of solving the list disjointness problem, which we are able to reduce through a chain of results to the known quantum difficulty of the element distinctness problem.

## 1 Introduction

The looming threat of quantum computers has, in recent years, inspired significant efforts to design and analyze the security of post-quantum cryptographic schemes. In the public-key setting, Shor [28] established polynomial time quantum algorithms for factoring and computing discrete logarithms – breaking the security of essentially all practically deployed primitives.

In contrast, the general expectation in the secret-key setting is that quantum computers will not completely break the security of the basic primitives. Quantum computers will, however, reduce the effective key length of secret-key primitives by half via Grover’s search algorithm [12]. Thus, a primitive like the AES-128 blockcipher which may be thought to have 128 bits of security against classical computers can provide no more than 64 bits of security against a quantum computer, which would be considered significantly lacking. Even more worrisome, it was relatively recently shown that quantum computers can break several secret-key constructions completely such as the Even-Mansour blockcipher [23] and CBC-MAC [17], if we grant the attacker fully quantum access to the cryptosystem.

This would not be the first time that we find ourselves using too short of a key. A similar issue had to be addressed when the DES blockcipher was widely used and its 56 bit keylength was considered insufficient. Following approaches considered at that time, we can either transition to using basic primitives which have longer keys (e.g. replacing AES-128 with AES-256) or design key-length extension techniques to address

the loss of concrete security due to quantum computers. In this paper we analyze the latter approach. We consider two key-length extension techniques, FX [20] and double encryption, and provide provable bounds against quantum attackers in *ideal* models.

Of broader and independent interest, our study of FX requires us to consider a hybrid quantum model which only allows for *classical* online access to the encrypted data, whereas offline computation is quantum. This model is often referred to as the “Q1 model” in the cryptanalysis literature [17,5], and is weaker than the stronger, fully-quantum, “Q2 model”, which allows for quantum online access. This is necessary in view of existing attacks in the Q2 model showing that FX is no more secure than the underlying cipher [24], but *also*, Q1 is arguably more realistic and less controversial than Q2. We observe that (in contrast to the plain model) ideal-model proofs in the Q1 model can be *harder* than those in the Q2 model, as we need to explicitly account for *measuring* the online queries to obtain improved bounds. To the best of our knowledge, this work is the first to introduce techniques to handle the simultaneous interaction between classical online queries and quantum offline ideal-model queries in Q1 proofs. In prior ideal-model Q1 proofs, e.g. [21,22,15,4,18], this interaction is handled essentially for free because the effects of online and offline queries on an attacker’s advantage can largely be analyzed separately. We stress that our result on double encryption considers the full Q2 model – and interestingly, there does not seem to be any security improvement by restricting adversaries to the Q1 model. To be self-explanatory we will often refer to the Q1 and Q2 models as the partially-quantum and fully-quantum models, respectively.

The remainder of this introduction provides a detailed overview of our results for these two constructions, and of the underlying techniques and challenges.

## 1.1 The FX Construction

The FX construction was originally introduced by Kilian and Rogaway [20] as a generalization of Rivest’s DESX construction. Consider a blockcipher  $E$  which uses a key  $K \in \{0, 1\}^k$  to encrypt messages  $M \in \{0, 1\}^n$ . Then the FX construction introduces “whitening” key  $K_2 \in \{0, 1\}^n$  which is xor-ed into the input and output of the blockcipher. Formally, this construction is defined by  $\text{FX}[E](K \parallel K_2, M) = E_K(M \oplus K_2) \oplus K_2$ . (Note that the Even-Mansour blockcipher [11] may be considered to be a special case of this construction where  $k = 0$ , i.e., the blockcipher is a single permutation.) This construction has negligible efficiency overhead as compared to using  $E$  directly.

Kilian and Rogaway proved this scheme secure against classical attacks in the ideal cipher model. In particular, they established that

$$\text{Adv}_{\text{FX}}^{\text{SPRP}}(\mathcal{A}) \leq pq/2^{k+n-1}.$$

Here  $\text{Adv}_{\text{FX}}^{\text{SPRP}}$  measures the advantage of  $\mathcal{A}$  in breaking the strong pseudorandom permutation (SPRP) security of FX while making at most  $p$  queries to the ideal cipher and at most  $q$  queries to the FX construction. Compared to the  $p/2^k$  bound achieved by  $E$  alone, this is a clear improvement so FX can be considered a successful key-length extension technique against classical attackers.

Is this construction equally effective in the face of quantum attackers? The answer is unfortunately negative. Leander and May [24], inspired by a quantum attack due to Kuwakado and Morii [23] that completely breaks Even-Mansour blockcipher, gave a quantum attack against FX, which shows that the whitening keys provide essentially no additional security over that achieved by  $E$  in isolation. Bonnetain, et al. [5] further reduced the number of *online* quantum queries in the attack. Roughly speaking,  $O(n)$  quantum queries to FX construction and  $O(n2^{k/2})$  local quantum computations of the blockcipher suffice to recover the secret encryption key. Note that, however, such attacks require full quantum access to both the ideal primitive and to the instance FX that is under attack, i.e. they are attacks in the fully-quantum model. The latter is rather strong and may be considered unrealistic. While we cannot prevent a quantum attacker from locally evaluating a blockcipher in quantum superposition, honest implementations of encryption will likely continue to be classical.<sup>3</sup>

<sup>3</sup> Some may argue that maintaining purely classical states, e.g., enforcing perfect measurements is also non-trivial physically. However, we deem maintaining coherent quantum superposition significantly more challenging.

PARTIALLY-QUANTUM MODEL. Because of the realistic concern of the fully-quantum model and the attacks therein that void key extension in FX, we turn to the partially-quantum model in which the attacker makes quantum queries to ideal primitives, but only *classical* queries to the cryptographic constructions.

In this model there has been extensive quantum cryptanalysis on FX and related constructions [14,5]. The best existing attack [5] recovers the key of FX using roughly  $2^{(k+n)/3}$  classical queries to the construction and  $2^{(k+n)/3}$  quantum queries to the ideal cipher, which is likely optimal in terms of recovering a key. However, to date, despite the active development in provable quantum security, we are not aware of SPRP or just PRF security analysis, which gives stronger security guarantees. Namely, it should not just be infeasible to retrieve a key, but also to merely distinguish the system from a truly random permutation (or function). We note that in the special case where the primitives are plain-model instantiations (e.g., non-random-oracle hash functions), with a bit of care many security reductions usually carry over to the quantum setting [25]. This is because the underlying primitives are hidden from the adversary, and hence the difficulty arising from the interaction of classical and quantum queries to two correlated oracles becomes irrelevant.

Our main contribution on FX is to prove, for the first time, indistinguishability security in the partially-quantum model, in two restricted ways. Although they do not establish the complete security, our security bounds are tight in their respective settings.<sup>4</sup>

NON-ADAPTIVE SECURITY. We first consider non-adaptive security where we restrict the adversary such that its classical queries to the FX construction (but not to the underlying ideal cipher) must be specified before execution has begun. We emphasize that non-adaptive security of a blockcipher suffices to prove adaptive security for many practical uses of blockciphers such as the various randomized or stateful encryption schemes (e.g. those based on counter mode or output feedback mode) in which an attacker would have no control over the inputs to the blockcipher.

In this setting the bound we prove is of the form

$$\text{Adv}_{\text{FX}}^{\text{sprp-na}}(\mathcal{A}) \leq O\left(\sqrt{p^2q/2^{k+n}}\right).$$

Supposing  $k = n = 128$  (as with AES-128), an attacker able to make  $p \approx 2^{64}$  queries to the ideal cipher could break security of **E** in isolation. But to attack FX, such an attacker with access to  $q \approx 2^{64}$  encryptions would need to make  $p \approx 2^{96}$  queries to the ideal cipher. In fact we can see from our bound that breaking the security with constant probability would require the order of  $\Omega(2^{(k+n)/3})$  queries in total, matching the bound given in the attacks mentioned above [5]. Hence our bound is tight.

To prove this bound we apply a one-way to hiding (O2H) theorem of Ambainis, Hamburg, and Unruh [3], an improved version of the original one in [32]. This result provides a clean methodology for bounding the probability that an attacker can distinguish between two functions drawn from closely related distributions given quantum access. The non-adaptive setting allows us to apply this result by sampling the outputs of FX ahead of time and then considering the ideal world in which the ideal cipher is chosen independently of these outputs and the real world in which we very carefully reprogram this ideal cipher to be consistent with the outputs chosen for FX. These two ideal ciphers differ only in the  $O(q)$  places where we need to reprogram.

ADAPTIVE SECURITY OF FFX. As a second approach towards understanding fully adaptive security of FX, we consider a variant construction (which we call FFX for “function FX”) that replaces the random permutation with a random function. In particular, suppose **F** is a function family which uses a key  $K \in \{0, 1\}^k$  on input messages  $M \in \{0, 1\}^n$  to produce outputs  $C \in \{0, 1\}^m$ . Then we define  $\text{FFX}[\mathbf{F}](K \parallel K_2, M) = \mathbf{F}_K(M \oplus K_2)$ .<sup>5</sup> For this construction we prove a bound of the form

$$\text{Adv}_{\text{FFX}}^{\text{prf}}(\mathcal{A}) \leq O\left(\sqrt{p^2q/2^{k+n}}\right).$$

<sup>4</sup> Throughout, when mentioning tightness, we mean it with respect to the resources required to achieve advantage around one. Our bounds all have roots over them which make them weaker for lower resource regimes. Removing these is an interesting direction for future work.

<sup>5</sup> Note we have removed the external xor with  $K_2$ . In FX this xor is necessary, but in our analysis it would not provide any benefit for FFX.

in the partially-quantum random oracle model. Note that this matches the bound we obtained for the non-adaptive security of FX. Since the same key-recovery attack [5] also applies here, it follows that our bounds is tight as well. Our proof combines two techniques of analyzing a quantum random oracle, the O2H theorem above and a simulation technique by Zhandry [35]. The two techniques usually serve distinct purposes. O2H is helpful to program a random oracle, whereas Zhandry’s technique is typically convenient for (compactly) maintaining a random oracle and providing some notion of “recording” the queries. In essence, in the two function distributions of O2H for which we aim to argue indistinguishability, we apply Zhandry’s technique to simulate the functions in a compact representation. As a result, analyzing the guessing game in O2H, which implies indistinguishability, becomes intuitive and much simplified. This way of combining them could also be useful elsewhere.

To build intuition for the approach of our proof, let us first consider one way to prove the security of this construction classically. The core idea is to use lazy sampling. In the ideal world, we can independently lazily sample a random function  $F : \{0, 1\}^k \times \{0, 1\}^n \rightarrow \{0, 1\}^m$  to respond to F queries and a random function  $T : \{0, 1\}^n \rightarrow \{0, 1\}^m$  to respond to FFX queries. These lazily random functions are stored in tables.

The real world can similarly be modeled by lazily sampling  $F$  and  $T$  to respond to the separate oracles. However, these oracles need to be kept consistent. So if the adversary ever queries  $M$  to FFX and  $(K, M \oplus K_2)$  to F, then the game should copy values between the two tables such that the same value is returned by both oracles. (Here  $K$  and  $K_2$  are the keys honestly sampled by the game.) Alternatively, we can think of the return value being stored only in the  $T$  table when such queries occur (rather than being copied into both tables) as long as we remember that this has happened. When represented in this manner, the two games only differ if the adversary makes such a pair of queries, where we think of the latter one as being “bad”. Thus a simple  $O(pq/2^{k+n})$  bound on the probability of making such a query bounds the advantage of the adversary.

In our quantum security proof we wish to proceed analogously. First, we find a way to represent the responses to oracle queries with two (superpositions over) tables that are independent in the ideal world and dependent in the real world (the dependency occurs only for particular “bad” inputs). Then (using the O2H theorem of Ambainis, Hamburg, and Unruh) we can bound the distinguishing advantage by the probability of an attacker finding a “bad” input. In applying this theorem we will jointly think of the security game and its adversary  $\mathcal{A}$  as a combined adversary  $\mathcal{A}'$  making queries to an oracle which takes in both the input of  $\mathcal{A}$  and the tables being stored by the game – processing them appropriately.

The required representation of the oracles via two tables is a highly non-trivial step in the quantum setting. For starters, the no-cloning theorem prevents us from simply recording queries made by the adversary. This has been a recurring source of difficulty for numerous prior papers such as [33,30,9]. We make use of the recent elegant techniques of Zhandry [35] which established that, by changing the perspective (e.g., to the Fourier domain), a random function can be represented by a table which is initialized to all zeros and then xor-ed into with each oracle query made by the adversary. This makes it straightforward to represent the ideal world as two separate tables. To represent the real world similarly, we exploit the fact that the queries to FFX are classical. To check if an input to FFX is “bad” we simply check if the corresponding entry of the random oracle’s table is non-zero. To check if an input to the random oracle is “bad” we check if it overlaps with prior queries to FFX which we were able to record because they were classical. For a “bad” input we then share the storage of the two tables and this is the only case where the behavior of the real world differs from that of the ideal world. These “bad” inputs may of course be part of a superposition query and it is only for the bad components of the superposition that the games differ.

DIFFICULTY OF EXTENDING TO FX. It is possible that this proof could be extended to work for normal FX given an analogous way to lazily represent a random permutation. Unfortunately, no such representation is known.

Czajkowski, et al. [8] extended Zhandry’s lazy sampling technique to a more general class of random functions, but this does not include permutations because of the correlation between the different outputs of a random permutation. Chevalier, et al. [6] provided a framework for recording queries to quantum oracles. What they provide (succinctly recording queries made to an externally provided function for purposes of later responding to inverse queries) is distinct from the lazy sampling of a permutation that we require.

Rosmanis [27] introduced a new technique for analyzing random permutations in a compressed manner and applied it to the question inverting a permutation (given only forward access to it). Extending this technique to analyze other problems is an interesting open question. Additional ideas seem needed to support actions based on the oracles queries that have been performed so far. This is essential in order to extend our proof for the function variant of FX to maintain consistency for the real world in the face of “bad” queries.

Recent work of Czaikowski [7] provided an *imperfect* lazy sampling technique for permutations and used it to prove indistinguishability of SHA3. They claim that their lazy sampling strategy cannot be distinguished from a random permutation with advantage better than  $O(q^{2.5}/2^n)$ . Unfortunately, this bound is too weak to be useful to our FX proof. For example, if  $k \geq n$  we already have  $O(q^2/2^n)$  security without key-length extension. Determining if it is possible to *perfectly* lazily sample a random permutation remains an interesting direction for future work.

## 1.2 Double Encryption

The other key-extension technique we consider is double encryption. Given a blockcipher  $E : \{0, 1\}^k \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  this is defined by  $DE[E](K_1 \parallel K_2, M) = E_{K_2}(E_{K_1}(M))$ . This construction requires more computational overhead than FX because it requires two separate application of the blockcipher with different keys. Classically, this construction is not considered to be a successful key-length extension technique because the meet-in-the-middle attack [10,26] shows that it can be broken in essentially the same amount of time as  $E$  alone.

However, this does not rule out of double encryption being an effectively key-length extension attack in the quantum setting. This is because it is not clear that the Grover search algorithm [12] used to half the effective keylength of blockciphers can be composed with the meet-in-the middle attack to unify their savings. The security of double encryption in the quantum setting was previously considered by Kaplan [16]. They related the key-recovery problem in double encryption to the claw-finding problem, and gave the tight quantum query bound  $\Theta(N^{2/3})$  for solving key recovery (here  $N = 2^k$  is the length of the lists in the claw-finding problem). This indicates that in the quantum setting double encryption is in fact useful (compare to  $N^{1/2}$ ), although key-recovery security is fairly weak.

We strengthen their security result by proving the SPRP security of double encryption, further confirming the effectiveness of double encryption as a key-extension scheme against quantum attacks. Our result is proven by a reduction to list disjointness. This is a worst-case decision problem measuring how well an algorithm can distinguish between a pair of lists with zero or *exactly* one element in common, which can be viewed as a decision version of the claw-finding problem. This reduction technique was originally used by Tessaro and Thiruvengadam [31] to establish a classical time-memory trade-off for the security of double encryption. We observe that their technique works for a quantum adversary.

This reduction induces  $N = 2^k$  in the underlying list disjointness problem. We then provide a chain of results showing that the known quantum hardness of element distinctness [1,34] (deciding if a list of  $N$  elements are all distinct) can be used to imply quantum hardness of solving list disjointness. Our result (ignoring log factors) implies that a highly successful attacker must make  $\Omega(2^{2k/3})$  oracle queries which is more than the  $\Omega(2^{k/2})$  queries needed to attack  $E$  used in isolation.

Our proof starts by modifying Zhandry’s [34] proof of the hardness of the search version element distinctness (finding a collision in a list), to show that a promise version on element distinctness (promising that there is exactly one collision) is hard. Then a simple reduction (randomly splitting the element distinctness list into two lists) shows that the search version of list disjointness is hard. We provide a binary search inspired algorithm showing that the decision version of list disjointness can be used to solve the search version, implying that the decision version must be hard. During our binary search we pad the lists we are considering with random elements to ensure that our lists maintain a fixed size which is necessary for our proof to go through.

The final bound we obtain for double encryption is of the form

$$\text{Adv}_{\text{DE}}^{\text{sprp}}(\mathcal{A}) \leq O\left(\sqrt[6]{(q \cdot k \lg k)^3 / 2^{2k}}\right).$$

The sixth root arises in this bound from the final step in our chain of results analyzing list disjointness. The binary search algorithm requires its underlying decision list disjointness algorithm to have relatively high advantage. To obtain this from a give algorithm with advantage  $\delta$  we need to amplify its advantage by running in on the order of  $1/\delta^2$  times. It is through making the number of queries depend on the square of  $\delta$  that our root arises.

### 1.3 Overview

In Section 2, we introduce preliminaries such as notation, basic cryptographic definitions, and some background on quantum computation that we will use throughout the paper. Following this, in Section 3 we consider the security of FX in the partially quantum setting. Non-adaptive SPRP security of FX is proven in Section 3.1 and adaptive PRF security of FFX is proven in Section 3.2. We conclude with Section 4 in which we prove the SPRP security of double encryption against fully quantum adaptive attacks.

## 2 Preliminaries

For  $n, m \in \mathbb{N}$ , we let  $[n] = \{1, \dots, n\}$  and  $[n..m] = \{n, n+1, \dots, m\}$ . The set of length  $n$  bit strings is denoted  $\{0, 1\}^n$ . We use  $\parallel$  to denote string concatenation. We let  $\text{Inj}(n, m)$  denote the set of injections  $f : [n] \rightarrow [m]$ .

We let  $y \leftarrow^s \mathcal{A}[O_1, \dots](x_1, \dots)$  denote the (randomized) execution of algorithm  $\mathcal{A}$  with input  $x_1, \dots$  and oracle access to  $O_1, \dots$  which produces output  $y$ . For different  $\mathcal{A}$  we will specify whether it can access its oracles in quantum superposition or only classically. If  $\mathcal{S}$  is a set, then  $y \leftarrow^s \mathcal{S}$  denotes randomly sampling  $y$  from  $\mathcal{S}$ .

We express security notions via pseudocode games. See Fig. 1 for some example games. In the definition of games, oracles will sometimes be specified by pseudocode with the following form.

$$\begin{array}{l} \text{Oracle } O(X_1, \dots : Z_1, \dots) \\ // \text{Code defining } X'_1, \dots \text{ and } Z'_1, \dots \\ \text{Return } (X'_1, \dots : Z'_1, \dots) \end{array}$$

This notation indicates that  $X_1, \dots$  are variables controlled by the adversary prior to the oracle query and  $Z_1, \dots$  are variables controlled by the game itself which the adversary cannot access. At the end of the execution of the oracle, these variables are overwritten with the values indicated in the return statement. Looking ahead, we will be focusing on quantum computation so this notation will be useful to make it explicit that  $O$  can be interpreted as a unitary acting on the registers  $X_1, \dots$  and  $Z_1, \dots$  (because  $O$  will be an efficiently computable and invertible permutation over these values). If  $\mathbf{H}$  is a function stored by the game, then oracle access to  $\mathbf{H}$  represents access to the oracle that on input  $(X, Y : \mathbf{H})$  returns  $(X, \mathbf{H}(X) \oplus Y : \mathbf{H})$ .

We define games as outputting boolean values and let  $\text{Pr}[\mathbf{G}]$  denote the probability that game  $\mathbf{G}$  returns **true**. When not otherwise indicated, variables are implicitly initialized to store all 0's.

If  $\mathcal{A}$  is an adversary expecting access to multiple oracles we say that it is *order consistent* if the order it will alternate between queries to these different oracles is a priori fixed before execution. Note that order consistency is immediate if, e.g.,  $\mathcal{A}$  is represented by a circuit where each oracle is modeled by a separate oracle gate, but is not immediate for other possible representations of an adversary.

**IDEAL MODELS.** In this work we will work in ideal models – specifically, the random oracle model or the ideal cipher model. Fix  $k, n, m \in \mathbb{N}$  (throughout this paper we will treat these parameters as having been fixed already). We let  $\text{Fcs}(k, n, m)$  be the set of all functions  $\mathbf{H} : \{0, 1\}^k \times \{0, 1\}^n \rightarrow \{0, 1\}^m$  and  $\text{lcs}(k, n) \subset \text{Fcs}(k, n, n)$  be the set of all functions  $\mathbf{E} : \{0, 1\}^k \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  such that  $\mathbf{E}(K, \cdot)$  is a permutation on  $\{0, 1\}^n$ . When convenient, we will write  $\mathbf{H}_K(x)$  in place of  $\mathbf{H}(K, x)$  for  $\mathbf{H} \in \text{Fcs}(n, m)$ . Similarly, we will write  $\mathbf{E}_K(x)$  for  $\mathbf{E}(K, x)$  and  $\mathbf{E}_K^{-1}(\cdot)$  for the inverse of  $\mathbf{E}_K(\cdot)$  when  $\mathbf{E} \in \text{lcs}(k, n)$ . When  $K = \varepsilon$  we omit the subscript to  $\mathbf{H}$  or  $\mathbf{E}$ .

In the random oracle model, honest algorithms and the adversary are given oracle access to a randomly chosen  $\mathbf{H} \in \text{Fcs}(k, n, m)$ . In the ideal cipher model, they are given oracle access to  $\mathbf{E}$  and  $\mathbf{E}^{-1}$  for  $\mathbf{E}$  chosen at random from  $\text{lcs}(k, n)$ . We refer to queries to these oracles as *primitive queries* and queries to all other oracles as *construction queries*.

Game $G_{F,b}^{\text{prf}}(\mathcal{A})$	$\text{Ev}(X, Y : \mathbf{H}, K, F)$
$\mathbf{H} \leftarrow_{\$} \text{Fcs}(k, n, m)$	$Y_1 \leftarrow \mathbf{F}[\mathbf{H}](K, X)$
$K \leftarrow_{\$} \{0, 1\}^{\text{F.kl}}$	$Y_0 \leftarrow F(X)$
$F \leftarrow_{\$} \text{Fcs}(0, \text{F.il}, \text{F.ol})$	Return $(X, Y_b \oplus Y : \mathbf{H}, K, F)$
$b' \leftarrow_{\$} \mathcal{A}[\text{Ev}, \mathbf{H}]$	
Return $b' = 1$	

  

Game $G_{E,b}^{\text{sprp}}(\mathcal{A})$	$\text{Ev}(X, Y : \mathbf{E}, K, P)$
$\mathbf{E} \leftarrow_{\$} \text{lcs}(k, n)$	$Y_1 \leftarrow \mathbf{E}[\mathbf{E}](K, X)$
$K \leftarrow_{\$} \{0, 1\}^{\text{E.kl}}$	$Y_0 \leftarrow P(X)$
$P \leftarrow_{\$} \text{lcs}(0, \text{E.bl})$	Return $(X, Y_b \oplus Y : \mathbf{E}, K, P)$
$b' \leftarrow_{\$} \mathcal{A}[\text{Ev}, \text{Inv}, \mathbf{E}, \mathbf{E}^{-1}]$	$\text{Inv}(X, Y : \mathbf{E}, K, P)$
Return $b' = 1$	$Y_1 \leftarrow \mathbf{E}^{-1}[\mathbf{E}](K, X)$
	$Y_0 \leftarrow P^{-1}(X)$
	Return $(X, Y_b \oplus Y : \mathbf{E}, K, P)$

**Fig. 1.** Security games measuring PRF security of a family of functions  $F$  and SPRP security of a blockcipher  $E$ .

FUNCTION FAMILY AND PSEUDORANDOMNESS. A function family  $F$  is an efficiently computable element of  $\text{Fcs}(\text{F.kl}, \text{F.il}, \text{F.ol})$ . If, furthermore,  $F \in \text{lcs}(\text{F.kl}, \text{F.il})$  and  $F^{-1}$  is efficiently computable then we say  $F$  is a blockcipher and let  $\text{F.bl} = \text{F.il}$ .

If  $F$  is a function family (constructed using oracle access to a function  $\mathbf{H} \in \text{Fcs}(k, n, m)$ ), then its security (in the random oracle model) as a *pseudorandom function* (PRF) is measured by the game  $G^{\text{prf}}$  shown in Fig. 1. In it, the adversary  $\mathcal{A}$  attempts to distinguish between a *real world* ( $b = 1$ ) where it is given oracle access to  $F$  with a random key  $K$  and an *ideal world* ( $b = 0$ ) where it is given access to a random function. We define the advantage function  $\text{Adv}_F^{\text{prf}}(\mathcal{A}) = \Pr[G_{F,1}^{\text{prf}}(\mathcal{A})] - \Pr[G_{F,0}^{\text{prf}}(\mathcal{A})]$ .

If  $E$  is a blockcipher (constructed using oracle access to a function  $\mathbf{E} \in \text{lcs}(k, n)$  and its inverse), then its security (in the ideal cipher model) as a *strong pseudorandom permutation* (SPRP) is measured by the game  $G^{\text{sprp}}$  shown in Fig. 1. In it, the adversary  $\mathcal{A}$  attempts to distinguish between a real world ( $b = 1$ ) where it is given oracle access to  $E, E^{-1}$  with a random key  $K$  and an ideal world ( $b = 0$ ) where it is given access to a random permutation. We define the advantage function  $\text{Adv}_E^{\text{sprp}}(\mathcal{A}) = \Pr[G_{E,1}^{\text{sprp}}(\mathcal{A})] - \Pr[G_{E,0}^{\text{sprp}}(\mathcal{A})]$ .

In some examples, we will restrict attention to *non-adaptive* SPRP security. In such cases our attention is restricted to attackers whose queries to  $\text{Ev}$  and  $\text{Inv}$  when relevant are a priori fixed before execution. That is,  $\mathcal{A}$  is a non-adaptive attacker which makes at most  $q$  classical, non-adaptive queries to  $\text{Ev}, \text{Inv}$  if there exists  $M_1, \dots, M_{q'}, Y_{q'+1}, \dots, Y_q \in \{0, 1\}^n$  such that  $\mathcal{A}$  only ever queries  $\text{Ev}$  on  $M_i$  for  $1 \leq i \leq q'$  and  $\text{Inv}$  on  $Y_i$  for  $q' + 1 \leq i \leq q$ . Then we write  $\text{Adv}_E^{\text{sprp-na}}(\mathcal{A})$  in place of  $\text{Adv}_E^{\text{sprp}}(\mathcal{A})$ .

## 2.1 Quantum Background

We assume the reader has basic familiarity with quantum computation. Quantum computation proceeds by performing unitary operations on registers which each contain a fixed number of qubits. We sometimes use  $\circ$  to denote composition of unitaries. Additionally, qubits may be measured in the computational basis. We will typically use the principle of deferred measurements to without loss of generality think of such measurements as being deferred until the end of computation.

The Hadamard transform  $\mathcal{H}$  acts on a bitstring  $x \in \{0, 1\}^n$  (for some  $n \in \mathbb{N}$ ) via  $\mathcal{H}|x\rangle = 1/\sqrt{2^n} \cdot \sum_{x'} (-1)^{x \cdot x'} |x'\rangle$ . Here  $\cdot$  denotes inner product modulo 2 and the summation is over  $x' \in \{0, 1\}^n$ . The Hadamard transform is its own inverse. We sometimes use the notation  $\mathcal{H}^{X_1, X_2, \dots}$  to denote the Hadamard transform applied to registers  $X_1, X_2, \dots$ .

Game $G_{\mathbf{D},b}^{\text{dist}}(\mathcal{A})$	Game $G_{\mathbf{D}}^{\text{guess}}(\mathcal{A})$
$(S, S', P_0, P'_0, P_1, P'_1, z) \leftarrow \mathbf{D}$	$(S, S', P_0, P'_0, P_1, P'_1, z) \leftarrow \mathbf{D}$
$b' \leftarrow \mathcal{A}[P_b, P'_b](z)$	$i \leftarrow \{1, \dots, q\}$
Return $b' = 1$	Run $\mathcal{A}[P_0, P'_0]$ until its $i$ -th query
	Measure the input $x$ to this query
	If the query is to $P_0$ then
	Return $x \in S$
	Else (the query is to $P'_0$ )
	Return $x \in S'$

**Fig. 2.** Games used for O2H Theorem 1.

We make use of the fact that if  $P$  is a permutation for which both  $P$  and  $P^{-1}$  can be efficiently implemented classically, then there is a comparable efficient quantumly computable unitary  $U_P$  which maps according to  $U_P |x\rangle = |P(x)\rangle$  for  $x \in \{0, 1\}^n$ . For simplicity, we often write  $P$  in place of  $U_P$ . If  $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$  is a function, we define the permutation  $f[\oplus](x, y) = (x, f(x) \oplus y)$ .

ONE-WAY TO HIDING. We will make use of (a slight variant of) a one-way to hiding (O2H) theorem of Ambainis, Hamburg, and Unruh [3]. The theorem will consider an adversary given oracles access either to permutations  $(P_0, P'_0)$  or permutations  $(P_1, P'_1)$ . It relates the advantage of the adversary in distinguishing between these two cases to the probability that the adversary can be used to find one of those points on which  $P_0$  differs from  $P_1$  or  $P'_0$  differs from  $P'_1$ . The result considers a distribution  $\mathbf{D}$  over  $(S, S', P_0, P'_0, P_1, P'_1, z)$  where  $S, S'$  are sets,  $P_0, P_1$  are permutations on the same domain,  $P'_0, P'_1$  are permutations on the same domain, and  $z \in \{0, 1\}^*$  is some auxiliary information. Such a  $\mathbf{D}$  is *valid* if  $P_0(x) = P_1(x)$  for all  $x \notin S$  and  $P'_0(x) = P'_1(x)$  for all  $x \notin S'$ . Now consider the game  $G_{\mathbf{D}}^{\text{D}}$  shown in Fig. 2. In it, an adversary  $\mathcal{A}$  is given  $z$  and tries to determine which of the oracle pairs it has access to. We define  $\text{Adv}_{\mathbf{D}}^{\text{dist}}(\mathcal{A}) = \Pr[G_{\mathbf{D},1}^{\text{dist}}(\mathcal{A})] - \Pr[G_{\mathbf{D},0}^{\text{dist}}(\mathcal{A})]$ .

The game  $G_{\mathbf{D}}^{\text{guess}}(\mathcal{A})$  in the same figure measures the ability of  $\mathcal{A}$  to query its oracles on inputs at which  $P_0$  and  $P_1$  (or  $P'_0$  and  $P'_1$ ) differ. It assumes that the adversary makes at most  $q$  oracle queries. The adversary is halted in its execution on making a random one of these queries and the input to this query is measured. If the input falls in the appropriate set  $S$  or  $S'$ , then the game returns **true**. Thus we can roughly think of this as a game in which  $\mathcal{A}$  is trying to guess a point on which the two oracles differ. We define  $\text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A}) = \Pr[G_{\mathbf{D}}^{\text{guess}}(\mathcal{A})]$ , which leads to a bound on  $\text{Adv}_{\mathbf{D}}^{\text{dist}}(\mathcal{A})$ .

**Theorem 1 ([3], Thm.3).** *Let  $\mathbf{D}$  be a valid distribution and  $\mathcal{A}$  be an adversary making at most  $q$  oracle queries. Then  $\text{Adv}_{\mathbf{D}}^{\text{dist}}(\mathcal{A}) \leq 2q\sqrt{\text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A})}$ .*

Our statement of the theorem differs from the result as given in [3] in that we consider arbitrary permutations, rather than permutations of the form  $f[\oplus]$  for some function  $f$ , and we provide the attacker with access to two oracles rather than one.<sup>6</sup> These are simply notational conveniences to match how we will be applying the theorem. The proof given in [3] suffices to establish this variant without requiring any meaningful modifications.

The most natural applications of this theorem would apply it to distributions  $\mathbf{D}$  for which the guessing advantage  $\text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A})$  is small for *any* efficient adversary  $\mathcal{A}$ . This will indeed be the case for our use of it in our Theorem 2. However, note that it can also be applied more broadly with a distribution  $\mathbf{D}$  where it is not necessarily difficult to guess inputs on which the oracles differ. We will do so at the end of our proof of Theorem 3. Here we will use a *deterministic*  $\mathbf{D}$  so, in particular, the sets  $S$  and  $S'$  are a priori fixed and not hard to query. The trick we will use to profitably apply the O2H result is to exploit knowledge of the

<sup>6</sup> Their result additionally allows the adversary to make oracle queries in parallel and bounds its advantage in terms of the “depth” of its oracle queries rather than the total number of queries. We omit this for simplicity.

particular form that  $\mathcal{A}$  will take (it will be a reduction adversary internally simulating the view of another adversary) to provide a useful bound on its guessing advantage  $\text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A})$ .

### 3 The FX Construction

The FX construction (originally introduced by Kilian and Rogaway [20] as a generalization of Rivest’s DESX construction) is a keylength extension for blockciphers. In this construction, an additional key is used which is xor-ed with input and the output of the blockcipher.<sup>7</sup> Formally, given a blockcipher  $\mathbf{E} \in \text{lcs}(\mathbf{E}.\text{kl}, \mathbf{E}.\text{bl})$ , the blockcipher  $\text{FX}[\mathbf{E}]$  is defined by  $\text{FX}[\mathbf{E}](K_1 \parallel K_2, x) = \mathbf{E}_{K_1}(x \oplus K_2) \oplus K_2$ . Here  $|K_1| = \mathbf{E}.\text{kl}$  and  $|K_2| = \mathbf{E}.\text{bl}$  so  $\text{FX}[\mathbf{E}].\text{kl} = \mathbf{E}.\text{kl} + \mathbf{E}.\text{bl}$  and  $\text{FX}[\mathbf{E}].\text{bl} = \mathbf{E}.\text{bl}$ . Its inverse can similarly be computed as  $\text{FX}[\mathbf{E}]^{-1}(K_1 \parallel K_2, x) = \mathbf{E}_{K_1}^{-1}(x \oplus K_2) \oplus K_2$ . Let  $k = \mathbf{E}.\text{kl}$  and  $n = \mathbf{E}.\text{bl}$ .

Kilian and Rogaway [19] analyzed the PRP security of FX against classical attacks, showing that  $\text{Adv}_{\text{FX}}^{\text{sprp}}(\mathcal{A}) \leq 2pq/2^{k+n}$  where  $q$  is the number of  $\text{EV}, \text{INV}$  queries and  $p$  is the number of  $\mathbf{E}, \mathbf{E}^{-1}$  queries made by  $\mathcal{A}$  (with  $\mathbf{E}$  modeled as an ideal cipher). In [24], Leander and May showed a quantum attack against the FX construction – establishing that the added whitening keys did not provide additionally security. This attack uses a clever combination of the quantum algorithms of Grover [12] and Simon [29]. It was inspired by an attack by Kuwakado and Morii [23] showing that the Even-Mansour blockcipher [11] provides no quantum security. Thus, it seems that  $\text{FX}[\mathbf{E}]$  does not provide meaningfully more security than  $\mathbf{E}$  against quantum attackers.

However, the attack of Leander and May requires quantum access to both the FX construction and the underlying blockcipher  $\mathbf{E}$ . This raises the question of whether the FX is actually an effective key-length extension technique in the partially-quantum setting where the adversary performs only classical queries to the construction oracles. In this section, we approach this question from two directions. First, in Section 3.1 we apply Theorem 1 with a careful representation of the real and ideal worlds to show that FX does indeed achieve improved security against non-adaptive attacks.

Analyzing the full adaptive security of FX against classical construction queries seems beyond the capabilities of current proof techniques. Accordingly, in Section 3.2, we consider a variant of FX in which a random oracle is used in place of the ideal cipher and prove its quantum PRF security. Here we apply a new reduction technique (built on the “sparse” quantum representation of a random function introduced by Zhandry [35] and Theorem 1, the O2H theorem from Ambainis, Hamburg, and Unruh [3]) to prove that this serves as an effective key-length extension technique in our setting. It seems likely that our technique could be extended to the normal FX construction, should an appropriate sparse quantum representation of random permutations be discovered.

#### 3.1 Security of FX Against Non-Adaptive Attacks

The following theorem bounds the security of the FX construction against non-adaptive attacks (in which the non-adaptive queries are all classical). This result is proven via a careful use of Theorem 1 in which the distribution  $\mathbf{D}$  is defined in terms of the non-adaptive queries that the adversary will make and defined so as to perfectly match the two worlds that  $\mathcal{A}$  is attempting to distinguish between.

**Theorem 2.** *Let  $\mathcal{A}$  be a quantum adversary which makes at most  $q$  classical, non-adaptive queries to  $\text{EV}, \text{INV}$  and consider  $\text{FX}[\cdot]$  with the underlying blockcipher modeled by an ideal cipher drawn from  $\text{lcs}(k, n)$ . Then*

$$\text{Adv}_{\text{FX}}^{\text{sprp-na}}(\mathcal{A}) \leq \sqrt{8p^2q/2^{k+n}},$$

where  $p$  is the number of quantum oracle queries that  $\mathcal{A}$  makes to the ideal cipher.

<sup>7</sup> Technically, the original definition of FX [20] uses distinct keys for xor-ing with the input and the output, but this would not provide any benefit in our concrete security analysis so we focus on the simplified construction.

```

Distribution D
// Step 1: Sample responses to construction queries
For  $i = 1, \dots, q'$  do
   $Y_i \leftarrow \{0, 1\}^n \setminus \{Y_1, \dots, Y_{i-1}\}$ 
   $T[M_i] \leftarrow Y_i; T^{-1}[Y_i] \leftarrow M_i$ 
For  $i = q' + 1, \dots, q$  do
  If  $T^{-1}[Y_i] \neq \perp$  then  $M_i \leftarrow T^{-1}[Y_i]$ 
  Else  $M_i \leftarrow \{0, 1\}^n \setminus \{M_1, \dots, M_{i-1}\}$ 
   $T[M_i] \leftarrow Y_i; T^{-1}[Y_i] \leftarrow M_i$ 
 $z \leftarrow (T, T^{-1})$ 
// Step 2: Sample  $f_0$  as independent ideal cipher
 $f_0 \leftarrow \text{lcs}(k, n)$ 
// Step 3: Reprogram  $f_1$  for consistency with construction queries
 $K_1 \leftarrow \{0, 1\}^k; K_2 \leftarrow \{0, 1\}^n$ 
 $\mathcal{I} \leftarrow \{M_i \oplus K_2 : 1 \leq i \leq q\}; \mathcal{O} \leftarrow \{Y_i \oplus K_2 : 1 \leq i \leq q\}$ 
 $\mathcal{I}' \leftarrow \{f_0^{-1}(K_1, y) : y \in \mathcal{O}\}; \mathcal{O}' \leftarrow \{f_0(K_1, x) : x \in \mathcal{I}\}$ 
 $S = \{(K_1, x) : x \in \mathcal{I} \cup \mathcal{I}'\}; S' = \{(K_1, y) : y \in \mathcal{O} \cup \mathcal{O}'\}$ 
For  $(K, x) \notin S$  do
   $f_1(K, x) \leftarrow f_0(K, x)$ 
For  $i = 1, \dots, q$  do
   $f_1(K_1, M_i \oplus K_2) \leftarrow Y_i \oplus K_2$ 
For  $x \in \mathcal{I}' \setminus \mathcal{I}$  do
   $f_1(K_1, x) \leftarrow \mathcal{O}' \setminus \{f_1(K_1, x) : x \in \mathcal{I} \cup \mathcal{I}', f_1(K_1, x) \neq \perp\}$ 
Return  $(S, S', f_0[\oplus], f_0^{-1}[\oplus], f_1[\oplus], f_1^{-1}[\oplus], z)$ 

```

**Fig. 3.** Distribution of oracles used in proof of Theorem 2.

*Proof.* We will use Theorem 1 to prove this result, so first we define a distribution **D**. Suppose that  $M_1, \dots, M_{q'} \in \{0, 1\}^n$  are the distinct queries  $\mathcal{A}$  will make to  $\text{Ev}$  and  $Y_{q'+1}, \dots, Y_q \in \{0, 1\}^n$  are the distinct queries that  $\mathcal{A}$  will make to  $\text{Inv}$ . The order in which these queries will be made does not matter. Then we define **D** as shown in Fig. 3. This distribution is valid because  $G_1$  is reprogrammed to differ from  $G_0$  by making inputs in  $S$  map to different values in  $S'$ .

We will show that the oracles output by this distribution (described in words momentarily) can be used to perfectly simulate the views expected by  $\mathcal{A}$ . In particular, let  $\mathcal{A}'$  be an adversary (for  $\mathbb{G}_{\mathbf{D}}^{\text{dist}}$ ) which runs  $\mathcal{A}$ , responding to  $\text{Ev}(M_i)$  queries with  $T[M_i]$ , responding to  $\text{Inv}(Y_i)$  queries with  $T^{-1}[Y_i]$ , and simulating  $\mathbf{E}, \mathbf{E}^{-1}$  with its own oracles  $f_b[\oplus], f_b^{-1}[\oplus]$ . When  $\mathcal{A}$  halts with output  $b$ , this adversary halts with the same output. We claim that (i)  $\Pr[\mathbb{G}_{\mathbf{F},1}^{\text{sprp}}(\mathcal{A})] = \Pr[\mathbb{G}_{\mathbf{D},1}^{\text{dist}}(\mathcal{A}')]$  and (ii)  $\Pr[\mathbb{G}_{\mathbf{F},0}^{\text{sprp}}(\mathcal{A})] = \Pr[\mathbb{G}_{\mathbf{D},0}^{\text{dist}}(\mathcal{A}')]$ . This gives  $\text{Adv}_{\mathbf{F}}^{\text{sprp}}(\mathcal{A}) = \text{Adv}_{\mathbf{D}}^{\text{dist}}(\mathcal{A}')$ .

Claim (ii) follows by noting that the view of  $\mathcal{A}$  is identical when run by  $\mathbb{G}_0^{\text{sprp}}$  or by  $\mathcal{A}'$  in  $\mathbb{G}_{\mathbf{D},0}^{\text{dist}}$ . In  $\mathbb{G}_0^{\text{sprp}}$ , its construction queries are answered with the random permutation  $F$ . When it is run by  $\mathcal{A}'$  in  $\mathbb{G}_{\mathbf{D},0}^{\text{dist}}$ , these queries are answered with the tables  $T$  and  $T^{-1}$  which can be viewed as having just lazily sampled enough of a random permutation to respond to the given queries. In both cases, its primitive oracle is an independently chosen ideal cipher.

Claim (i), follows by noting that the view of  $\mathcal{A}$  is identical when run by  $\mathbb{G}_1^{\text{sprp}}$  or by  $\mathcal{A}'$  in  $\mathbb{G}_{\mathbf{D},1}^{\text{dist}}$ . In  $\mathbb{G}_1^{\text{sprp}}$ , construction queries are answered by the FX construction using the ideal cipher and keys  $K_1, K_2$ . In the distribution, we first sample the responses to the construction queries and then construct the ideal cipher  $f_1$  by picking  $K_1$  and  $K_2$  and setting  $f_1$  to equal  $f_0$  except for places where we reprogram it to be consistent with these construction queries. The construction will map  $M_i$  to  $Y_i$  for each  $i$ , which means that the condition  $f_1(K_1, M_i \oplus K_2) = Y_i \oplus K_2$  should hold. These inputs and outputs are stored in the sets  $\mathcal{I}$  and  $\mathcal{O}$ . The sets  $\mathcal{I}'$  and  $\mathcal{O}'$  store the inputs mapping to  $\mathcal{O}$  and outputs mapped to  $\mathcal{I}$  by  $f_0(K_1, \cdot)$ , respectively. Thus while

making the above condition hold, we additionally reprogram  $f_1$  so that elements of  $\mathcal{I}' \setminus \mathcal{I}$  map to (random, non-repeating) elements of  $\mathcal{O}' \setminus \mathcal{O}$ .

In particular, the uniformity of  $f_0$  ensures that the map induced by  $f_1(K_1, \cdot)$  between  $\{0, 1\}^n \setminus (\mathcal{I} \cup \mathcal{I}')$  and  $\{0, 1\}^n \setminus (\mathcal{O} \cup \mathcal{O}')$  is a random bijection. The last for loop samples a random bijection between  $\mathcal{I}' \setminus \mathcal{I}$  and  $\mathcal{O}' \setminus \mathcal{O}$ . Because there are no biases in which values fall into these two cases among those of  $\{0, 1\}^n \setminus \mathcal{I}$  and  $\{0, 1\}^n \setminus \mathcal{O}$ , this means the map between these two sets is a uniform bijection as desired. A more detailed probability analysis of Claim (i) is given in Appendix A.

Applying the bound on  $\text{Adv}_{\mathbf{D}}^{\text{dist}}(\mathcal{A}')$  from Theorem 1 gives us

$$\text{Adv}_{\mathbf{F}}^{\text{sprp}}(\mathcal{A}) \leq 2p\sqrt{\Pr[\mathbf{G}_{\mathbf{D}}^{\text{guess}}(\mathcal{A}')]}$$

so we complete the proof by bounding this probability. Let  $\text{fw}$  denote the event that the  $i$ -th query of  $\mathcal{A}'$  in  $\mathbf{H}_0^{\mathbf{D}}$  is to  $f_0$  and let  $(K, x)$  denote the measured value of this query so that

$$\Pr[\mathbf{H}_{\mathbf{D}}^{\text{guess}}(\mathcal{A}')] = \Pr[\text{fw}] \cdot \Pr[(K, x) \in S \mid \text{fw}] + \Pr[\neg \text{fw}] \cdot \Pr[(K, x) \in S' \mid \neg \text{fw}].$$

A union bound over the different elements of  $S$  gives

$$\begin{aligned} \Pr[(K, x) \in S \mid \text{fw}] &\leq \sum_{j=1}^q \Pr[K = K_1 \wedge x \oplus M_j = K_2 \mid \text{fw}] \\ &\quad + \sum_{j=1}^q \Pr[K = K_1 \wedge G_0(K_1, x) \oplus Y_j = K_2 \mid \text{fw}]. \end{aligned}$$

However, note that the view of  $\mathcal{A}$  in  $\mathbf{H}_0^{\mathbf{D}}$  is independent of  $K_1$  and  $K_2$  so we get that

$$\Pr[(K, x) \in S \mid \text{fw}] \leq 2q/2^{k+n}.$$

Applying analogous analysis to the  $\neg \text{fw}$  case gives

$$\Pr[(K, x) \in S' \mid \neg \text{fw}] \leq 2q/2^{k+n}$$

and hence  $\Pr[\mathbf{H}_0^{\mathbf{D}}(\mathcal{A}')] \leq 2q/2^{k+n}$ . Plugging this into our earlier inequality gives the stated bound.  $\square$

### 3.2 Adaptive Security of FFX

In this section we will prove the security of FFX (a variant of FX using a random oracle in place of the ideal cipher) against quantum adversaries making strictly classical queries to the construction.

Formally, given a function family  $\mathbf{F} \in \text{Fcs}(\mathbf{F.kl}, \mathbf{F.il}, \mathbf{F.ol})$ , we model the FFX construction by the function family  $\text{FFX}[\mathbf{F}]$  by  $\text{FFX}[\mathbf{F}](K_1 \parallel K_2, x) = \mathbf{F}_{K_1}(x \oplus K_2)$ .<sup>8</sup> Here  $|K_1| = \mathbf{F.kl}$  and  $|K_2| = \mathbf{F.il}$  so  $\text{FFX}[\mathbf{F}].\text{kl} = \mathbf{F.kl} + \mathbf{F.il}$ ,  $\text{FFX}[\mathbf{F}].\text{il} = \mathbf{F.il}$ , and  $\text{FFX}[\mathbf{F}].\text{ol} = \mathbf{F.ol}$ . Let  $k = \mathbf{F.kl}$ ,  $n = \mathbf{F.il}$ , and  $m = \mathbf{F.ol}$ .

**Theorem 3.** *Let  $\mathcal{A}$  be an order consistent quantum adversary which makes classical queries to  $\text{Ev}$  and consider  $\text{FFX}[\cdot]$  with the underlying function family modeled by a random oracle drawn from  $\text{Fcs}(k, n, m)$ . Then*

$$\text{Adv}_{\text{FFX}}^{\text{prf}}(\mathcal{A}) \leq \sqrt{\frac{8(p+q)pq}{2^{k+n}}},$$

where  $p$  is the number of quantum oracle queries that  $\mathcal{A}$  makes to the random oracle and  $q$  is the number of queries it makes to  $\text{Ev}$ .

<sup>8</sup> The outer xor by  $K_2$  used in FX is omitted because it is unnecessary for our analysis.

We can reasonably assume that  $p > q$  so the dominant behavior of the above expression is  $O\left(\sqrt{p^2q/2^{k+n}}\right)$ .

The proof of this result proceeds via a sequence of hybrids which gradually transition from the real world of  $G_{\text{FFX}}^{\text{prf}}$  to the ideal world. Crucial to this sequence of hybrids are the technique of Zhandry [35] which, by viewing a space under dual bases, allows one to simulate a random function using a sparse representation table and to “record” the queries to the function. For the ideal world, we can represent the random oracle and the random function underlying Ev independently using such sparse representation tables. With some careful modification, we are also able to represent the real world’s random oracle using a similar pair of sparse representation tables as if it were two separate functions. However, in this case, the tables will be slightly non-independent in that if the adversary queries Ev on an input  $x$  and the random oracle on  $(K_1, x \oplus K_2)$  then the results of the latter query is stored in the Ev table, rather than the random oracle table. Beyond this minor consistency check (which we are only able to implement because the queries to Ev are classical and so can be stored by simulation), the corresponding games are identical. Having done this rewriting, we can carefully apply Theorem 1 to bound the ability of an adversary to distinguish between the two worlds by its ability to trigger this consistency check.

As mentioned in Section 2.1, our application of Theorem 1 here is somewhat atypical. Our distribution over functions  $\mathbf{D}$  will be deterministic, but we are able to still extract a meaningful bound from this by taking advantage of our knowledge of the particular behavior of the adversary we apply Theorem 1 with.

*Proof.* In this proof we will consider a sequence of hybrid games  $H_0$  through  $H_9$ . Of these games we will establish the following claims.

1.  $\Pr\left[G_{\text{FFX},0}^{\text{prf}}(\mathcal{A})\right] = \Pr[H_0] = \Pr[H_1] = \Pr[H_2] = \Pr[H_3]$
2.  $\Pr\left[G_{\text{FFX},1}^{\text{prf}}(\mathcal{A})\right] = \Pr[H_9] = \Pr[H_8] = \Pr[H_7] = \Pr[H_6] = \Pr[H_5] = \Pr[H_4]$
3.  $\Pr[H_4] - \Pr[H_3] \leq \sqrt{8(p+q)pq/2^{\text{F.kl}+\text{F.il}}}$

Combining these claims gives the desired result.

In formally defining our hybrids we write the computation to be performed using the following quantum registers.

- $W$ : The workspace of  $\mathcal{A}$ . The adversary’s final output is written into  $W[1]$ .
- $K$ : The  $k$ -qubit register (representing the function *key/index*)  $\mathcal{A}$  uses when making oracle queries to the random oracle.
- $X$ : The  $n$ -qubit register (representing function inputs) used when making oracle queries to the random oracle or Ev.
- $Y$ : The  $m$ -qubit register (representing function outputs) into which the results of oracle queries are written.
- $H$ : The  $2^{k+n} \cdot m$ -qubit register which stores the function defining the random oracle (initially via its truth table).
- $F$ : The  $2^n \cdot m$ -qubit register which stores the function defining by Ev.
- $K_1$ : The  $k$ -qubit register which stores the first key of the construction.
- $K_2$ : The  $n$ -qubit register which stores the second key of the construction.
- $I$ : The  $\lceil \log p \rceil$ -qubit register which tracks how many Ev queries  $\mathcal{A}$  has made.
- $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_p)$ : The  $p$   $n$ -qubit registers used to store the classical queries that  $\mathcal{A}$  makes to Ev.

We start by changing our perspective. A quantum algorithm that makes classical queries to Ev can be modeled by thinking of a quantum algorithm that measures its  $X$  register immediately before the query. (Because the behavior of Ev is completely classical at this point, we do not need to measure the  $Y$  register as well.) Measuring the register  $X$  is indistinguishable from using a CNOT operation to copy it into a separate register (i.e. xor-ing  $X$  into the previously empty register  $\mathbf{X}_I$  that will never again be modified). By incorporating this CNOT operation into the behavior of our hybrid game, we treat  $\mathcal{A}$  as an attacker that makes fully quantum queries to its oracles in the hybrid game. We think of  $\mathcal{A}$  as deferring all of measurements until the end of its computation. Because all that matters is its final output  $W[1]$  we can have the game

<p>Games <math>H_0, H_1</math></p> <p><math>\mathbf{H} \leftarrow \text{Fcs}(k, n, m)</math></p> <p><math>\mathbf{F} \leftarrow \text{Fcs}(0, n, m)</math></p> <p><math> H, F\rangle \leftarrow  \mathbf{H}, \mathbf{F}\rangle</math></p> <p><math> H, F\rangle \leftarrow \mathcal{H}  0^{2^{k+n} \cdot m}, 0^{2^n \cdot m}\rangle</math></p> <p>Run <math>\mathcal{A}[\text{Ev}, \text{Ro}]</math></p> <p>Measure <math>W[1], H, F</math></p> <p>Return <math>W[1] = 1</math></p>	<p><math>\text{Ev}(X, Y : I, \mathbf{X}, F)</math></p> <p><math>I \leftarrow I + 1 \bmod 2^{ I }</math></p> <p><math>\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X</math></p> <p><math>Y \leftarrow F(X) \oplus Y</math></p> <p>Return <math>(X, Y : I, \mathbf{X}, F)</math></p> <p><math>\text{Ro}(K, X, Y : H)</math></p> <p><math>Y \leftarrow H_K(X) \oplus Y</math></p> <p>Return <math>(K, X, Y : H)</math></p>
<p>Games <math>H_2, H_3</math></p> <p><math> H, F\rangle \leftarrow \mathcal{H}  0^{2^{k+n} \cdot m}, 0^{2^n \cdot m}\rangle</math></p> <p>Run <math>\mathcal{A}[\mathcal{H}^{Y,F} \circ \text{FEV} \circ \mathcal{H}^{Y,F}, \mathcal{H}^{Y,H} \circ \text{FRO} \circ \mathcal{H}^{Y,H}]</math></p> <p><math> H, F\rangle \leftarrow  0^{2^{k+n} \cdot m}, 0^{2^n \cdot m}\rangle</math></p> <p>Run <math>\mathcal{A}[\mathcal{H}^Y \circ \text{FEV} \circ \mathcal{H}^Y, \mathcal{H}^Y \circ \text{FRO} \circ \mathcal{H}^Y]</math></p> <p><math> H, F\rangle \leftarrow \mathcal{H}  H, F\rangle</math></p> <p>Measure <math>W[1], H, F</math></p> <p>Return <math>W[1] = 1</math></p>	<p><math>\text{FEV}(X, Y : I, \mathbf{X}, F)</math></p> <p><math>I \leftarrow I + 1 \bmod 2^{ I }</math></p> <p><math>\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X</math></p> <p><math>F(X) \leftarrow F(X) \oplus Y</math></p> <p>Return <math>(X, Y : I, \mathbf{X}, F)</math></p> <p><math>\text{FRO}(K, X, Y : H)</math></p> <p><math>H_K(X) \leftarrow H_K(X) \oplus Y</math></p> <p>Return <math>(K, X, Y : H)</math></p>

**Fig. 4.** Hybrid games  $H_0$  through  $H_3$  for the proof of Theorem 3 which are equivalent to the ideal world of  $G_{\text{FFX}}^{\text{prf}}$ . Highlighted or boxed code is only included in the correspondingly highlighted or boxed game.

measure just that register and assume that  $\mathcal{A}$  does not internally make any measurements. The principle of deferred measurement ensures that the various changes discussed here do not change the behavior of  $\mathcal{A}$ . This perspective change lets us use purely quantum analysis, rather than mixing quantum and classical.

CLAIM 1. We start by considering the hybrids  $H_0$  through  $H_3$ , defined in Fig. 4 which are all identical to the ideal world of  $G_{\text{FFX}}^{\text{prf}}$ . In these transitions we are applying the ideas of Zhandry [35] to transition to representing the random functions stored in  $H$  and  $F$  by an all zeros table which is updated whenever the adversary makes a query.

The hybrid  $H_0$  is mostly just  $G_{\text{FFX}}^{\text{prf}}$  rewritten to use the registers indicated above. So  $\Pr[G_{\text{FFX},0}^{\text{prf}}(\mathcal{A}) = \text{Pr}[H_0]]$  holds.

Next consider  $H_1$  which differs from  $H_0$  only in the grey highlighted code which initializes  $H$  and  $F$  in the uniform superposition and then measures them at the end of execution. (Recall that the Hadamard transform applied to the all zeros state gives the uniform superposition.) Note that these register control, but are unaffected by the oracles EV and RO. Because they are never modified while  $\mathcal{A}$  is executing, the principle of deferred measurement tells us that this modification is undetectable by  $\mathcal{A}$ , giving  $\Pr[H_0] = \Pr[H_1]$

Next consider  $H_2$  which contains the boxed, but not the highlighted, code. This game uses the oracles FEV and FRO, the Fourier versions of EV and RO, which xor the  $Y$  value of  $\mathcal{A}$ 's query into the the register  $F$  or  $H$ . Note that  $\mathcal{A}$ 's access to these oracles is mitigated by  $\mathcal{H}^{Y,F}$  on each query. The superscript here indicate that the Hadamard transform is being applied to the registers  $Y$  and  $F$ . We have that  $\mathcal{H}^{Y,F} \circ \text{FEV} \circ \mathcal{H}^{Y,F} = \text{EV}$  and  $\mathcal{H}^{Y,H} \circ \text{FRO} \circ \mathcal{H}^{Y,H} = \text{RO}$  both hold.<sup>9</sup> So  $\Pr[H_1] = \Pr[H_2]$  because the adversary's oracles are identical.

Next consider  $H_3$  which contains the highlighted, but not the boxed, code. For this transition, recall that  $\mathcal{H} \circ \mathcal{H}$  is the identity operator. So to transition to this game we can cancel the  $\mathcal{H}$  operator used to initialize  $H$  with the  $\mathcal{H}^H$  operator applied before  $\mathcal{A}$ 's first FRO oracle query. Similarly, we can cancel the  $\mathcal{H}^H$  operation performed after any (non-final) FRO query with the  $\mathcal{H}^H$  operation performed before the next FRO query.

<sup>9</sup> This follows as a consequence of the following. Let  $U_{\oplus}$  and  $U'_{\oplus}$  be the unitaries which for  $y, z \in \{0, 1\}$  are defined by  $U_{\oplus} |y, z\rangle = |y \oplus z, z\rangle$  and  $U'_{\oplus} |y, z\rangle = |y, y \oplus z\rangle$ . Then  $\mathcal{H} \circ U_{\oplus} \circ \mathcal{H} = U'_{\oplus}$ .

<p>Games <math>H_9, H_8</math></p> <p><math>\mathbf{H} \leftarrow_s \text{Fcs}(k, n, m)</math></p> <p><math>\mathbf{K}_1 \leftarrow_s \{0, 1\}^k</math></p> <p><math>\mathbf{K}_2 \leftarrow_s \{0, 1\}^n</math></p> <p><math> H, K_1, K_2\rangle \leftarrow  \mathbf{H}, \mathbf{K}_1, \mathbf{K}_2\rangle</math></p> <p><math> H, K_1, K_2\rangle \leftarrow \mathcal{H}  0^{2^{k+n} \cdot m}, 0^k, 0^n\rangle</math></p> <p>Run <math>\mathcal{A}[\text{Ev}, \text{Ro}]</math></p> <p>Measure <math>W[1], H, K_1, K_2</math></p> <p>Return <math>W[1] = 1</math></p>	<p><math>\text{Ev}(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>I \leftarrow I + 1 \pmod{2^{ I }}</math></p> <p><math>\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X</math></p> <p><math>Y \leftarrow H_{K_1}(X \oplus K_2) \oplus Y</math></p> <p>Return <math>(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>\text{Ro}(K, X, Y : H)</math></p> <p><math>Y \leftarrow H_K(X) \oplus Y</math></p> <p>Return <math>(K, X, Y : H)</math></p>
<p>Games <math>H_7, H_6</math></p> <p><math> H, K_1, K_2\rangle \leftarrow \mathcal{H}  0^{2^{k+n} \cdot m}, 0^k, 0^n\rangle</math></p> <p><math>O \leftarrow \mathcal{H}^{Y, H} \circ \text{FEV} \circ \mathcal{H}^{Y, H}</math></p> <p><math>O' \leftarrow \mathcal{H}^{Y, H} \circ \text{FRO} \circ \mathcal{H}^{Y, H}</math></p> <p>Run <math>\mathcal{A}[O, O']</math></p> <p><math> K_1, K_2\rangle \leftarrow \mathcal{H}  0^k, 0^n\rangle</math></p> <p><math> H\rangle \leftarrow  0^{2^{k+n} \cdot m}\rangle</math></p> <p>Run <math>\mathcal{A}[\mathcal{H}^Y \circ \text{FEV} \circ \mathcal{H}^Y, \mathcal{H}^Y \circ \text{FRO} \circ \mathcal{H}^Y]</math></p> <p><math> H\rangle \leftarrow \mathcal{H} H\rangle</math></p> <p>Measure <math>W[1], H, K_1, K_2</math></p> <p>Return <math>W[1] = 1</math></p>	<p><math>\text{FEV}(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>I \leftarrow I + 1 \pmod{2^{ I }}</math></p> <p><math>\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X</math></p> <p><math>H_{K_1}(X \oplus K_2) \leftarrow H_{K_1}(X \oplus K_2) \oplus Y</math></p> <p>Return <math>(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>\text{FRO}(K, X, Y : H)</math></p> <p><math>H_K(X) \leftarrow H_K(X) \oplus Y</math></p> <p>Return <math>(K, X, Y : H)</math></p>

**Fig. 5.** Hybrid games  $H_9$  through  $H_6$  for the proof of Theorem 3 which are equivalent to the real world of  $\mathbb{G}_{\text{FFX}}^{\text{prf}}$ . Highlighted or boxed code is only included in the correspondingly highlighted or boxed game.

Finally, the  $\mathcal{H}^H$  operation that would be performed after the final FRO query is instead delayed to be performed immediately before  $H$  is measured. (We could have omitted this operation and measurement entirely because all that matters at that point is the measurement of  $W[1]$ .) The  $\mathcal{H}$  operators on  $F$  are similarly changed. Because  $\mathcal{A}$  does not have access to the  $H$  and  $F$  registers, we can indeed commute the  $\mathcal{H}$  operators with  $\mathcal{A}$  in this manner without changing behavior. Hence  $\Pr[H_2] = \Pr[H_3]$ , as desired. Note that  $H$  and  $F$  independently store tables which are initialized to all zeros and then written into by the adversary's queries.

CLAIM 2. We now consider the hybrids  $H_4$  through  $H_9$  (starting from  $H_9$ ), which are defined in Fig. 5 and Fig. 6 using similar ideas as in the transition from  $H_0$  through  $H_3$ . As we will justify, these games are all equivalent to the real world of  $\mathbb{G}_{\text{FFX}}^{\text{prf}}$ .

First  $H_9$  rewrote the real world of  $\mathbb{G}_{\text{FFX}}^{\text{prf}}$  to use our specified registers and to record queries into  $\mathbf{X}$  in FEV. Then in  $H_8$ , rather than sampling  $H$ ,  $K_1$ , and  $K_2$  uniformly at the beginning of the game we put them in the uniform superposition and measure them at the end of the game. For  $H_7$  we replace our oracles that xor into the adversary's  $Y$  register with oracles that xor into the  $H$  register using Hadamard operations, some of which we then cancel out to transition to  $H_6$ . The same arguments from Claim 1 of why these sorts of modifications do not change the behavior of the game apply here and so  $\Pr[\mathbb{G}_{\text{FFX},1}^{\text{prf}}(\mathcal{A})] = \Pr[H_9] = \Pr[H_8] = \Pr[H_7] = \Pr[H_6]$ .

Our next transitions are designed to make the current game's oracles identical with those of  $H_3$ , except on some "bad" inputs. In  $H_6$  we have a single all zeros table  $H$  which gets written into by queries that  $\mathcal{A}$  makes to either of its oracles, while in  $H_3$  the oracles separately wrote into either  $H$  or  $F$ . For  $H_5$  we will similarly separate the single table  $H$  into separate tables  $H$  and  $F$ . However, we cannot keep them completely

<p>Games <math>\boxed{\mathbf{H}_5}, \mathbf{H}_4</math></p> <p><math> K_1, K_2\rangle \leftarrow \mathcal{H}  0^k, 0^n\rangle</math></p> <p><math> H, F\rangle \leftarrow  0^{2^{k+n} \cdot m}, 0^{2^n \cdot m}\rangle</math></p> <p><math>O \leftarrow \mathcal{H}^Y \circ \mathcal{T} \circ \text{FEV} \circ \mathcal{T} \circ \mathcal{H}^Y</math></p> <p><math>O' \leftarrow \mathcal{H}^Y \circ \mathcal{T} \circ \text{FRO} \circ \mathcal{T} \circ \mathcal{H}^Y</math></p> <p>Run <math>\mathcal{A}[O, O']</math></p> <p>Run <math>\mathcal{A}[\mathcal{H}^Y \circ \text{FEV}' \circ \mathcal{H}^Y, \mathcal{H}^Y \circ \text{FRO}' \circ \mathcal{H}^Y]</math></p> <p><math> H, F, I, \mathbf{X}, K_1, K_2\rangle \leftarrow \mathcal{T}  H, F, I, \mathbf{X}, K_1, K_2\rangle</math></p> <p><math> H\rangle \leftarrow \mathcal{H}  H\rangle</math></p> <p>Measure <math>W[1], H, K_1, K_2</math></p> <p>Return <math>W[1] = 1</math></p> <p><math>\text{FRO}'(K, X, Y : H, F, I, \mathbf{X}, K_1, K_2)</math></p> <p>If <math>K = K_1</math> and <math>X \oplus K_2 \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\}</math> then</p> <p style="padding-left: 20px;">// <b>Input is bad</b></p> <p style="padding-left: 20px;"><math>F(X \oplus K_2) \leftarrow F(X \oplus K_2) \oplus Y</math></p> <p>Else</p> <p style="padding-left: 20px;"><math>H_K(X) \leftarrow H_K(X) \oplus Y</math></p> <p>Return <math>(K, X, Y : H, F, I, \mathbf{X}, K_1, K_2)</math></p>	<p><math>\text{FEV}(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>I \leftarrow I + 1 \pmod{2^{ I }}</math></p> <p><math>\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X</math></p> <p><math>H_{K_1}(X \oplus K_2) \leftarrow H_{K_1}(X \oplus K_2) \oplus Y</math></p> <p>Return <math>(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>\text{FRO}(K, X, Y : H)</math></p> <p><math>H_K(X) \leftarrow H_K(X) \oplus Y</math></p> <p>Return <math>(K, X, Y : H)</math></p> <p><math>\text{FEV}'(X, Y : H, F, I, \mathbf{X}, K_1, K_2)</math></p> <p><math>I \leftarrow I + 1 \pmod{2^{ I }}</math></p> <p><math>\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X</math></p> <p>If <math>H_{K_1}(X \oplus K_2) \neq 0^m</math> then</p> <p style="padding-left: 20px;">// <b>Input is bad</b></p> <p style="padding-left: 20px;"><math>F'(X) \leftarrow F(X)</math></p> <p style="padding-left: 20px;"><math>F(X) \leftarrow H_{K_1}(X \oplus K_2)</math></p> <p style="padding-left: 20px;"><math>H_{K_1}(X \oplus K_2) \leftarrow F'(X)</math></p> <p style="padding-left: 20px;"><math>F(X) \leftarrow F(X) \oplus Y</math></p> <p>Return <math>(X, Y : H, I, \mathbf{X}, K_1, K_2)</math></p>
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**Fig. 6.** Hybrid games  $\mathbf{H}_5$  and  $\mathbf{H}_4$  for the proof of Theorem 3 which are equivalent to the real world of  $\mathbf{G}_{\text{FFX}}^{\text{prf}}$ . Highlighted or boxed code is only included in the correspondingly highlighted or boxed game. Unitary  $\mathcal{T}$  is define in the text.

independent, because if the adversary queries FEV with  $X = x$  and FRO with  $(K, X) = (K_1, x \oplus K_2)$  then both of these operations would be writing into the same table location in  $\mathbf{H}_6$ . Consider the following unitary  $\mathcal{T}$  which acts on registers  $H$  and  $F$  and is controlled by the registers  $I, \mathbf{X}, K_1$ , and  $K_2$ . We will think of this unitary as transitioning us between a representation of  $H$  as a single table (with an all-zero  $F$  table) and a representation of it divided between  $H$  and  $F$ .

$$\begin{array}{l}
\mathcal{T}(H, F, I, \mathbf{X}, K_1, K_2) \\
\text{For } x \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\} \text{ do} \\
\quad F'(x) \leftarrow F(x) \\
\quad F(x) \leftarrow H_{K_1}(x \oplus K_2) \\
\quad H_{K_1}(x \oplus K_2) \leftarrow F'(x) \\
\text{Return } (H, F, I, \mathbf{X}, K_1, K_2)
\end{array}$$

In words,  $\mathcal{T}$  swaps  $F(x)$  and  $H_{K_1}(x \oplus K_2)$  for each  $x$  that has been previous queried to FEV (as stored by  $\mathbf{X}$  and  $I$ ). Note that  $\mathcal{T}$  is its own inverse. In  $\mathbf{H}_5$  we (i) initialize the table  $F$  as all zeros, (ii) perform  $\mathcal{T}$  before and after each oracle query, and (iii) perform  $\mathcal{T}$  after  $\mathcal{A}$  has executed. We verify that  $H$  has the same value in  $\mathbf{H}_5$  that it would have had in  $\mathbf{H}_6$  during each oracle query and at the end before measurement. The application of  $\mathcal{T}$  before the first oracle query does nothing (because  $I = 0$ ) so  $H$  is all zeros for this query as required. As we've seen previously with  $\mathcal{H}$ , we can commute  $\mathcal{T}$  with the operations of  $\mathcal{A}$  because  $\mathcal{T}$  only acts on registers outside of the adversary's control. We can similarly commute  $\mathcal{T}$  with  $\mathcal{H}^Y$ . Hence the  $\mathcal{T}$  operation after every non-final oracle query can be seen to cancel with the  $\mathcal{T}$  operation before the following oracle query. The  $\mathcal{T}$  operation after the final oracle query cancels with the  $\mathcal{T}$  operation performed after  $\mathcal{A}$  halts execution. Hence,  $\Pr[\mathbf{H}_6] = \Pr[\mathbf{H}_5]$  as claimed.

For the transition to  $\mathbf{H}_4$  let us dig into how our two-table representation in  $\mathbf{H}_5$  works so that we can incorporate the behavior of  $\mathcal{T}$  directly into the oracles. For simplicity of notation in discussion, let  $\tilde{H}(x)$  denote  $H_{K_1}(x \oplus K_2)$ . First note that in between oracle queries the two tables representation of  $\mathbf{H}_5$  will satisfy

Games $\tilde{\mathbf{H}}_3, \tilde{\mathbf{H}}_4$	$\text{FEV}(X, Y : H, F, I, \mathbf{X}, K_1, K_2)$
$ K_1, K_2\rangle \leftarrow \mathcal{H}  0^k, 0^n\rangle$	$I \leftarrow I + 1 \pmod{2^{ I }}$
$ H, F\rangle \leftarrow  0^{2^{k+n} \cdot m}, 0^{2^n \cdot m}\rangle$	$\mathbf{X}_I \leftarrow \mathbf{X}_I \oplus X$
Run $\mathcal{A}[\mathcal{H}^Y \circ \text{FEV} \circ \mathcal{H}^Y, \mathcal{H}^Y \circ \text{FRO} \circ \mathcal{H}^Y]$	If $H_{K_1}(X \oplus K_2) \neq 0^m$ then
Measure $W[1]$	// <b>Input is bad</b>
Return $W[1] = 1$	$F'(X) \leftarrow F(X)$
$\text{FRO}(K, X, Y : H, F, I, \mathbf{X}, K_1, K_2)$	$F(X) \leftarrow H_{K_1}(X \oplus K_2)$
If $K = K_1$ and $X \oplus K_2 \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$ then	$H_{K_1}(X \oplus K_2) \leftarrow F'(X)$
// <b>Input is bad</b>	$F(X) \leftarrow F(X) \oplus Y$
$F(X \oplus K_2) \leftarrow F(X \oplus K_2) \oplus Y$	Return $(X, Y : H, I, \mathbf{X}, K_1, K_2)$
$H_K(X) \leftarrow H_K(X) \oplus Y$	
Else	
$H_K(X) \leftarrow H_K(X) \oplus Y$	
Return $(K, X, Y : H, F, I, \mathbf{X}, K_1, K_2)$	

**Fig. 7.** Hybrid games  $\tilde{\mathbf{H}}_3$  and  $\tilde{\mathbf{H}}_4$  for the proof of Theorem 3 which are rewritten versions of  $\mathbf{H}_3$  and  $\mathbf{H}_4$  to emphasize that their oracles are identical-until-bad. Highlighted or boxed code is only included in the correspondingly highlighted or boxed game.

the property that for each  $x \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$  we will have  $\tilde{H}(x) = 0^{F \cdot \text{ol}}$  and for all other  $x$  we will have that  $F(x) = 0^{F \cdot \text{ol}}$ .<sup>10</sup> This is the case because after each query we have applied  $\mathcal{T}$  to an  $H$  which contains the same values it would have in  $\mathbf{H}_6$  and an  $F$  which is all zeros.

Now consider when a  $\mathcal{T} \circ \text{FEV} \circ \mathcal{T}$  query is executed with some  $X$  and  $Y$ . If  $X \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$ , then  $F(X)$  and  $\tilde{H}(X)$  are swapped,  $Y$  is xored into  $\tilde{H}(X)$ , and  $F(X)$  and  $\tilde{H}(X)$  are swapped back. Equivalently, we could have xored  $Y$  directly into  $F(X)$  and skipped the swapping around. If  $X \notin \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$ , then  $Y$  is xored into  $\tilde{H}(X)$  before  $F(X)$  and  $\tilde{H}(X)$  are swapped. If  $\tilde{H}(X) = 0^{F \cdot \text{ol}}$  beforehand, then we could equivalently have xored  $Y$  directly into  $F(X)$  and skipped the swapping around (because  $F(X) = 0^{F \cdot \text{ol}}$  must have held from our assumption on  $X$ ). If  $\tilde{H}(X) \neq 0^{F \cdot \text{ol}}$  beforehand, then we could equivalently could have swapped  $\tilde{H}(X)$  and  $F(X)$  first, then xored  $Y$  into  $F(X)$ . The equivalent behavior we described is exactly the behavior captured by the oracle  $\text{FEV}'$  which is used in  $\mathbf{H}_4$  in place of  $\mathcal{T} \circ \text{FEV} \circ \mathcal{T}$ . It checks if  $\tilde{H}(X) \neq 0$ , performing a swap if so. Then  $Y$  is xored into  $F(X)$ . Note that  $\tilde{H}(X) \neq 0$  implies  $X \notin \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$  from our earlier observation, so we omitted the check of the latter.

Similarly, consider when a  $\mathcal{T} \circ \text{FRO} \circ \mathcal{T}$  query is executed with some  $K, X, Y$ . Any swapping done by  $\mathcal{T}$  uses  $H_{K_1}$ , so when  $K \neq K_1$  this just xors  $Y$  into  $H_K(X)$ . If  $X \oplus K_2$  is not in  $\{\mathbf{X}_1, \dots, \mathbf{X}_I\}$ , then  $H_K(X)$  is unaffected by the swapping so again this just xors  $Y$  into  $H_K(X)$ . When  $K = K_1$  and  $X \oplus K_2 \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$ , first  $H_K(X)$  would have been swapped with  $F(X \oplus K_2)$ , then  $Y$  would be xored into  $H_K(X)$ , then  $H_K(X)$  and  $F(X \oplus K_2)$  would be swapped back. Equivalently, we could just have xored  $Y$  into  $F(X \oplus K_2)$  and skipped the swapping. This behavior we described is exactly the behavior captured by the oracle  $\text{FRO}'$  which is used in  $\mathbf{H}_4$  in place of  $\mathcal{T} \circ \text{FRO} \circ \mathcal{T}$ .

We have just described that on the inputs we care about  $\text{FEV}'$  behaves identically to  $\mathcal{T} \circ \text{FEV} \circ \mathcal{T}$  and  $\text{FRO}'$  behaves identically to  $\mathcal{T} \circ \text{FRO} \circ \mathcal{T}$ . Hence  $\Pr[\mathbf{H}_5] = \Pr[\mathbf{H}_4]$ , completing this claim.

**CLAIM 3.** To compare hybrids  $\mathbf{H}_3$  and  $\mathbf{H}_4$  we will note their oracles only differ on a small number of inputs (in particular those labelled as bad by comments in our code) and then apply Theorem 1 to bound the difference between them. To aid in this we have rewritten them as  $\tilde{\mathbf{H}}_3$  and  $\tilde{\mathbf{H}}_4$  in Fig. 7. For both we have removed some operations performed after  $\mathcal{A}$  halted its execution because these operations on registers other

<sup>10</sup> More precisely, the corresponding registers hold superpositions over tables satisfying the properties we discuss.

than  $W[1]$  cannot affect the probability that it is measured to equal 1. So we have  $\Pr[\tilde{H}_3] = \Pr[H_3]$  and  $\Pr[\tilde{H}_4] = \Pr[H_4]$ .

Let  $\text{FEV}_3$  and  $\text{FRO}_3$  be the permutations defining the corresponding oracle in  $\tilde{H}_3$ . Define  $\text{FEV}_4$  and  $\text{FRO}_4$  analogously. These permutations differ only on the inputs we referred to as bad. So let  $S$  denote the set of bad  $X, H, F, I, \mathbf{X}, K_1, K_2$  for  $\text{FEV}$  (i.e. those for which  $H_{K_1}(X \oplus K_2) \neq 0^m$ ). Let  $S'$  denote the set of bad  $K, X, H, F, I, \mathbf{X}, K_1, K_2$  for  $\text{FRO}$  (i.e. those for which  $K = K_1$  and  $X \oplus K \in \{\mathbf{X}_1, \dots, \mathbf{X}_I\}$ ). Let  $\mathbf{D}$  denote the distribution which always outputs  $(S, S', \text{FEV}_3, \text{FRO}_3, \text{FEV}_4, \text{FRO}_4, \varepsilon)$ . Clearly this is a valid distribution by our choice of  $S$  and  $S'$ .

Now we can define an adversary  $\mathcal{A}'$  for  $\mathbf{G}_{\mathbf{D},b}^{\text{dist}}$  which simulates the view of  $\mathcal{A}$  in  $\tilde{H}_{3+b}$  by locally running the code of that hybrid except for during oracle queries when it uses its  $f_b$  oracle to simulate  $\text{FEV}$  and  $f'_b$  oracle to simulate  $\text{FRO}$ . Because the simulation of these views are perfect we have that

$$\Pr[\tilde{H}_3] - \Pr[\tilde{H}_4] = \text{Adv}_{\mathbf{D}}^{\text{dist}}(\mathcal{A}') \leq 2(p+q)\sqrt{\text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A}' )}$$

where the inequality follows from Theorem 1, noting that  $\mathcal{A}'$  makes  $p+q$  oracle queries.

To complete the proof we bound  $\text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A}')$ . In the following probability calculation we use  $x$  and  $i$  to denote random variables taking on the values the corresponding variables have at the end of an execution of  $\mathbf{G}_{\mathbf{D}}^{\text{guess}}(\mathcal{A})$ . Let  $S$  denote a random variable which equals  $S$  if the measured query is to  $f_0$  and equals  $S'$  otherwise. Then conditioning over each possible value of  $i$  gives

$$\begin{aligned} \text{Adv}_{\mathbf{D}}^{\text{guess}}(\mathcal{A}') = \Pr[x \in S] &= \sum_{j=1}^{p+q} \Pr[x \in S \mid i = j] \Pr[i = j] \\ &= (p+q)^{-1} \sum_{j=1}^{p+q} \Pr[x \in S \mid i = j]. \end{aligned}$$

Because  $\mathcal{A}$  is order consistent we can pick disjoint sets  $E$  and  $R$  with  $E \cup R = \{1, \dots, p+q\}$  such that  $i \in E$  means the  $i$ -th query is to  $\mathcal{A}$ 's  $\text{FEV}$  oracle and  $i \in R$  means that the  $i$ -th query is to its  $\text{FRO}$  oracle. Note that  $|E| = q$  and  $|R| = p$ . The view of  $\mathcal{A}$  (when run by  $\mathcal{A}'$ ) in  $\mathbf{G}_{\mathbf{D}}^{\text{guess}}$  matches its view in  $\tilde{H}_3$  so, in particular, it is independent of  $K_1$  and  $K_2$ . Hence we can think of these keys being chosen at random at the end of execution when analyzing the probability of  $x \in S$ .

For a  $\text{FRO}$  query, the check for bad inputs is if  $K_1 = K$  and  $K_2 \in \{X \oplus \mathbf{X}_1, \dots, X \oplus \mathbf{X}_I\}$ . The variable  $I$  is counting the number of  $\text{FEV}$  queries made so far, so  $I < q$ . By a union bound,

$$\Pr[x \in S' \mid i = j] \leq q/2^{\text{F.kl+F.il}}$$

when  $j \in R$ .

For a  $\text{FEV}$  query, the check for bad inputs is if  $H_{K_1}(X \oplus K_2)$  is non-zero. In  $\tilde{H}_3$ , each query to  $\text{FRO}$  can make a single entry of  $H$  non-zero so it will never have more than  $p$  non-zero entries. By a union bound,

$$\Pr[x \in S \mid i = j] \leq p/2^{\text{F.kl+F.il}}$$

when  $j \in E$ .

The proof is then completed by noting

$$\begin{aligned} \sum_{j=1}^{p+q} \Pr[x \in S \mid i = j] &= \sum_{j \in R} \Pr[x \in S' \mid i = j] + \sum_{j \in E} \Pr[x \in S \mid i = j] \\ &\leq p(q/2^{\text{F.kl+F.il}}) + q(p/2^{\text{F.kl+F.il}}) = 2pq/2^{\text{F.kl+F.il}} \end{aligned}$$

and plugging in to our earlier expression.  $\square$

Problem	Witness	Promise
ED	$x \neq y$ s.t. $L(x) = L(y)$	-
1ED	$x \neq y$ s.t. $L(x) = L(y)$	At most one witness.
1LD	$x, y$ s.t. $L_0(x) = L_1(y)$	At most one witness. Injective $L_0, L_1$ .

**Fig. 8.** Summary of the element distinctness and list disjointness problems we consider.

## 4 Double Encryption

In this section we prove the security of the double encryption key-length extension technique against fully quantum attacks. Our proof first reduces this to the ability of a quantum algorithm to solve the list disjointness problem and then extends known query lower bounds for element distinctness to list disjointness (with some modifications).

The double encryption blockcipher is constructed via two sequential application of an underlying blockcipher. Formally, given a blockcipher  $E \in \text{lcs}(\text{E.kl}, \text{E.bl})$ , we define the double encryption blockcipher  $\text{DE}[E]$  by  $\text{DE}[E](K_1 \parallel K_2, x) = E_{K_2}(E_{K_1}(x))$ . Here  $|K_1| = |K_2| = \text{E.kl}$  so  $\text{DE}[E].\text{kl} = 2\text{E.kl}$  and  $\text{DE}[E].\text{bl} = \text{E.bl}$ . Its inverse can be computed as  $\text{DE}[E]^{-1}(K_1 \parallel K_2, x) = E_{K_1}^{-1}(E_{K_2}^{-1}(x))$ .

Classically, the meet-in-the-middle attack [10,26] shows that this construction achieves essentially the same security a single encryption. In the quantum setting, this construction was recently considered by Kaplan [16]. They gave an attack and matching security bound for the key-recovery security of double encryption. This leaves the question of whether their security result can be extended to cover full SPRP security, which we resolve by the main theorem of this section. This theorem is proven via a reduction technique of Tessaro and Thiruvengadam [31] which they used to establish a (classical) time-memory tradeoff for the security of double encryption, by reducing its security to the list disjointness problem and conjecturing a time-memory tradeoff for that problem.

**PROBLEMS AND LANGUAGES.** In addition to the list disjointness problem (1LD), we will also consider two versions of the element distinctness problem (ED, 1ED). In general, a problem  $\mathcal{P}$  specifies a relation  $\mathcal{R}$  on set on instances  $\mathcal{I}$  (i.e.  $\mathcal{R}$  is function which maps an instance  $L \in \mathcal{I}$  and witness  $w$  to a decision  $\mathcal{R}(L, w) \in \{0, 1\}$ ). This relation induces a language  $\mathcal{L} = \{L \in \mathcal{I} : \exists w, \mathcal{R}(L, w) = 1\}$ . Rather than think of instances as bit strings, we will think of them as functions (to which decision and search algorithms are given oracle access). To restrict attention to functions of specific sizes we let  $\mathcal{L}(D, R) = \mathcal{L} \cap \mathcal{I}(D, R)$  where  $\mathcal{I}(D, R)$  denotes the restriction of  $\mathcal{I}$  to functions  $L : [D] \rightarrow [R]$ , where  $D \leq R$ . To discuss instances not in the language we let  $\mathcal{L}' = \mathcal{I} \setminus \mathcal{L}$  and  $\mathcal{L}'(D, R) = \mathcal{I}(D, R) \setminus \mathcal{L}$ .

Problems have decision and search versions. The goal of a decision algorithm is to output 1 (representing “acceptance”) on instances in the language and 0 (representing “rejection”) otherwise. Relevant quantities are the minimum probability  $P^1$  of accepting an instance in the language, the maximum probability  $P^0$  of accepting an instance not in the language, and the error rater  $E$  which are formally defined by

$$P_{D,R}^1(\mathcal{A}) = \min_{L \in \mathcal{L}(D,R)} \Pr[\mathcal{A}[L] = 1], \quad P_{D,R}^0(\mathcal{A}) = \max_{L \in \mathcal{L}'(D,R)} \Pr[\mathcal{A}[L] = 1]$$

$$E_{D,R}(\mathcal{A}) = \max\{1 - P_{D,R}^1(\mathcal{A}), P_{D,R}^0(\mathcal{A})\}.$$

We define the decision PB advantage of  $\mathcal{A}$  by  $\text{Adv}_{D,R}^{\text{PB}}(\mathcal{A}) = P_{D,R}^1(\mathcal{A}) - P_{D,R}^0(\mathcal{A})$ . In non-cryptographic contexts, instead of looking at the difference in probability that inputs that are in or out of the language are accepted, one often looks at how far these are each from 1/2. This motivates the definition  $\text{Adv}_{D,R}^{\text{PB-d}}(\mathcal{A}) = \min\{2P_{D,R}^1(\mathcal{A}) - 1, 1 - 2P_{D,R}^0(\mathcal{A})\} = 1 - 2E_{D,R}(\mathcal{A})$ .

The goal of a search algorithm is to output a witness for the instance. We define its advantage to be the minimum probability it succeeds, i.e.,  $\text{Adv}_{D,R}^{\text{PB-s}}(\mathcal{A}) = \min_{L \in \mathcal{L}(D,R)} \Pr[\mathcal{R}(L, \mathcal{A}[L]) = 1]$ .

**EXAMPLE PROBLEMS.** The *list disjointness* problem asks how well an algorithm can distinguish between the case that is give (oracle access to) two lists which are disjoint or have one element in common. In particular,

we interpret an instance  $L$  as the two functions  $L_0, L_1 : [[D/2]] \rightarrow [R]$  defined by  $L_b(x) = L(x + b[D/2])$ . Let  $\mathcal{S}_n$  denote the set of  $L$  for which  $L_0$  and  $L_1$  are injective and which have  $n$  elements in common, i.e. for which  $|\{L_0(1), \dots, L_0([D/2])\} \cap \{L_1(1), \dots, L_1([D/2])\}| = n$ . Then 1LD is defined by the relation  $\mathcal{R}$  which on input  $(L, (x, y))$  returns 1 iff  $L_0(x) = L_1(y)$  and the instance set  $\mathcal{I} = \mathcal{S}_0 \cup \mathcal{S}_1$  (i.e., the promise that there is at most one element in common and that the lists are individually injective). The search version of list disjointness is sometimes referred to as claw-finding.

The *element distinctness* problem asks how well an algorithm can detect whether all the elements in a list are distinct. Let  $\mathcal{S}'_n$  denote the set of  $L$  which have  $n$  collision pairs, i.e. for which  $|\{\{x, y\} : x \neq y, L(x) = L(y)\}| = n$ . Then ED is defined by the relation  $\mathcal{R}$  which on input  $(L, (x, y))$  returns 1 iff  $x \neq y$  and  $L(x) = L(y)$  with the instance set  $\mathcal{I} = \bigcup_{n=0}^{\infty} \mathcal{S}'_n$  consisting of all functions. We let 1ED denote restricting ED to  $\mathcal{I} = \mathcal{S}'_0 \cup \mathcal{S}'_1$  (i.e., the promise that there is at most one repetition in the list).

#### 4.1 Security result

The following theorem establishes the security of double encryption against quantum attacks. It shows that an attacker achieving constant advantage must make  $\Omega(2^{2k/3})$  oracle queries (ignoring log terms). Our bound is not tight for a lower parameter regimes, though future work may establish better bounds for list disjointness in these regimes.

**Theorem 4.** *Consider  $\text{DE}[\cdot]$  with the underlying blockcipher modeled by an ideal cipher drawn from  $\text{lcs}(k, n)$ . Let  $\mathcal{A}$  be a quantum adversary which makes at most  $q$  queries to the ideal cipher. Then*

$$\text{Adv}_{\text{DE}}^{\text{srp}}(\mathcal{A}) \leq 11 \sqrt[6]{(q \cdot k \lg k)^3 / 2^{2k} + 1/2^k}.$$

As mentioned earlier, our proof works by first reducing the security of double encryption against quantum queries to the security of the list disjointness problem against quantum queries. This is captured in the following theorem which we prove now.

**Theorem 5.** *Consider  $\text{DE}[\cdot]$  with the underlying blockcipher modeled by an ideal cipher drawn from  $\text{lcs}(k, n)$ . Let  $\mathcal{A}$  be a quantum adversary which makes at most  $q$  queries to the ideal cipher. Then for any  $R \geq 2^k$  we can construct  $\mathcal{A}'$  making at most  $q$  oracle queries such that*

$$\text{Adv}_{\text{DE}}^{\text{srp}}(\mathcal{A}) \leq \text{Adv}_{2^k, R}^{\text{1LD-d}}(\mathcal{A}') + 1/2^k.$$

We state and prove a bound on  $\text{Adv}^{\text{1LD-d}}$  in Section 4.2. Our proof applies the same reduction technique as Tessaro and Thiruvengadam [31], we are verifying that it works quantumly as well.

*Proof.* For  $b \in \{0, 1\}$ , let  $\mathbf{H}_b$  be defined to be identical to  $\mathbf{G}_{\text{DE}, b}^{\text{srp}}$  except that  $K_2$  is chosen uniformly from  $\{0, 1\}^k \setminus \{K_1\}$  rather than from  $\{0, 1\}^k$ . This has no effect when  $b = 0$  because the keys are not used, so  $\Pr[\mathbf{G}_{\text{DE}, 0}^{\text{srp}}(\mathcal{A})] = \Pr[\mathbf{H}_0]$ . When  $b = 1$  there was only a  $1/2^k$  chance that  $K_2$  would have equalled  $K_1$  so  $\Pr[\mathbf{G}_{\text{DE}, 1}^{\text{srp}}(\mathcal{A})] \leq \Pr[\mathbf{H}_1] + 1/2^k$ .

Now we define a decision algorithm  $\mathcal{A}'$  for 1LD which uses its input lists to simulate a view for  $\mathcal{A}$ . When the lists are disjoint  $\mathcal{A}'$ 's view will perfectly match that of  $\mathbf{H}_0$  and when the lists have exactly one element in common  $\mathcal{A}'$ 's view will perfectly match that of  $\mathbf{H}_1$ . Hence we have  $\Pr[\mathbf{H}_1] = \min_{(L, L') \in \mathcal{L}_{1, k}^{\kappa, k'}} \Pr[\mathbf{G}_{L, L'}^{\text{ld}}(\mathcal{A})]$  and  $\Pr[\mathbf{H}_0] = \max_{(L, L') \in \mathcal{L}_{0, k}^{\kappa, k'}} \Pr[\mathbf{G}_{L, L'}^{\text{ld}}(\mathcal{A})]$ , so  $\Pr[\mathbf{H}_1] - \Pr[\mathbf{H}_0] = \text{Adv}_{2^k, k'}^{\text{ld}}(\mathcal{A}')$  which gives the claimed bound.

The adversary  $\mathcal{A}'$  is defined in Fig. 9. It samples a permutation  $\rho$  on  $\{0, 1\}^k$ , a permutation  $\pi$  on  $\{0, 1\}^n$ , and a cipher  $F$ . The permutation  $\rho$  is used to provide a random map from the keys  $K \in \{0, 1\}^k$  to the elements of the lists  $L_0$  and  $L_1$ . We will interpret  $\rho(K)$  as a tuple  $(i, j)$  with  $i \in \{0, 1\}$  and  $j \in [2^k/2]$ . Then  $K$  gets mapped to  $L_i(j)$ . Thus we are considering either the case that none of the keys map to the same element or that a single pair of them map to the same element.

Adversary  $\mathcal{A}'$ 's queries to EV and INV are answered using  $\pi$  and  $\pi^{-1}$ . Its queries to the ideal cipher are more complicated and are handled by the oracles IC and INV. We can verify that these oracles define permutations

Adversary $\mathcal{A}'[L]$	IC( $K, X, Y : \rho, \pi, F$ )	INV( $K, X, Y : \rho, \pi, F$ )
$\rho \leftarrow \text{lcs}(0, k)$	$(i, j) \leftarrow \rho(K)$	$(i, j) \leftarrow \rho(K)$
$\pi \leftarrow \text{lcs}(0, n)$	If $i = 0$ then	If $i = 0$ then
$F \leftarrow \text{lcs}(\lceil \lg R \rceil, n)$	$Y \leftarrow Y \oplus F_{L_0(j)}(X)$	$Y \leftarrow Y \oplus F_{L_0(j)}^{-1}(X)$
$b' \leftarrow \mathcal{A}[\pi, \pi^{-1}, \text{IC}, \text{INV}]$	Else	Else
Return $b'$	$Y \leftarrow Y \oplus \pi(F_{L_1(j)}^{-1}(X))$	$Y \leftarrow Y \oplus F_{L_1(j)}(\pi^{-1}(X))$
	Return $(K, X, Y : \rho, \pi, F)$	Return $(K, X, Y : \rho, \pi, F)$

**Fig. 9.** Reduction adversary used in proof of Theorem 5.

on bitstring inputs, so  $\mathcal{A}'$  is a well defined quantum adversary. Consider a key  $K$  and interpret  $\rho(K)$  as a tuple  $(i, j)$  as described above. If  $i = 0$ , then ideal cipher queries for it are answered as if  $\mathbf{E}_K(\cdot) = F_{L_0(j)}(\cdot)$ . If  $i = 1$ , then ideal cipher queries for it are answered as if  $\mathbf{E}_K(\cdot) = \pi(F_{L_1(j)}^{-1}(\cdot))$ .<sup>11</sup> If the list element  $K$  is mapped to is not mapped to by any other keys, then the indexing into  $F$  ensures that  $\mathbf{E}_K(\cdot)$  is independent of  $\pi$  and  $\mathbf{E}_{K'}$  for all other  $K'$ . If  $K$  and  $K'$  map to the same list element (and  $i = 0$  for  $K$ ), then  $\mathbf{E}_K(\cdot)$  and  $\mathbf{E}'_{K'}(\cdot)$  are random permutations conditioned on  $\mathbf{E}'_{K'}(\mathbf{E}_K(\cdot)) = \pi(\cdot)$  and independent of all other  $\mathbf{E}_{K''}(\cdot)$ .

In  $\mathbf{H}_0$ , the permutation of EV and INV is independent of each  $\mathbf{E}_K(\cdot)$  which are themselves independent of each other. So this perfectly matches the view presented to  $\mathcal{A}$  by  $\mathcal{A}'$  when the lists are disjoint. In  $\mathbf{H}_1$ , each  $\mathbf{E}_K(\cdot)$  is pairwise independent and the permutation of EV and INV is defined to equal  $\mathbf{E}_{K_2}(\mathbf{E}_{K_1}(\cdot))$ . This perfectly matches the view presented to  $\mathcal{A}$  by  $\mathcal{A}'$  when the lists have one element in common because we can think of it as just having changed the order in which the permutations  $\mathbf{E}_{K_2}(\mathbf{E}_{K_1}(\cdot))$ ,  $\mathbf{E}_{K_1}(\cdot)$ , and  $\mathbf{E}_{K_2}(\cdot)$  were sampled.  $\square$

## 4.2 The Hardness of List Disjointness

If  $\mathcal{A}$  is an algorithm making at most  $q$  classical oracle queries, then it is not hard to prove that  $\text{Adv}_{D,R}^{\text{1LD}}(\mathcal{A}) \leq q/D$ . If, instead,  $\mathcal{A}$  makes at most  $q$  quantum oracle queries, the correct bound is less straightforward. In this section, we will prove the following result.

**Theorem 6.** *If  $\mathcal{A}$  is a quantum algorithm making at most  $q$  queries to its oracle and  $D \geq 32$  is a power of 2, then*

$$\text{Adv}_{D,3D^2}^{\text{1LD}}(\mathcal{A}) \leq 11 \sqrt[6]{(q \cdot \lg D \cdot \lg \lg D)^3 / D^2}.$$

We restrict attention to the case that  $D$  is a power of 2 only for notational simplicity in the proof. Essentially the same bound for more general  $D$  follows from the same techniques.

Ambanis’s  $\mathcal{O}(N^{2/3})$  query algorithm for element distinctness [2] can be used to solve list disjointness and hence shows this is tight (up to logarithmic factors) for attackers achieving constant advantage. The sixth root does degrade the quality of the bound for lower parameter regimes. An interesting question we leave open is whether this result could be proven without the sixth root or the logarithmic factors.

**PROOF SKETCH.** The starting point for our reduction is that  $\Omega(N^{2/3})$  lower bounds are known both the search and decision versions of ED [1,34].<sup>12</sup> By slightly modifying Zhandry’s [34] technique for proving this, we instead get a bound on the hardness of 1ED-s. Next, a simple reduction (split the list in half at random) shows that 1LD-s is as hard as 1ED-s.

Then a “binary search” style reduction shows that 1LD-d is as hard as 1LD-s. In the reduction, the 1LD-s algorithm repeatedly splits its lists in half and uses the 1LD-d algorithm to determine which pair of lists contains the non-disjoint entries. However, we need our reduction to work by running the 1LD-d algorithm on a particular fixed size of list (the particular size we showed 1LD-d is hard for) rather than running it on

<sup>11</sup> When using output of  $L$  as keys for  $F$  we are identifying the elements of  $[R]$  with elements of  $\{0, 1\}^{\lceil \lg R \rceil}$  in the standard manner.

<sup>12</sup> In the proof we actually work with the advantage upper bounds, rather than the corresponding query lower bounds.

numerous shrinking sizes. We achieve this by padding the lists with random elements. The choice of  $R = 3D^2$  was made so that with good probability these random elements do not overlap with the actual list. This padding adds the  $\lg D$  term to our bound.

Finally a generic technique allows us to relate the hardness of 1LD and 1LD-d. Given an algorithm with high 1LD advantage we can run it multiple times to get a precise estimate of how frequently it is outputting 1 and use that to determine what we want to output. This last step is the primary cause of the sixth root in our bound; it required running the 1LD algorithm on the order of  $1/\delta^2$  times to get a precise enough estimate, where  $\delta$  is the advantage of the 1LD algorithm. This squaring of  $\delta$  in the query complexity of our 1LD-d algorithm (together with the fact that the query complexity is cubed in our 1ED-s bound) ultimately causes the sixth root.

CONSTITUENT LEMMAS. In the rest of the section we will state and prove lemmas corresponding to each of the step of the proof described above. In Section 4.3 we apply them one at a time to obtain the specific bound claimed by Theorem 6.

**Lemma 1 (1ED-s is hard).** *If  $\mathcal{A}_{1ED-s}$  is a quantum algorithm for 1ED-s making at most  $q$  queries to its oracle and  $D \geq 32$ , then  $\text{Adv}_{D,3D^2}^{\text{ED-s}}(\mathcal{A}_{1ED-s}) \leq 9 \cdot (q+2)^3/D^2$ .*

*Proof.* In [34], Zhandry shows that no  $Q$ -query algorithm can distinguish between a random function and a random injective function with domain  $[D]$  and codomain  $[R]$  with advantage better than  $(\pi^2/3)Q^3/R$ , as long as  $D \leq R$ . We could build a distinguisher  $\mathcal{B}$  that on input  $L : [D] \rightarrow [R]$  runs  $(x, y) \leftarrow \mathcal{A}[L]$ . If  $(x, y)$  is a collision (i.e.,  $x \neq y$  and  $L(x) = L(y)$ ), then  $\mathcal{B}$  outputs 1. It checks this by making two additional  $L$  queries, so  $Q = q + 2$ . Otherwise it outputs zero. Clearly,  $(x, y)$  cannot be a collision when  $L$  is an injection so it will always output zero. Hence, if  $\text{1coll}$  denotes the event that  $L$  contains exactly one collision we obtain a bound of

$$\Pr[\text{1coll}] \cdot \text{Adv}_{D,3D^2}^{\text{ED-s}}(\mathcal{A}_{1ED-s}) \leq (\pi^2/3)(q+2)^3/R. \quad (1)$$

It remains to lower bound  $\Pr[\text{1coll}]$ . Note there are  $\binom{D}{2}$  possible pairs of inputs that could be collisions. Each pair has a  $1/R$  chance of colliding. By a union bound, the probability any other input has the same output as the collision is at most  $(D-2)/R$ , so there is at least a  $1 - (D-2)/R$  probability this does not occur. Given the above there are  $(D-2)$  inputs sampled from  $R-1$  possible values, so by a union bound the probability none of them collide is at least  $1 - \binom{D-2}{2}/(R-1)$ . This gives

$$\Pr[\text{1coll}] \geq \frac{D(D-1)}{2R} \cdot \left(1 - \frac{D-2}{R}\right) \cdot \left(1 - \frac{(D-2)(D-3)}{2(R-1)}\right).$$

Now setting  $R = 3D^2$  and applying simple bounds (e.g.  $D-2 < D-1 < D$  and  $(D-3)/(R-1) < (D-2)/R$ ) gives,

$$\Pr[\text{1coll}] \geq \frac{(1-1/D)}{6} \cdot \left(1 - \frac{1}{3D}\right) \cdot \left(1 - \frac{1}{6}\right)$$

Plugging this lower bound into equation 1, re-arranging, applying the bound  $D \geq 32$ , and rounding up to the nearest whole number gives the claimed bound  $\text{Adv}_{D,3D^2}^{\text{ED-s}}(\mathcal{A}_{1ED-s}) \leq 9(q+2)^3/D^3$ .  $\square$

**Lemma 2 (1ED-s hard  $\Rightarrow$  1LD-s hard).** *Let  $D$  be even. If  $\mathcal{A}_{1LD-s}$  is a quantum algorithm for 1LD-s making at most  $q$  queries to its oracle, then there is an algorithm  $\mathcal{A}_{1ED-s}$  (described in the proof) such that  $\text{Adv}_{D,R}^{\text{LD-s}}(\mathcal{A}_{1LD-s}) \leq 2\text{Adv}_{D,R}^{\text{ED-s}}(\mathcal{A}_{1ED-s})$ . Algorithm  $\mathcal{A}_{1ED-s}$  makes at most  $q$  queries to its oracle.*

*Proof.* On input a list  $L : [D] \rightarrow [R]$ , the algorithm  $\mathcal{A}_{1ED-s}$  will pick a random permutation  $\pi : [D] \rightarrow [D]$ . It runs  $\mathcal{A}_{1LD-s}[L \circ \pi]$  and then, on receiving output  $(x, y) \in [D/2]^2$ , returns  $(\pi^{-1}(x), \pi^{-1}(y + D/2))$ . The permutation  $\pi$  serves the role of splitting  $L$  into two sublists for  $\mathcal{A}_{1LD-s}$  at random. As long as the original collision of  $L$  doesn't end up being put into the same sublist (which has probability less than  $1/2$ ),  $\mathcal{A}_{1LD-s}$  will be run on a valid 1LD-s instance and  $\mathcal{A}_{1ED-s}$  will succeed whenever  $\mathcal{A}_{1LD-s}$  does. The claim follows.  $\square$

Algorithm  $\mathcal{A}_{1LD-s}[L_0, L_1]$   
 $L' \leftarrow \text{Inj}(D, R); L'_0 \parallel L'_1 \leftarrow L$   
 $i \leftarrow 0; L_0^0 \leftarrow L_0; L_1^0 \leftarrow L_1$   
Repeat  
   $i \leftarrow i + 1$   
   $L_{0,l} \parallel L_{0,r} \leftarrow L_0^{i-1}$   
   $L_{1,l} \parallel L_{1,r} \leftarrow L_1^{i-1}$   
   $(j^*, k^*) \leftarrow (r, r)$   
  For  $(j, k) \in \{(l, l), (l, r), (r, l)\}$  do  
    //  $f \square g(x) = f(x)$  for  $x \in \text{Dom}(f)$  else  $g(x)$   
     $b \leftarrow \mathcal{A}_{1LD-d}[L_{0,j} \square L'_0, L_{1,k} \square L'_1]$   
    If  $b = 1$  then  $(j^*, k^*) \leftarrow (j, k)$   
   $L_0^i \leftarrow L_{0,j^*}; L_1^i \leftarrow L_{1,k^*}$   
Until  $|\text{Dom}(L_0^i)| = |\text{Dom}(L_1^i)| = 1$   
Pick  $(x, y) \in \text{Dom}(L_0^i) \times \text{Dom}(L_1^i)$   
Return  $(x, y)$

**Fig. 10.** Reduction algorithm  $\mathcal{A}_{1LD-s}$  for Lemma 3. For notational convenience we write the two lists as separate input, rather than combined into a single list.

**Lemma 3 (1LD-s hard  $\Rightarrow$  1LD-d hard).** *Let  $D$  be a power of two. If  $\mathcal{A}_{1LD}$  is a quantum algorithm for 1LD making at most  $q$  queries to its oracle, then there is an 1LD-s algorithm  $\mathcal{A}_{1LD-s}$  (described in the proof) such that*

$$\text{Adv}_{D,R}^{1LD-s}(\mathcal{A}_{1LD-s}) \geq 1 - D^2/R - 1.5(\lg D - 2)(1 - \text{Adv}_{D,R}^{1LD-d}(\mathcal{A}_{1LD-d})).$$

*Algorithm  $\mathcal{A}_{1LD-s}$  makes at most  $3q \lg D$  queries to its oracle.*

This theorem's bound is vacuous if  $\text{Adv}_{D,R}^{1LD-d}(\mathcal{A}_{1LD-d})$  is too small. When applying the result we will have obtained  $\mathcal{A}_{1LD-d}$  by amplifying the advantage of another adversary to ensure it is sufficiently large.

*Proof.* The algorithm  $\mathcal{A}_{1LD-s}$  is given in Fig. 10. The intuition behind this algorithm is as follows. It wants to use the decision algorithm  $\mathcal{A}_{1LD-d}$  to perform a binary search to find the overlap between  $L_0$  and  $L_0$ . It runs for  $\lg D/2 - 1$  rounds. In the  $i$ -th round, it splits the current left list  $L_0^{i-1}$  into two sublists  $L_{0,l}, L_{0,r}$  and the current right lists  $L_1^{i-1}$  into two sublists  $L_{1,l}, L_{1,r}$ .<sup>13</sup> If  $L_0^{i-1}$  and  $L_1^{i-1}$  have an element in common, then one of the pairs of sublists  $L_{0,j}$  and  $L_{1,k}$  for  $j, k \in \{l, r\}$  must have an element in common. The decision algorithm  $\mathcal{A}_{1LD-d}$  is run on different pairs to determine which contains the overlap. We recurse with chosen pair, until we are left with lists that have singleton domains at which point we presume the entries therein give the overlap.

Because we need to run the decision algorithm  $\mathcal{A}_{1LD-d}$  on fixed size inputs we pad the sublists to be of a fixed size using lists  $L'_0$  and  $L'_1$  (the two halves of an injection  $L'$  that we sampled locally). As long as the image of  $L'$  does not overlap with the images of  $L_0$  and  $L_1$ , this padding does not introduce any additional elements repetitions between or within the list input to  $\mathcal{A}_{1LD-d}$ . By a union bound, overlaps occurs with probability at most  $D^2/R$ .

Conditioned on such overlaps not occurring,  $\mathcal{A}_{1LD-s}$  will output the correct result if  $\mathcal{A}'_{1LD-d}$  always answers correctly. To bound the probability of  $\mathcal{A}'_{1LD-d}$  erring we can note that it is run  $3 \cdot (\lg D - 2)$  times and, each time, has at most a  $(1 - \text{Adv}_{D,R}^{1LD-d}(\mathcal{A}'_{1LD-d}))/2$  chance of error and apply a union bound.

Put together we have

$$\text{Adv}_{D,R}^{1LD-s}(\mathcal{A}_{1LD-s}) \geq 1 - D^2/R - 1.5(\lg D - 2)(1 - \text{Adv}_{D,R}^{1LD-d}(\mathcal{A}_{1LD-d})).$$

<sup>13</sup> In code,  $f \parallel g \leftarrow h$  for  $h$  with domain  $\text{Dom}(h) = \{n, n+1, \dots, m\}$  defines  $f$  to be the restriction of  $h$  to domain  $\text{Dom}(f) = \{n, n+1, \dots, \lfloor (n+m)/2 \rfloor\}$  and  $g$  to be the restriction of  $h$  to domain  $\text{Dom}(g) = \text{Dom}(h) \setminus \text{Dom}(f)$ .

That  $\mathcal{A}_{1\text{LD-s}}$  makes at most  $3q(\lg D - 2)$  oracle queries is clear.  $\square$

**Lemma 4 (PB-d hard  $\Rightarrow$  PB hard).** *Let PB be any problem. Suppose  $\mathcal{A}_{\text{PB}}$  is a quantum algorithm for PB making at most  $q$  queries to its oracle, with  $\text{Adv}_{D,R}^{\text{PB}}(\mathcal{A}_{\text{PB}}) = \delta > 0$ . Then for any  $t \in \mathbb{N}$  there is an algorithm  $\mathcal{A}_{\text{PB-d}}$  (described in the proof) such that  $\text{Adv}_{D,R}^{\text{PB-d}}(\mathcal{A}_{\text{PB-d}}) > 1 - 2/2^t$ . Algorithm  $\mathcal{A}_{\text{PB-d}}$  makes  $q \cdot \lceil 4.5(t+1) \ln 2/\delta^2 \rceil$  queries to its oracle.*

*Proof.* For compactness, let  $p^1 = P_{D,R}^1(\mathcal{A}_{\text{PB}})$  denote the minimum probability that  $\mathcal{A}_{\text{PB}}$  outputs 1 on instances in the language,  $p^0 = P_{D,R}^0(\mathcal{A}_{\text{PB}})$  denote the maximum probability that  $\mathcal{A}_{\text{PB}}$  outputs 1 on instances not in the language, and  $\delta = \text{Adv}_{D,R}^{\text{PB}}(\mathcal{A}_{\text{PB}}) = p^1 - p^0$ . We define  $\mathcal{A}_{\text{PB-d}}$  to be the algorithm that, on input  $L$ , runs  $n$  independent copies of  $\mathcal{A}_{\text{PB}}[L]$  (with  $n = \lceil 4.5(t+1) \ln 2/\delta^2 \rceil$ ) and calculates the average  $p$  of all the values output by  $\mathcal{A}_{\text{PB}}[L]$ . (Think of this as an estimate of the probability  $\mathcal{A}_{\text{PB}}$  outputs 1 on this input.) If  $p < p^0 + \delta/2$ ,  $\mathcal{A}_{\text{PB-d}}$  outputs 0, otherwise it outputs 1.

Let  $X_i$  denote the output of the  $i$ -th execution of  $\mathcal{A}_{\text{PB-d}}$ . An inequality of Hoeffding [13] bounds how far the average of independent random variables  $0 \leq X_i \leq 1$  can differ from the expectation by

$$\Pr \left[ \left| \sum_{i=1}^n X_i/n - \mathbb{E} \left[ \sum_{i=1}^n X_i/n \right] \right| > \varepsilon \right] < 2e^{-2\varepsilon^2 n}$$

for any  $\varepsilon > 0$ . Let  $p'$  denote the expected value of  $p$  (which is also the expected value of  $X_i$ ). If  $L$  is in the language, then  $p^1 \leq p$ . Otherwise  $p^0 \geq p$ . In either case,  $\mathcal{A}_{\text{PB-d}}$  will output the correct answer if  $p$  does not differ from  $p'$  by more than  $\delta/3$  (because this is strictly less than  $\delta/2$ ). Applying the above inequality tells us that  $\Pr[|p - p'| > \delta/3] < 2e^{-2\delta^2 n/9} \leq 2^{-t}$ . (The value of  $n$  was chosen to make this last inequality hold.)

Hence  $P_{D,R}^1(\mathcal{A}_{\text{PB-d}}) > 1 - 2^{-t}$ ,  $P_{D,R}^0(\mathcal{A}_{\text{PB-d}}) < 2^{-t}$ , and  $\text{Adv}_{D,R}^{\text{PB-d}}(\mathcal{A}_{\text{PB-d}}) > 1 - 2/2^t$ . The bound on  $\mathcal{A}_{\text{PB-d}}$ 's number of queries is clear.  $\square$

### 4.3 Proof of Theorem 6

Let  $\mathcal{A}_{1\text{LD}}$  be our given list disjointness adversary which makes  $q$  oracle queries and has advantage  $\delta = \text{Adv}_{D,R}^{1\text{LD}}(\mathcal{A}_{1\text{LD}}) > 0$ . If we apply Lemma 4 with  $t = \lg(12 \lg D)$ , then we get an adversary  $\mathcal{A}_{1\text{LD-d}}$  which makes  $q_{1\text{LD-d}} = q \cdot \lceil 4.5(t+1) \ln 2/\delta^2 \rceil < (10q \lg \lg D)/\delta^2$  oracle queries. (We used here the assumption  $D \geq 32$  to simplify constants.) This adversary has advantage

$$\text{Adv}_{D,3D^2}^{1\text{LD-d}}(\mathcal{A}_{1\text{LD-d}}) > 1 - 2/2^t = 1 - 1/(6 \lg(D)).$$

Next applying Lemma 3, gives us  $\mathcal{A}_{1\text{LD-s}}$  which makes fewer than  $q_{1\text{LD-s}} = 30q(\lg D)(\lg \lg D)/\delta^2$  oracle queries and has advantage

$$\begin{aligned} \text{Adv}_{D,R}^{1\text{LD-s}}(\mathcal{A}_{1\text{LD-s}}) &\geq 1 - D^2/R - 1.5(\lg D - 2)(1 - \text{Adv}_{D,R}^{1\text{LD-d}}(\mathcal{A}_{1\text{LD-d}})) \\ &> 1 - 1/3 - 1.5(\lg D - 2)(6 \lg D)^{-1} > 1 - 1/3 - 1/4 = 5/12. \end{aligned}$$

Lemmas 1 and 2 together bound this advantage from the other direction. In particular, we get that

$$5/12 < \text{Adv}_{D,R}^{1\text{LD-s}}(\mathcal{A}_{1\text{LD-s}}) \leq 18 \cdot (q_{1\text{LD-s}} + 2)^3/D^2.$$

Using the assumption  $D \geq 32$  we can bound  $q_{1\text{LD-s}} + 2$  by  $31q(\lg D)(\lg \lg D)/\delta^2$ . Plugging this in and solving for  $\delta$  gives our claimed bound of

$$\delta < 11 \sqrt[6]{(q \cdot \lg D \cdot \lg \lg D)^3/D^2}.$$

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## References

1. S. Aaronson and Y. Shi. Quantum lower bounds for the collision and the element distinctness problems. *Journal of the ACM (JACM)*, 51(4):595–605, 2004.
2. A. Ambainis. Quantum walk algorithm for element distinctness. *SIAM Journal on Computing*, 37(1):210–239, 2007.
3. A. Ambainis, M. Hamburg, and D. Unruh. Quantum security proofs using semi-classical oracles. In A. Boldyreva and D. Micciancio, editors, *CRYPTO 2019, Part II*, volume 11693 of *LNCS*, pages 269–295. Springer, Heidelberg, Aug. 2019.
4. N. Bindel, M. Hamburg, K. Hövelmanns, A. Hülsing, and E. Persichetti. Tighter proofs of cca security in the quantum random oracle model. In *Theory of Cryptography Conference*, pages 61–90. Springer, 2019.
5. X. Bonnetain, A. Hosoyamada, M. Naya-Plasencia, Y. Sasaki, and A. Schrottenloher. Quantum attacks without superposition queries: the offline Simon’s algorithm. In *Advances in Cryptology – ASIACRYPT*, pages 552–583. Springer, 2019.
6. C. Chevalier, E. Ebrahimi, and Q.-H. Vu. On security notions for encryption in a quantum world. Cryptology ePrint Archive, Report 2020/237, 2020. <https://eprint.iacr.org/2020/237>.
7. J. Czajkowski. Quantum indistinguishability of sha-3. Cryptology ePrint Archive, 2021. <http://eprint.iacr.org/2021/192>.
8. J. Czajkowski, C. Majenz, C. Schaffner, and S. Zur. Quantum lazy sampling and game-playing proofs for quantum indistinguishability. Cryptology ePrint Archive, Report 2019/428, 2019. <https://eprint.iacr.org/2019/428>.
9. Ö. Dagdelen, M. Fischlin, and T. Gagliardoni. The Fiat-Shamir transformation in a quantum world. In K. Sako and P. Sarkar, editors, *ASIACRYPT 2013, Part II*, volume 8270 of *LNCS*, pages 62–81. Springer, Heidelberg, Dec. 2013.
10. W. Diffie and M. Hellman. Exhaustive cryptanalysis of the nbs data encryption standard. *IEEE Computer*. 1977.
11. S. Even and Y. Mansour. A construction of a cipher from a single pseudorandom permutation. *Journal of Cryptology*, 10(3):151–162, June 1997.
12. L. K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing*, STOC ’96, pages 212–219, New York, NY, USA, 1996. Association for Computing Machinery.
13. W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.
14. A. Hosoyamada and Y. Sasaki. Cryptanalysis against symmetric-key schemes with online classical queries and offline quantum computations. In *Cryptographers’ Track at the RSA Conference*, pages 198–218. Springer, 2018.
15. K. Hövelmanns, E. Kiltz, S. Schäge, and D. Unruh. Generic authenticated key exchange in the quantum random oracle model. In *IACR International Conference on Public-Key Cryptography*, pages 389–422. Springer, 2020.
16. M. Kaplan. Quantum attacks against iterated block ciphers. *arXiv preprint arXiv:1410.1434*, 2014.
17. M. Kaplan, G. Leurent, A. Leverrier, and M. Naya-Plasencia. Quantum differential and linear cryptanalysis. *IACR Trans. Symm. Cryptol.*, 2016(1):71–94, 2016. <https://tosc.iacr.org/index.php/ToSC/article/view/536>.
18. S. Katsumata, S. Yamada, and T. Yamakawa. Tighter security proofs for GPV-IBE in the quantum random oracle model. *Journal of Cryptology*, 34(1):1–46, 2021.
19. J. Kilian and P. Rogaway. How to protect DES against exhaustive key search. In N. Kobitz, editor, *CRYPTO’96*, volume 1109 of *LNCS*, pages 252–267. Springer, Heidelberg, Aug. 1996.
20. J. Kilian and P. Rogaway. How to protect DES against exhaustive key search (an analysis of DESX). *Journal of Cryptology*, 14(1):17–35, Jan. 2001.
21. E. Kiltz, V. Lyubashevsky, and C. Schaffner. A concrete treatment of Fiat-Shamir signatures in the quantum random-oracle model. In J. B. Nielsen and V. Rijmen, editors, *EUROCRYPT 2018, Part III*, volume 10822 of *LNCS*, pages 552–586. Springer, Heidelberg, Apr. / May 2018.
22. V. Kuchta, A. Sakzad, D. Stehlé, R. Steinfeld, and S.-F. Sun. Measure-rewind-measure: tighter quantum random oracle model proofs for one-way to hiding and cca security. In *Advances in Cryptology – EUROCRYPT 2020*, pages 703–728. Springer, 2020.
23. H. Kuwakado and M. Morii. Security on the quantum-type even-mansour cipher. In *2012 International Symposium on Information Theory and its Applications*, pages 312–316, 2012.
24. G. Leander and A. May. Grover meets simon - quantumly attacking the FX-construction. In T. Takagi and T. Peyrin, editors, *ASIACRYPT 2017, Part II*, volume 10625 of *LNCS*, pages 161–178. Springer, Heidelberg, Dec. 2017.

25. B. Mennink and A. Szeponiec. XOR of PRPs in a quantum world. In T. Lange and T. Takagi, editors, *Post-Quantum Cryptography - 8th International Workshop, PQCrypto 2017*, pages 367–383. Springer, Heidelberg, 2017.
26. R. C. Merkle and M. E. Hellman. On the security of multiple encryption. *Communications of the ACM*, 24(7):465–467, 1981.
27. A. Rosmanis. Tight bounds for inverting permutations via compressed oracle arguments. *arXiv preprint arXiv:2103.08975*, 2021.
28. P. W. Shor. Algorithms for quantum computation: Discrete logarithms and factoring. In *35th FOCS*, pages 124–134. IEEE Computer Society Press, Nov. 1994.
29. D. R. Simon. On the power of quantum computation. *SIAM journal on computing*, 26(5):1474–1483, 1997.
30. E. E. Targhi and D. Unruh. Post-quantum security of the Fujisaki-Okamoto and OAEP transforms. In M. Hirt and A. D. Smith, editors, *TCC 2016-B, Part II*, volume 9986 of *LNCS*, pages 192–216. Springer, Heidelberg, Oct. / Nov. 2016.
31. S. Tessaro and A. Thiruvengadam. Provable time-memory trade-offs: Symmetric cryptography against memory-bounded adversaries. In A. Beimel and S. Dziembowski, editors, *TCC 2018, Part I*, volume 11239 of *LNCS*, pages 3–32. Springer, Heidelberg, Nov. 2018.
32. D. Unruh. Revocable quantum timed-release encryption. *J. ACM*, 62(6), Dec. 2015.
33. M. Zhandry. How to construct quantum random functions. In *53rd FOCS*, pages 679–687. IEEE Computer Society Press, Oct. 2012.
34. M. Zhandry. A note on the quantum collision and set equality problems. *Quantum Information and Computation*, 15(7& 8), 2015.
35. M. Zhandry. How to record quantum queries, and applications to quantum indistinguishability. In A. Boldyreva and D. Micciancio, editors, *CRYPTO 2019, Part II*, volume 11693 of *LNCS*, pages 239–268. Springer, Heidelberg, Aug. 2019.

## A Probabilistic Analysis for Proof of Theorem 2

In this appendix we give a detailed probability analysis of Claim (i) from the proof of Theorem 2.

The view of  $\mathcal{A}$  when run by  $\mathcal{A}'$  in  $\mathbf{G}_{\mathbf{D},1}^{\text{dist}}$  is determined by  $f_1$ ,  $f_1^{-1}$ , and  $z = (T, T^{-1})$ . These are chosen such that  $T[M_i] = f_1(K_1, M_i \oplus K_2) \oplus K_2$  for  $i = 1, \dots, q'$  and  $T^{-1}[Y_i] = f_1^{-1}(K_1, Y_i \oplus K_2) \oplus K_2$  for  $i = q' + 1, \dots, q$ . Consequently to ensure this matches the view from  $\mathbf{G}_{\mathbf{F},1}^{\text{sprp}}$  we need to show that  $\mathbf{D}$  chooses  $(f_1, K_1, K_2)$  uniformly from  $\text{lcs}(k, n) \times \{0, 1\}^k \times \{0, 1\}^n$ . It is clear that  $K_1$  and  $K_2$  are uniform. Furthermore,  $f_1(K, \cdot) = f_0(K, \cdot)$  for  $K \neq K_1$  so these are sampled correctly. Hence we can think of these values as fixed and argue that the distribution induced over  $f_1(K_1, \cdot)$  is uniform over all permutations on  $\{0, 1\}^n$ .

Let  $\mathbf{f} \in \text{lcs}(k, n)$ ,  $N = 2^n$ , and  $E$  denote the event that  $f_1(K_1, \cdot) = \mathbf{f}(\cdot)$ . Let  $E_1$  denote the event that  $T[M_i] = \mathbf{f}(M_i \oplus K_2) \oplus K_2$  for  $i = 1, \dots, q$  after Step 1. The second for loop of Step 3 programs  $f_0$  to satisfy  $f_1(K_1, M_i \oplus K_2) = T[M_i] \oplus K_2$  so  $E_1$  is a necessary condition for  $E$ . Note that  $E_1$  requires  $Q$  values to have been sampled correctly in Step 1's for loops where  $Q = |\{\mathbf{f}(M_i \oplus K_2) \oplus K_2 : 1 \leq i \leq q'\} \cup \{\mathbf{f}(Y_i \oplus K_2) \oplus K_2 : q' \leq i \leq q\}|$ . Hence,

$$\Pr[E] = \Pr[E|E_1] \cdot \frac{(N - Q)!}{N!}.$$

Let  $E_2$  denote the event that Step 2 samples an  $f_0$  which is consistent with  $\mathbf{f}$ . That is to say, that  $f_0$  is chosen such that  $\mathbf{f}(\mathcal{I} \cup \mathcal{I}') = \mathcal{O} \cup \mathcal{O}'$  and  $\mathbf{f}(x) = f_0(K_1, x)$  for  $x \notin \mathcal{I} \cup \mathcal{I}'$ . The first for loop in Step 3 programs  $f_1(K_1, x) = f_0(K_1, x)$  for  $x \notin \mathcal{I} \cup \mathcal{I}'$ . The second and third for loop in Step 3 programs  $f_1(K_1, \mathcal{I} \cup \mathcal{I}')$  to have values in  $\mathcal{O} \cup \mathcal{O}'$  so  $E_2$  is a necessary condition for  $E$ . Let  $M(f_0) = |\mathcal{I} \setminus \mathcal{I}'| = |\mathcal{O} \setminus \mathcal{O}'|$  and let  $E_2^m$  denote the event that  $M(f_0) = m$ .

$$\Pr[E|E_1] = \sum_m \Pr[E_2^m|E_1] \cdot \Pr[E_2|E_1, E_2^m] \cdot \Pr[E|E_1, E_2^m, E_2].$$

We can think of Step 2 as lazily sampling  $f_0(K_1, \cdot)$  by first sampling  $f_0(K_1, x)$  for  $x \in \mathcal{I}$ , then sampling  $f_0^{-1}(K_1, y)$  for  $y \in \mathcal{O} \setminus \mathcal{O}'$ , and then sampling  $f_0(K_1, x)$  for  $x \notin \mathcal{I} \cup \mathcal{I}'$ . The event  $E_2$  requires that the  $m$

values sampled in this second step are the elements of  $\mathbf{f}^{-1}(\mathcal{O}' \setminus \mathcal{O})$  and that on the third step  $\mathbf{f}(x)$  is sampled for  $f_0(K_1, x)$  for each  $x \notin \mathcal{I} \cup \mathcal{I}'$ . Hence,

$$\Pr[E_2 | E_1, E_2^m] = \frac{1}{\binom{N-Q}{m}} \cdot \frac{1}{(N-Q-m)!}.$$

For  $E$  to occur (conditioned on  $E_1$ ,  $E_2^m$ , and  $E_2$ ) the third for loop in Step 3 must sample  $m$  values correctly, so  $\Pr[E | E_1, E_2^m, E_2] = 1/m!$ . Putting everything together we have

$$\Pr[E] = \frac{(N-Q)!}{N!} \sum_m \Pr[E_2^m | E_1] \cdot \frac{m! \cdot (N-Q-m)!}{(N-Q)!} \cdot \frac{1}{(N-Q-m)!} \cdot \frac{1}{m!} = \frac{1}{N!}.$$

So  $f_1$  is uniformly distributed, as desired.