Limits on the Adaptive Security of Yao's Garbling

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Abstract

Yao's garbling scheme is one of the most fundamental cryptographic constructions. Lindell and Pinkas (Journal of Cryptograhy 2009) gave a formal proof of security in the *selective* setting where the adversary chooses the challenge inputs before seeing the garbled circuit assuming secure symmetric-key encryption (and hence one-way functions). This was followed by results, both positive and negative, concerning its security in the, stronger, *adaptive* setting. Applebaum et al. (Crypto 2013) showed that it cannot satisfy adaptive security as is, due to a simple incompressibility argument. Jafargholi and Wichs (TCC 2017) considered a natural adaptation of Yao's scheme (where the output mapping is sent in the *online phase*, together with the garbled input) that circumvents this negative result, and proved that it is adaptively secure, at least for shallow circuits. In particular, they showed that for the class of circuits of depth δ, the loss in security is at most exponential in δ. The above results all concern the *simulation-based* notion of security.

In this work, we show that the upper bound of Jafargholi and Wichs is basically optimal in a strong sense. As our main result, we show that there exists a family of Boolean circuits, one for each depth $\delta \in \mathbb{N}$, such that *any* black-box reduction proving the adaptive *indistinguishability* of the natural adaptation of Yao's scheme from any symmetric-key encryption has to lose a factor that is exponential in $\sqrt{\delta}$. Since indistinguishability is a weaker notion than simulation, our bound also applies to adaptive simulation.

To establish our results, we build on the recent approach of Kamath et al. (Eprint 2021), which uses pebbling lower bounds in conjunction with oracle separations to prove fine-grained lower bounds on loss in cryptographic security.

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1 Introduction

A garbling scheme allows one to garble a circuit C and an input x such that only the output $\mathsf{C}(x)$ can be learned while everything else – besides some leakage such as the size or topology of the circuit – remains hidden. It was originally used by Yao as a means to achieve secure function-evaluation [Yao82, Yao86]. Despite its huge impact on cryptography, it was formally defined as a stand-alone primitive only much later by Bellare, Hoang and Rogaway [BHR12b]. In addition to a syntactic definition, they propose two different security notions for garbling schemes: simulatability and indistinguishability. They show the equivalence of the two definitions[1] in the presence of a *selective* adversary, which sends the circuit and input to be garbled in one shot. In contrast, for the more general case in which the adversary first – in an *offline* phase – chooses a circuit C and then (after receiving its garbling) – in the *online* phase – *adaptively* chooses its input x, the notion of indistinguishability turns out to be strictly weaker than simulatability. Many applications require security in such an adaptive setting, and for the sake of efficiency the cost during the online phase is to be kept minimal.

Prior work on security. Whilst there exist several constructions of provably-secure (even in the adaptive sense) garbling schemes (see Section 1.3), a feature of Yao's scheme (and variants thereof) is that security can be proven under the minimal assumption of one-way functions. At the same time, this scheme offers almost-optimal online complexity, with the size of the garbled input being linear in the input-size, and independent of the output- as well as circuit-size. A formal security proof of Yao's scheme in the *selective* setting was given by Lindell and Pinkas [LP09]. There exists a generic approach to reduce adaptive security to selective security: the adaptive reduction simply guesses the input x and then runs the selective reduction on the adaptive adversary. This, unfortunately, leads to a loss in security that is exponential in $|x|$. Furthermore, Applebaum et al. [AIKW13] showed that the *online complexity* of any adaptively-simulatable garbling scheme must exceed the output-size of the circuit, thereby proving a first limitation of Yao's scheme.

All of this led Jafargholi and Wichs [JW16] to consider a natural adaptation of Yao's garbling scheme (described in Section 1.1), where the mapping of output labels to output bits is sent in the online phase as part of the garbled input (see below for the construction). The negative result by Applebaum et al. does not apply to this adaptation of Yao's garbling scheme since its online complexity exceeds the output size. Therefore, this adaptation is the natural version of Yao's garbling scheme for the case of adaptive security, and is the scheme that we consider in this work and will simply refer to as "Yao's garbling" from now on. Jafargholi and Wichs [JW16] were able to show that it satisfies adaptive security for a wide class of circuits, including $\mathbf{NC}^1$ circuits. More precisely, they prove adaptive security of Yao's

[1] In the security game for simulatability, the simulator has to simulate $\tilde{\mathsf{C}}$ given only the output $y = \mathsf{C}(x)$ and some *leakage* $\Phi(\mathsf{C})$. While equivalence of selective simulatability and selective indistinguishability holds for the most natural leakage functions (e.g. the size or topology of C), it *does not* hold for arbitrary leakage functions Φ.

garbling via a black-box reduction to the IND-CPA security of the underlying symmetric-key encryption (SKE) scheme with a loss in security that is exponential in the *depth* of the circuit. Their proof employs a specially tailored *pebble game* on graphs, and can be seen as an application of the *piecewise-guessing framework* of Jafargholi et al. [JKK+17]. Since our work concerns the optimality of this proof, let's look at it in a bit more detail.

1.1 Yao's Scheme and Adaptive Indistinguishability

Let's first informally recall Yao's garbling scheme. A circuit $\mathsf{C} : \{0,1\}^n \to \{0,1\}^\ell$ is garbled in the offline phase as follows:

1. For each wire w in C, choose a pair of secret keys $k_w^0, k_w^1 \leftarrow \mathsf{Gen}(1^\lambda)$ for a SKE $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$.

2. For every gate $g : \{0,1\} \times \{0,1\} \to \{0,1\}$ with left input wire u, right input wire v, and output wire w, compute a garbling table $\tilde{\boldsymbol{g}}$ consisting of the following four ciphertexts (in a random order).

$$\begin{aligned} c_1 &:= \mathsf{Enc}_{k_u^0}(\mathsf{Enc}_{k_v^0}(k_w^{g(0,0)})) \qquad & c_2 &:= \mathsf{Enc}_{k_u^1}(\mathsf{Enc}_{k_v^0}(k_w^{g(1,0)})) \\ c_3 &:= \mathsf{Enc}_{k_u^0}(\mathsf{Enc}_{k_v^1}(k_w^{g(0,1)})) & c_4 &:= \mathsf{Enc}_{k_u^1}(\mathsf{Enc}_{k_v^1}(k_w^{g(1,1)})) \end{aligned} \tag{1}$$

3. If C has s wires and output wires denoted by $w_{s-\ell+1}, \ldots, w_s$, assemble the output mapping $\{k_w^b \to b\}_{i \in [s-\ell+1,s],\, b \in \{0,1\}}$.

The garbled circuit $\tilde{\mathsf{C}}$ consists of all the garbling tables $\tilde{\boldsymbol{g}}$ as well as the output mapping. To garble an input $x = (b_1, \ldots, b_n)$ in the online phase, simply set

$$\tilde{x} := (k_{w_1}^{b_1}, \ldots, k_{w_n}^{b_n})$$

where w_i denotes the ith input wire. The only difference in the variant from [JW16] is that the sending of the output mapping is moved to the online phase, which leads to an increase in the online complexity to linear in the input- *and* output-size.

To evaluate the garbled circuit on the garbled input, one requires the following *special property* of the SKE: For each ciphertext $c \leftarrow \mathsf{Enc}_k(m)$ there exists a unique key – namely k – such that decryption doesn't fail. Evaluation of the garbled circuit given the garbled input then works starting from the gates at the lowest level by simply trying which of the four ciphertexts can be decrypted using the two given input keys. This allows to recover exactly one of the two keys associated to the output wire of the respective gate and in the end the output mapping is used to map the sequence of revealed output keys to an output string $y \in \{0,1\}^\ell$.

Adaptive indistinguishability. A garbling scheme is adaptively indistinguishable if no efficient adversary can succeed in the following experiment[2] with non-negligible advantage:

1. The adversary submits a circuit C to the challenger, who responds with $\tilde{\mathsf{C}}$.
2. The adversary then submits a pair of inputs (x_0, x_1).
3. The challenger flips a coin b and responds with $\tilde{x}_b$.
4. The adversary wins if it guesses the bit b correctly.

In the following, we will refer to the two games for $b = 0$ and $b = 1$ as the "left" and "right" games, respectively.

To prove adaptive indistinguishability[3] of Yao's scheme for an arbitrary SKE (satisfying the special property), Jafargholi and Wichs construct a black-box reduction from the IND-CPA security of the SKE. More precisely, they proceed by a hybrid argument, where they define a sequence of hybrid games interpolating between the left and the right game such that each pair of subsequent hybrid games only differs in a single ciphertext (in the garbling table) and can be proven indistinguishable by relying on the IND-CPA security of the SKE.

The loss in security incurred by such a reduction then depends on the length of the sequence and the amount of information required to simulate the hybrid games. To end up with a meaningful security guarantee, thus, the sequence of hybrid games must not be too long and it must be possible to simulate any of the hybrid games without relying on too much information, particularly the knowledge of the entire input. Jafargholi and Wichs design such a sequence of hybrid games by using an appropriate pebble game on the topology graph underlying the circuit. In that game, a pebble on a gate indicates that the gate is not honestly garbled (as in Equation (1)) but is, instead, garbled in some *input-dependent* mode. The pebble rules, which dictate when a pebble can be placed on or removed from a vertex, guarantee that two subsequent hybrids can be proven indistinguishable, and the loss in security directly relates to the number of pebbles on the graph.

Keeping this proof technique in mind, the main idea of this work is to turn a pebble lower bound (w.r.t. an appropriate pebble game) into a lower bound on the security loss inherent to any black-box reduction of adaptive indistinguishability of Yao's scheme. Such an approach was recently adopted by Kamath et al. [KKPW21], also in the context of adaptive security but for primitives that are of a different flavour (e.g., multi-cast encryption). However, the case of garbled circuits turns out very different for several reasons we will highlight later (see Section 2.5).

[2] In fact, we define a *weaker* security notion than indistinguishability as defined in [BHR12b]; according to their definition the adversary can choose *two* circuits $\mathsf{C}_0, \mathsf{C}_1$ of the same topology and inputs x_0, x_1 such that $\mathsf{C}_0(x_0) = \mathsf{C}_1(x_1)$. Aiming at a lower bound on the gap between the security of Yao's scheme and the security of the underlying SKE, the additional restriction we put on our adversary only strengthens our results.

[3] To be precise, [JW16] prove the stronger security notion of simulatability, which implies indistinguishability.

1.2 Our Results

We prove a lower bound on the loss in security incurred by any black-box reduction proving adaptive indistinguishability of Yao's garbling scheme [JW16] from IND-CPA security of the SKE scheme. This immediately implies a similar lower bound with respect to the (stronger) more common security notion of adaptive simulatability. Our lower bound is subexponential in the depth d of the circuit, hence almost matches the best known upper bound from [JW16].

Theorem (main, Theorem 1)**.** *Any black-box reduction from adaptive indistinguishability (and thus also simulatability) of Yao's garbling scheme on the class of circuits with input length n and depth $\delta \leq 2n$ to the IND-CPA security of the underlying SKE loses at least a factor $\mathsf{loss} = \frac{1}{q} \cdot 2^{\sqrt{\delta}/61}$, where q denotes the number of times the reduction rewinds the adversary.*

Two remarks concerning the theorem are in order. Firstly, we are proving a negation of the statement in [JW16], which upper bounds loss for *every* graph in a class. Therefore, when we say that the class of circuits above loses at least a factor loss, we mean that there *exists* some circuit G in that class such that any reduction loses by that factor (and not that every circuit in that class loses by that factor). The design of this circuit G is one of the main technical contributions of this work. The second remark concerns the design of this circuit G. In addition to some structural properties that we will come to later, we design G to output the constant bit 0. This implies that the output mapping can easily be guessed by a reduction, and therefore the difference, in this case, between Yao's original scheme and [JW16] is only marginal.

Comparison with Applebaum et al. [AIKW13]. The result in [AIKW13] rules out adaptively-simulatable randomised encodings with online complexity less than the output-size of the function it encodes. Since Yao's garbling is one instantiation of randomised encodings, their result immediately rules out its adaptive simulatability. However, [AIKW13] does not apply to our setting for three reasons. Firstly, their result only applies to the original construction of Yao's garbled circuits where the garbled input can be smaller than the output size. In this work we consider the adaptation of Yao's garbling scheme [JW16] where the output mapping is sent in the online phase, hence the online complexity always exceeds the output size. Secondly, their result applies to circuits with large output, while our result holds even for Boolean circuits with outputs of length 1. Finally, their result only applies to simulation security, while our result even holds for indistinguishability.

Comparison with Hemenway et al. [HJO+16]. We would like to emphasise that our lower bound only holds for the *specific* construction of Yao's garbled circuits, and it does not rule out other constructions, even potentially from one-way functions. In fact, the construction of Hemenway et al. already circumvents our result and it is instructive to see how. On a high level, their idea (similar to [BHR12a]) is to take Yao's garbling scheme and then encrypt all the resulting garbling tables with an additional layer of "somewhere equivocal" encryption on top. This change allows them to prove adaptive security with only

a *polynomial* loss in security (at the cost of increased online complexity). The intuitive reason why our approach does not apply to this construction is that the additional layer of encryption somehow "blurs out" all the details about the individual garbling tables, on which our argument depends (see Section 2.4).

1.3 Further Related Work on Adaptive Security

Adaptive security for garbled circuits. The problem of constructing adaptively-secure garbling schemes was first raised by Bellare, Hoang and Rogaway in [BHR12a]; they gave a first adaptively-secure construction in the random oracle model, which bypasses the lower bound of Applebaum et al. [AIKW13]. Bellare, Hoang and Keelveedhi [BHK13] then proved the previous scheme adaptively-secure in the standard model, but under non-standard assumptions on hash functions. Further constructions from various assumption followed: Boneh et al. [BGG+14] constructed an adaptively-secure scheme from the learning with errors (LWE) assumption, where the online complexity depends on the depth of the circuit family. Ananth and Sahai [AS16] constructed an optimal garbling scheme from iO. Later, Ananth and Lombardi [AL18] constructed succinct garbling schemes from functional encryption. In [JSW17], Jafargholi et al. relax the simulation-based security to *indistinguishability* and show how to construct adaptively-secure garbling schemes from the minimal assumption of one-way functions, where the online complexity only depends on the pebble complexity and the input-size, but is *independent* of the output-size. A particularly strong result in this area was due to Garg and Srinivasan [GS18], who constructed adaptively-secure garbling with near optimal online complexity that can be based on standard assumptions such as the computational Diffie-Hellman (CDH), the factoring, or the LWE assumption. While this list is far from complete, we finally mention a recent work by Jafargholi and Oechsner [JO20] who analyze adaptive security of several practical garbling schemes. They give positive as well as negative results, and argue why the techniques from [JW16] cannot be applied to certain garbling schemes.

Adaptive security for other graph-based games. Jafargholi et al. gave a framework for proving adaptive security [JKK+17], also known as *piecewise guessing* technique. Beside several applications to other *graph-based* security games, this framework also comprises the reduction from [JW16] as a special case. Kamath et al. [KKPW21] considered optimality of this approach for certain graph-based games which arise in the context of e.g., multicast encryption, continuous group key agreement, and constrained PRF. They gave non-trivial fine-grained lower bounds on the loss in adaptive security incurred by (oblivious) reductions via *pebble lower bounds*.

2 Technical Overview

We aim to prove *fine-grained* lower bounds on loss in security incurred by black-box reductions in a setting where a primitive F is used in a protocol Π^F. In our case F is SKE

and Π^F is Yao's garbling scheme using the SKE. In order to bound loss, the loss in security incurred by any efficient black-box reduction R that breaks F when given black-box access to an adversary that breaks Π^F (i.e., from F to Π^F), we must show that for *every* R, there exists

- an instance F (not necessarily efficiently-implementable) of F and
- an adversary A (not necessarily efficient) that breaks Π^{F}

such that loss in security incurred by R in breaking F is at least loss.[4] We next describe how the instance and the adversary are defined in our setting.

2.1 Our Oracles

We define two oracles $\mathcal{F}$ and $\mathcal{A}$ implementing an ideal SKE and an adversary, respectively, such that

- the SKE scheme $\mathcal{F} = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ satisfies IND-CPA security *information-theoretically*,
- the (inefficient) adversary $\mathcal{A}$ breaks indistinguishability of the garbling scheme $\Pi^{\mathcal{F}}$, but is not helpful in breaking the IND-CPA security of $\mathcal{F}$.

Ideal encryption. We will define the ideal SKE oracle $\mathcal{F}$ such that Enc is defined through a random expanding function (which is injective with overwhelming probability). Since the security of $\mathcal{F}$ is information-theoretic, any advantage against IND-CPA which a reduction with oracle access to $\mathcal{F}$ and $\mathcal{A}$ obtains must stem (almost) entirely from the interaction with $\mathcal{A}$. This is true since the reduction can only make polynomially many queries and thus the probability that the answer to one of its oracle queries coincides with the IND-CPA challenge is negligible. On the other hand, a computationally unbounded adversary using an unlimited number of queries can break the scheme and (thanks to injectivity) perfectly recover messages and secret keys from any ciphertext.

The adversary. As for the (inefficient) adversary $\mathcal{A}$, we define a so-called *threshold* adversary which does the following in the indistinguishability game:

1. $\mathcal{A}$ chooses a particular circuit G (see Section 2.3) which has constant output (bit) 0 and sends G to the challenger.
2. After receiving the garbled circuit $\tilde{\mathsf{G}}$, $\mathcal{A}$ chooses garbling inputs x_0 and x_1 uniformly at random and sends them to the challenger. Note that $\mathsf{G}(x_0) = \mathsf{G}(x_1)$ trivially holds since G has constant output.

[4] This is obtained by simply negating the definition of a black-box reduction: *there exists an efficient reduction* R *for every implementation (not necessarily efficient)* F *of* F *and for every (not necessarily efficient) adversary* A *that breaks* Π^{F} *such that the loss in security is at most* loss.

3. On receipt of the garbled input $\tilde{x}_b$ along with an output mapping, $\mathcal{A}$ first runs some initial checks on $(\tilde{\mathsf{G}}, \tilde{x}_b)$ to verify that the garbling has the correct syntax, and then *extracts a pebble configuration* $\mathcal{P}$ on G (see Section 2.4). That is, every gate in G is either assigned a pebble or not, depending on the content of its garbling table in $\tilde{\mathsf{G}}$ and the garbled input $\tilde{x}_b$. To compute this mapping, the inefficient adversary $\mathcal{A}$ simply breaks the underlying encryption by brute force. Finally, $\mathcal{A}$ outputs 0 (denoting 'left') if the extracted pebble configuration is *good* (defined later through some pebble game), and 1 (denoting 'right') otherwise.

By design, the left indistinguishability game (where $b = 0$) will correspond to a good configuration, whereas the right game will not. Therefore the above adversary is a valid distinguisher for the indistinguishability game (Lemma 6). Moreover, $\mathcal{A}$ concentrates all its distinguishing advantage at the *threshold* of good and bad configurations (hence the name). Therefore, intuitively speaking, for any reduction to exploit $\mathcal{A}$'s distinguishing advantage, it must somehow embed its own (IND-CPA) challenge at the threshold. All the technicality in proving our main theorem goes into formalising this intuition, which we summarise next in Section 2.2.

2.2 High-Level Idea

To prove a lower bound on loss (Theorem 1), we construct a *punctured* adversary $\mathcal{A}[c^*]$ (see Section 4.5) which behaves similar to $\mathcal{A}$ *except* when it comes to the hardcoded challenge ciphertext $c^* \leftarrow \mathsf{Enc}_{k^*}(m)$ (for some arbitrary message m). We aim to puncture $\mathcal{A}[c^*]$ such that it never decrypts c^* but instead just proceeds by assuming that c^* decrypts to the all-0 string, and hence cannot be of any help to a reduction that aims to break c^*. However, we have to be careful here since the reduction embedding c^* in $\tilde{\mathsf{G}}$ will also embed other ciphertexts under key k^* (which it can derive through querying its IND-CPA encryption oracle Enc_{k^*}), and hence $\mathcal{A}[c^*]$ would learn the key k^* when brute-force decrypting these ciphertexts. We solve this issue by endowing $\mathcal{A}[c^*]$ with a decryption oracle Dec_{k^*} that allows to find and decrypt those ciphertexts under k^*. Since our ideal encryption scheme actually satisfies the stronger notion of IND-CCA security, this decryption oracle is of no help to the reduction.

The core of our lower bound is now to define the circuit G and the notion of *good* pebble configurations such that the following holds:

- Our threshold adversary $\mathcal{A}$ indeed breaks the garbling scheme.
- It is hard to distinguish $\mathcal{A}$ from $\mathcal{A}[c^*]$.

For the latter property, note that any efficient reduction R can only distinguish $\mathcal{A}$ from $\mathcal{A}[c^*]$ if their outputs differ, which only happens if they extract different pebbling configurations $\mathcal{P} \neq \mathcal{P}^*$ such that one of them is good and the other bad. Thus, to bound the success probability of R, it suffices to establish the following two properties:

1. The pebbling configurations $\mathcal{P}$ and $\mathcal{P}^*$ extracted by $\mathcal{A}$ and $\mathcal{A}[c^*]$ (in the same execution of the game, using the same randomness) differ by at most one valid pebbling move in some natural pebble game[5], where a pebble can be placed on or removed from a gate if at least one of its parent gates carries a pebble.

2. It is hard for any reduction to produce $(\tilde{\mathsf{G}}, \tilde{x})$ such that $\mathcal{A}$ extracts a *threshold* configuration, i.e. a pebble configuration that is good but can be switched to a bad configuration within one valid pebbling move.

Intuitively, pebbles on gates in the circuit represent malformed gates, i.e., gates whose garbling table is different from the honest garbling table. When considering circuits consisting only of non-constant gates, the pebbling rule in Property 1 captures the fact that a reduction cannot produce ciphertexts encrypting the key k^* under which its challenge ciphertext $c^* \leftarrow \mathsf{Enc}_{k^*}(m)$ (for some arbitrary m) was encrypted. Hence, in order to embed c^* at a gate, the reduction has to first output a malformed garbling (not encoding k^*) for its predecessor gate. Now, to see why Property 1 holds – i.e., the pebbling configurations $\mathcal{P}$ and $\mathcal{P}^*$ extracted by $\mathcal{A}$ and $\mathcal{A}[c^*]$ follow the same dynamics – note that the behaviour of $\mathcal{A}$ and $\mathcal{A}[c^*]$ can only differ if k^* is not encrypted in any ciphertext.

The tricky part of our proof is to establish Property 2 which, on a high level, works as follows. For a reduction R to simulate a threshold configuration we first force it to maul – and hence pebble – several gates. Then, for this mauling to go 'undetected' we force R to correctly guess the value of these gates when G is evaluated at x_0. This, intuitively, will be the source of its loss. To this end, we design our circuit G to consist of two blocks[6], $\mathsf{G}^{\oplus}$ and $\mathsf{G}^{\wedge}$. Looking ahead, whether there is a pebble on a gate in $\mathsf{G}^{\oplus}$ will be *independent* of the input and correspond to R's attempt at guessing x_0 (this relies on the properties of XOR gates). The pebbles on $\mathsf{G}^{\wedge}$, in contrast, will be extractable with respect to the input garbling $\tilde{x}_b$ and indicate whether or not the guesses on x_0 in the $\mathsf{G}^{\oplus}$ block were correct (this relies on the properties of AND gates). Moreover, by definition:

- In case of a proper garbling of (G, x_0) (i.e., the left game), the adversary $\mathcal{A}$ will not extract any pebble on $\mathsf{G}^{\oplus}$ or $\mathsf{G}^{\wedge}$.

- In case of a proper garbling of (G, x_1) (i.e., the right game), on the other hand, the adversary $\mathcal{A}$ will not extract any pebbles on $\mathsf{G}^{\oplus}$, but will extract some pebbles on $\mathsf{G}^{\wedge}$ (since $x_1 \neq x_0$).

Accordingly, we define the *good* predicate such that the empty configuration is good, whereas any configuration containing a pebble on $\mathsf{G}^{\wedge}$ is bad, and therefore the above ensures that $\mathcal{A}$ breaks the security of the garbling scheme. Furthermore, the threshold configurations contain many pebbles on $\mathsf{G}^{\oplus}$, but no pebbles on $\mathsf{G}^{\wedge}$. In other words, threshold configurations require R to make many guesses about x_0 and all of them need to be correct, which is unlikely to occur. This establishes Property 2.

[5] In Section 4.3 we actually consider a much more finegrained pebble game, where different types of pebbles represent different garbling modes of a gate. For this exposition, it suffices to focus on this simplified game.

[6] For this high-level overview, we ignore the third block G^0 consisting of a binary tree of AND gates, whose sole purpose is to guarantee constant 0 (bit) output.

2.3 The Circuit G and the Good Predicate

The design of topology of the circuit $\mathsf{G}^{\oplus}$ is such that it has high pebbling complexity with respect to our pebble game: i.e., every valid pebbling sequence starting from the *initial* empty configuration and reaching a *final* configuration that has a pebble on an output gate of $\mathsf{G}^{\oplus}$, must contain a "heavy" configuration with many, say d, pebbles. To guarantee that threshold configurations contain many pebbles, we define the good configurations as those that are reachable with $d-1$ pebbles following valid pebbling moves. Since $\mathsf{G}^{\wedge}$ will (topologically) succeed $\mathsf{G}^{\oplus}$ in G, any configuration with a pebble on $\mathsf{G}^{\wedge}$ is in particular bad (since an output gate of $\mathsf{G}^{\oplus}$ must have been pebbled first). At the same time, to allow for our "control mechanism", we construct G so that each gate g in $\mathsf{G}^{\oplus}$ has a 'companion' successor gate in $\mathsf{G}^{\wedge}$ that helps check correctness of g's output. Thus for each AND gate in $\mathsf{G}^{\wedge}$, one of the inputs comes from the output of $\mathsf{G}^{\oplus}$ and the other from the output of its companion gate (see Figure 1). This fixes the topology of G and we choose the type of gate as to enforce Property 2, as explained below.

- The $\mathsf{G}^{\oplus}$ circuit is composed only of XOR gates, since these gates allow us to maintain *high entropy* (of the input), and hence guarantee that it is hard to guess the outputs of the pebbled gates in $\mathsf{G}^{\oplus}$ (see Section 4.2). Furthermore, XOR gates are *symmetric* with respect to their input in the sense that from the garbling table alone even an inefficient adversary cannot distinguish which keys are associated with which bits. This property allows $\mathcal{A}$ to extract the pebbling configuration of $\mathsf{G}^{\oplus}$ just from $\tilde{\mathsf{G}}$, independently of the input (see next section).

- The $\mathsf{G}^{\wedge}$ circuit, on the other hand, is composed of AND gates. Since AND gates are *asymmetric* (since only $(1,1)$ maps to 1, while all three other input pairs map to 0), we can use them to detect errors in the $\mathsf{G}^{\oplus}$ circuit: i.e., looking at a garbling table of an AND gate our adversary $\mathcal{A}$ can exploit this asymmetry to easily associate keys to bits. Thus, whenever during evaluation of $\tilde{\mathsf{G}}$ on input $\tilde{x}$ the adversary $\mathcal{A}$ receives wrong input keys for a (properly garbled) AND gate, $\mathcal{A}$ considers this gate as malformed and associates it with a pebble. (The case of AND gates which are not properly garbled is rather technical and we refer the reader to Section 4.4.)

2.4 Extracting the Pebble Configuration

Since it is central to the working of our adversary $\mathcal{A}$ (and is a somewhat subtle matter), here we provide a high-level description of the extraction mechanism.[7] First of all, recall that pebbles on $\mathsf{G}^{\oplus}$ and $\mathsf{G}^{\wedge}$ have different meanings: a pebbled XOR gate indicates that its garbling table is *malformed* whereas a pebbled AND gate indicates that R's guess for the companion

[7] In Section 4.4 we consider a more general extraction mechanism that can be extended to arbitrary gates and assigns different types of pebbles, representing the "distance" of a garbling table $\tilde{\boldsymbol{g}}'$ for a gate g from an honest garbling table $\tilde{\boldsymbol{g}}$. For ease of exposition, here we consider a simplified pebble game and only discuss how to extract pebbles for XOR and AND gates, where a pebble in this simplified game would correspond to different sets of pebbles for XOR and AND gates in the more fine-grained pebble game.

XOR gate is *wrong*. This, coupled with the fact that the gates have differently-structured gate tables (i.e., symmetric vs. asymmetric) means that the extraction mechanism for the two gates (and hence the blocks) is also different. In particular, as we will see, the pebble status of an XOR gate is something that can inferred solely from the garbled circuit $\tilde{\mathsf{G}}$ (and thus can be done in the offline phase) whereas the pebble status of an AND gate is something that also depends on the garbled input $\tilde{x}$ and is necessarily done in the online phase. Let's look at how the respective extraction is carried out. First, given $\tilde{\mathsf{G}}$, $\mathcal{A}$ extracts a key pair for each wire in G from the encryptions associated with its *successor* gates, or the output mapping; if this cannot be done uniquely, $\mathcal{A}$ aborts and outputs 1 (we refer to Section 4.4 for more details). In the following, for a gate g, let u and v denote the input wires, w the output wire, and k_u, k'_u, k_v, k'_v, k_w, k'_w the corresponding keys associated with these wires.

- If g is an XOR gate, then the honest garbling table of g can be derived from Equation (1) as

$$\begin{array}{ll} \mathsf{Enc}_{k_u}(\mathsf{Enc}_{k_v}(k_w)) & \mathsf{Enc}_{k'_u}(\mathsf{Enc}_{k_v}(k'_w)) \\ \mathsf{Enc}_{k_u}(\mathsf{Enc}_{k'_v}(k'_w)) & \mathsf{Enc}_{k'_u}(\mathsf{Enc}_{k'_v}(k_w)). \end{array}$$

  Whenever a garbling table $\tilde{\boldsymbol{g}}$ differs from this representation (i.e., not symmetric), $\mathcal{A}$ assigns g a pebble and this assignment is *independent* of the bits running over the wires u, v, w and the keys revealed during evaluation. Thus, $\mathcal{A}$ can extract pebbles on $\mathsf{G}^{\oplus}$ already *before* it chose the inputs x_0, x_1, in particular independently of $\tilde{x}$.

- For an AND gate g, on the other hand, the garbling table of g consists of four ciphertexts derived from Equation (1) as

$$\begin{array}{ll} \mathsf{Enc}_{k_u}(\mathsf{Enc}_{k_v}(k_w)) & \mathsf{Enc}_{k'_u}(\mathsf{Enc}_{k_v}(k_w)) \\ \mathsf{Enc}_{k_u}(\mathsf{Enc}_{k'_v}(k_w)) & \mathsf{Enc}_{k'_u}(\mathsf{Enc}_{k'_v}(k'_w)). \end{array}$$

  Since the roles of the keys are *asymmetric*, the pebble extraction will depend on the bits b_u, b_v, b_w running over the wires and the keys k^r_u, k^r_v, k^r_w revealed during evaluation. A first attempt would be to simply map keys to bits as $k_u, k_v, k_w \to 0$ and $k'_u, k'_v, k'_w \to 1$, and assign g a pebble if $k^r_\eta \not\to b_\eta$ for some $\eta \in \{u, v, w\}$. Unfortunately, this simple idea does not work since a reduction R might embed its challenge ciphertext $c^* \leftarrow \mathsf{Enc}_{k^*}(m)$ in the garbling of an AND gate (recall from Section 2.3 that the gates in $\mathsf{G}^{\wedge}$ receive one input from an output gate of $\mathsf{G}^{\oplus}$ and the other input from their companion gate *within* the circuit $\mathsf{G}^{\oplus}$). Now, if R embeds the challenge key k^* at an output wire of $\mathsf{G}^{\oplus}$, it must pebble an output gate in $\mathsf{G}^{\oplus}$, hence end up with a bad pebbling configuration independently of c^*. However, this is not true if R embeds k^* at the other input wire of the AND gate. Thus, $\mathcal{A}$ must not extract a pebble for a garbling table that can be derived from an honest garbling table by embedding a challenge key at this wire. We show in Section 4.4 that such malformed garblings of AND gates either involve guessing the input bits or they can still be used for our "control mechanism".

2.5 Comparison with [KKPW21]

While both, [KKPW21] and our work, model choices made by a reduction by putting pebbles on a graph structure, the analogy basically ends there. In [KKPW21] an interactive game between a "builder" and a "pebbler" is considered in which the builder chooses edges and the pebbler decides adaptively whether to pebble them. The goal of the pebbler is to get into a "good" configuration, and the difficulty for the reduction (playing the role of the pebbler) there lies in the fact that the graph is only revealed edge-by-edge. In contrast, in this work the graph structure is initially known and the game has just two rounds. The difficulty for the reduction here comes from having to guess the bits running over a subset of wires during evaluation of the circuit. None of the main ideas from [KKPW21] seem applicable in this setting and vice versa. For example, most of the results in [KKPW21] are restricted to the limited class of so-called oblivious reductions, while our setting doesn't share the difficulties encountered in [KKPW21]; in particular, our result holds for arbitrary black-box reductions.

3 Preliminaries

3.1 Notation and Definitions

For integers $m, n \in \mathbb{N}$ with $m < n$, let $[n] := \{1, 2, \ldots, n\}$, $[n]_0 := \{0, 1, \ldots, n\}$, and $[m, n] := \{m, m+1, \ldots, n\}$. For two sets $\mathcal{S}, \mathcal{S}'$ we write $\mathcal{S} \subset \mathcal{S}'$ if $\mathcal{S}$ is a (not necessarily strict) subset of $\mathcal{S}'$. Furthermore, let log be always base 2. We recap the widely used notions of IND-CPA and IND-CCA security of a symmetric encryption (SKE scheme): Let $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ be a symmetric encryption scheme with $\mathcal{K}$ the image of Gen, $\mathsf{Enc}\colon \mathcal{K} \times \mathcal{M} \to \mathcal{C}$, $\mathsf{Dec}\colon \mathcal{K} \times \mathcal{C} \to \mathcal{M}$ and we assume $\mathcal{K}, \mathcal{C} \subseteq \mathcal{M}$ (i.e., we can encrypt keys and ciphertexts). We assume that the scheme is correct, i.e.,

$$\forall k \in \mathcal{K}, m \in \mathcal{M} \ : \ \mathsf{Pr}[\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = m] = 1$$

and that it is ε-indistinguishable under chosen-plaintext attack (IND-CPA):

Definition 1 (IND-CPA)**.** The game is played between a challenger (either G_0 or G_1) and an adversary on the symmetric encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$. The challenger chooses the challenge key $k \leftarrow \mathsf{Gen}(1^\lambda)$ for a security parameter λ. The adversary can make two types of queries:

- Encryption queries $(\texttt{encrypt}, m)$, $m \in \mathcal{M}$: the challenger returns $\mathsf{Enc}_k(m)$.
- One challenge query $(\texttt{challenge}, m_0, m_1)$, $m_0, m_1 \in \mathcal{M}$: the challenger when simulating G_b returns the challenge ciphertext $c^* \leftarrow \mathsf{Enc}_k(m_b)$.

An encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is said to be IND-CPA secure, if no PPT adversary can distinguish G_0 from G_1 with non-negligible probability (in λ).

Definition 2 (IND-CCA)**.** The games G_0 and G_1 are defined similar to Definition 1 except that the adversary can make an additional type of queries:

- Decryption queries $(\texttt{decrypt}, c)$, $c \in \mathcal{C}$: the challenger returns $m := \mathsf{Dec}_k(c)$ if $c \neq c^*$, otherwise it returns $\perp$.

An encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is said to be IND-CCA, if no PPT adversary can distinguish G_0 from G_1 with non-negligible probability (in λ).

3.2 Garbling schemes

The definitions are taken mostly from [JSW17]; more details can be found in [BHR12b].

Definition 3. A garbling scheme $\mathbf{GC}$ is a tuple of PPT algorithms $(\mathsf{GCircuit}, \mathsf{GInput}, \mathsf{GEval})$ with syntax and semantics defined as follows.

$(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(1^\lambda, \mathsf{C})$**.** On inputs a security parameter λ and a circuit $\mathsf{C} : \{0,1\}^n \to \{0,1\}^\ell$, the *garble-circuit* algorithm $\mathsf{GCircuit}$ outputs the *garbled circuit* $\tilde{\mathsf{C}}$ and *key* K.

$\tilde{x} \leftarrow \mathsf{GInput}(K, x)$**.** On input an input $x \in \{0,1\}^n$ and key K, the *garble-input* algorithm GInput outputs $\tilde{x}$.

$y = \mathsf{GEval}(\tilde{\mathsf{C}}, \tilde{x})$**.** On input a garbled circuit $\tilde{\mathsf{C}}$ and a garbled input $\tilde{x}$, the *evaluate* algorithm GEval outputs $y \in \{0,1\}^\ell$.

3.2.1 Correctness.

There is a negligible function $\epsilon = \epsilon(\lambda)$ such that for any $\lambda \in \mathbb{N}$, any circuit C and input x it holds that

$$\mathsf{Pr}\left[\mathsf{C}(x) = \mathsf{GEval}(\tilde{\mathsf{C}}, \tilde{x})\right] = 1 - \epsilon(\lambda),$$

where $(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(1^\lambda, \mathsf{C})$, $\tilde{x} \leftarrow \mathsf{GInput}(K, x)$.

We recall two different security notions: the stronger and typically considered notion based on simulatability, and a weaker indistinguishability based notion which was considered in [JSW17]. It is easy to see that simulatability implies indistinguishability (cf. [BHR12b]).

Definition 4 (Adaptive Simulatability.)**.** A garbling scheme $\mathbf{GC}$ is (ϵ, T)-adaptively-simulatable for a class of circuits $\mathcal{C}$, if there exists a PPT time simulator $\mathsf{S} = (\mathsf{SCircuit}, \mathsf{SInput})$ such that, for any probabilistic adversary A of size $T = T(\lambda)$,

$$\left|\mathsf{Pr}\left[\mathsf{F}_{\mathsf{A},\mathbf{GC},\mathsf{S}}(1^\lambda, 0) = 1\right] - \mathsf{Pr}\left[\mathsf{F}_{\mathsf{A},\mathbf{GC},\mathsf{S}}(1^\lambda, 1) = 1\right]\right| \leq \epsilon(\lambda),$$

where the experiment $\mathsf{F}_{\mathsf{A},\mathbf{GC},\mathsf{S}}(1^\lambda, b)$ is defined as follows:

1. The adversary A specifies a circuit $\mathsf{C} \in \mathcal{C}$ with underlying graph structure $\Phi(\mathsf{C})$ and gets $\tilde{\mathsf{C}}$ created as follows:
   - if $b = 0$: $(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(1^\lambda, C)$,

- if $b = 1$: $(\tilde{\mathsf{C}}, z) \leftarrow \mathsf{SCircuit}(1^\lambda, \Phi(C))$.

2. The adversary A specifies x and gets $\tilde{x}$ created as follows:
   - if $b = 0$, $\tilde{x} \leftarrow \mathsf{GInput}(k, x)$,
   - if $b = 1$, $\tilde{x} \leftarrow \mathsf{SInput}(C(x), z)$.
3. Finally, the adversary outputs a bit b', which is the output of the experiment.

Definition 5 (Adaptive Indistinguishability)**.** A garbling scheme $\mathbf{GC}$ is (ϵ, T)-adaptively-indistinguishable for a class of circuits $\mathcal{C}$, if for any probabilistic adversary A of size $T = T(\lambda)$,

$$\left|\mathsf{Pr}\left[\mathsf{Game}_{\mathsf{A},\mathbf{GC}}(1^\lambda, 0) = 1\right] - \mathsf{Pr}\left[\mathsf{Game}_{\mathsf{A},\mathbf{GC}}(1^\lambda, 1) = 1\right]\right| \leq \epsilon(\lambda).$$

where the experiment $\mathsf{Game}_{\mathsf{A},\mathbf{GC},\mathsf{S}}(1^\lambda, b)$ is defined as follows:

1. A selects a circuits $\mathsf{C} \in \mathcal{C}$ and receives $\tilde{\mathsf{C}}$, where $(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(1^\lambda, \mathsf{C})$.
2. A specifies x_0, x_1 such that $\mathsf{C}(x_0) = \mathsf{C}(x_1)$ and receives $\tilde{x}_b \leftarrow \mathsf{GInput}(x_b, K)$.
3. Finally, A outputs a bit b', which is the output of the experiment.

In the indistinguishability game as defined in [BHR12b] the adversary can select *two* circuits $\mathsf{C}_0, \mathsf{C}_1$ of the same topology and receives a garbling $\tilde{\mathsf{C}}_b$ of one of them. The choice of input x_0, x_1 is then restricted to satisfy $\mathsf{C}_0(x_0) = \mathsf{C}_1(x_1)$. Our notion of indistinguishability is clearly weaker, which strengthens our lower bound.

3.3 Yao's garbled circuit

In Algorithm 1 we describe the variant [JW16] of Yao's garbling scheme $\Pi^{\mathcal{F}}$ based on a symmetric encryption scheme $\mathcal{F}$ with the special property defined below. Recall that in contrast to the original scheme, here the output map is sent along with the garbled input in the online phase.

Definition 6 (Special Property of Encryption)**.** We say an encryption scheme $\mathcal{F} = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ satisfies the special property if for every security parameter λ, every key $k \leftarrow \mathsf{Gen}(1^\lambda)$, every message $m \in \mathcal{M}$, and encryption $c \leftarrow \mathsf{Enc}_k(m)$ it holds $\mathsf{Dec}_{k'}(c) = \perp$ for all $k' \neq k$.

4 Lower bound for Yao's Garbling Scheme

Let Π denote the variant of Yao's garbling scheme as analysed in [JW16]. In this section, we aim to prove a lower bound on the loss in security involved when reducing adaptive indistinguishability of Π to the IND-CPA security of the underlying symmetric encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$. As explained in the introduction, we follow the approach in [KKPW21]

Algorithm 1 Yao's garbling scheme $\Pi^{\mathcal{F}}$ with access to a symmetric-key encryption scheme $\mathcal{F} := (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$.

1: **procedure** $\mathsf{GCircuit}(1^\lambda, \mathsf{C})$
input:
1. Security parameter λ in unary
2. Circuit C with S gates, W wires and input and output sizes n and ℓ

2: **for** $j \in [1, W]$ and $b \in \{0,1\}$ **do** $k_j^b \leftarrow \mathsf{Gen}(1^\lambda)$ **end for** ▷ Sample wire keys
3: Set $f = \left\{(k_j^0 \to 0, k_j^1 \to 1)\right\}_{j \in [W-\ell+1, W]}$ ▷ Set output map
4: **for** $j \in [1, S]$ **do** ▷ Garble gates
5: Let (g_j, u, v, w) denote the j-th gate in C
6: $\tilde{g}_j = \left\{\mathsf{Enc}_{k_u^{b_u}}(\mathsf{Enc}_{k_v^{b_v}}(k_w^{g_j(b_u,b_v)}))\right\}_{b_u,b_v \in \{0,1\}}$
7: **end for**
8: **return** $(\tilde{\mathsf{C}}, K = (\boldsymbol{k}, f))$ where $\tilde{\mathsf{C}} := \{\tilde{g}_j\}_{j \in [1,S]}$ and $\boldsymbol{k} := \left\{(k_j^0, k_j^1)\right\}_{j \in [1,n]}$
9: **end procedure**

10: **procedure** $\mathsf{GInput}(K, x)$
input:
1. Garbling key K parsed as $(\boldsymbol{k}, f)$ as in Lines 3 and 8
2. Input $x \in \{0,1\}^n$ with bit decomposition $\{x_1, \ldots, x_n\}$

11: Set $\boldsymbol{k}_x := \left\{k_j^{x_j}\right\}_{j \in [1,n]}$ ▷ Select input keys from $\boldsymbol{k}$
12: **return** $\tilde{x} := (\boldsymbol{k}_x, f)$
13: **end procedure**

14: **procedure** $\mathsf{GEval}(\tilde{\mathsf{C}}, \tilde{x})$
input:
1. Garbled circuit $\tilde{\mathsf{C}}$ parsed as $\{\tilde{g}_j\}_{j \in [1,S]}$
2. Garbled input $\tilde{x}$ parsed as $(\boldsymbol{k}_x, f)$

15: Parse $\boldsymbol{k}_x$ as $\{k_1, \ldots, k_n\}$
16: **for** $j \in [1, S]$ **do** ▷ Decode circuit
17: Let (u, v, w) denote the wires of g_j ▷ The topology is assumed to be public
18: Parse $\tilde{g}_j$ as $(\{c_1, \ldots, c_4\})$ ▷ Parse the gate table
19: **for** $l \in [1, 4]$ **do** ▷ Decrypt each double ciphertext till successful
20: Let $m := \mathsf{Dec}_{k_u}(\mathsf{Dec}_{k_v}(c_l))$
21: **if** $m \neq \perp$ **then** set $k_w := m$
22: **end for**
23: **end for**
24: Parse f as $\left\{(k_j^0 \to 0, k_j^1 \to 1)\right\}_{j \in [W-\ell+1, W]}$
25: **for** $j \in [W - \ell + 1, W]$ **do** ▷ Decode output
26: Set $y_j = 0$ if $k_j = k_j^0$; else set $y_j = 1$
27: **end for**
28: **return** y
29: **end procedure**

and define two oracles $\mathcal{F}$ and $\mathcal{A}$ implementing an ideal SKE scheme and an adversary, respectively, such that $\mathcal{A}$ is not helpful in breaking IND-CPA security of $\mathcal{F}$. For the precise description of $\mathcal{F}$ we refer to Section 4.5. The (inefficient) threshold adversary $\mathcal{A}$ we define as follows:

1. On input the security parameter in unary, 1^λ, the adversary $\mathcal{A}$ chooses a circuit G with input size $n = \Theta(\lambda)$, constant output, and depth $\delta(d) \in O(n)$ for a parameter d. The circuit G consists of three parts, i.e., $\mathsf{G} = \mathsf{G}^0 \circ \mathsf{G}^\wedge \circ \mathsf{G}^\oplus$; for precise description of G, see Section 4.1. $\mathcal{A}$ sends G to the challenger.

2. After receiving $\tilde{\mathsf{G}}$, the adversary $\mathcal{A}$ chooses $x_0, x_1 \leftarrow \{0,1\}^n$ uniformly at random. Note that $\mathsf{G}(x_0) = \mathsf{G}(x_1)$ trivially holds since G has constant output. $\mathcal{A}$ sends x_0, x_1 to the challenger.

3. On receipt of $\tilde{x}_b = (k_1, \ldots, k_n)$ along with an output mapping, $\mathcal{A}$ extracts a *pebbling configuration* on the graph $G \setminus G^0$ corresponding to $\mathsf{G}^\wedge \circ \mathsf{G}^\oplus$ as described in Section 4.4. $\mathcal{A}$ outputs $b' = 0$ if the pebbling configuration is *good* as per Definition 8, and $b' = 1$ otherwise.

We will first provide a precise definition of the candidate circuit G in Section 4.1 and then show the following two properties of this circuit: First, in Section 4.2 we will prove that if a large subset of gates in $\mathsf{G}^\oplus$ is malformed, then on uniformly random input some of these gates will not evaluate correctly. Second, in Section 4.3, we introduce a new pebbling game on DAGs and prove a pebbling lower bound on the graph G underlying G. The definition of *good* pebbling configurations in Definition 8 then gives a cut in the configuration graph of $G \setminus G^0$ w.r.t. this new pebbling game. Having proven these properties of the circuit, in Section 4.4 we will then describe a mapping from garbled circuit/input pair $(\tilde{\mathsf{G}}, \tilde{x})$ to a pebbling configuration on $G \setminus G^0$. This mapping together with the cut in the configuration graph will guarantee that the threshold adversary $\mathcal{A}$ indeed breaks indistinguishability of the garbling scheme $\Pi^{\mathcal{F}}$. Finally, in Section 4.5, it then remains to combine these results. First, we will essentially prove that any black-box reduction proving adaptive indistinguishability of the garbling scheme based on the IND-CPA security of the underlying encryption scheme must *follow the pebbling rules*, i.e., it must define two hybrid games such that the extracted pebbling configurations differ by one valid pebbling move; this step will crucially rely on our choice of gates. The pebbling lower bound from Section 4.3 then implies that any threshold configuration contains many pebbles on $\mathsf{G}^\oplus$. We will then use the result from Section 4.2 as well as the technical fix from Section 4.4 concerning equivocation of keys to show that the simulation of any garbling $(\tilde{\mathsf{G}}, \tilde{x})$ that is mapped to a threshold configuration requires to guess many input bits. This will allow us to state our final theorem.

4.1 The Circuit

We construct a family of circuits $\mathsf{G} := \{\mathsf{G}_d\}_{d \in \mathbb{N}}$ and show that the loss in security for G_d is exponential in $\sqrt{d}$, where the parameter d is linear in the depth of the circuit. The circuit

is designed keeping our high-level idea in mind, we denote its underlying graph by G_d. The circuit $\mathsf{G}_d := \mathsf{G}^0_d \circ \mathsf{G}^\wedge_d \circ \mathsf{G}^\oplus_d$ consists of the three blocks $\mathsf{G}^\oplus_d$, $\mathsf{G}^\wedge_d$ and G^0_d, with underlying graphs denoted by $G^\oplus_d$, $G^\wedge_d$ and G^0_d, respectively. The graph $G^\oplus_d$ (see Figure 2.(b)) is a so-called *tower graph* [DKW11], and is obtained from so-called pyramid graphs of depth d (see Figure 2.(a)).

- $\mathsf{G}^\oplus_d$ is obtained from $G^\oplus_d$ by substituting each vertex with an XOR gate as shown in Figure 2. On a high level, the pyramid structure ensures high pebbling complexity whereas the XOR gates preserve (most) entropy in the input, which makes it hard for a reduction to obtain correct evaluation of pebbled gates.

- G^0_d consists of a binary tree of AND gates and its sole role is to set the output of the circuit G to constant 0 (Lemma 1).[8]

- $\mathsf{G}^\wedge_d$ sits in between the $\mathsf{G}^\oplus_d$ and G^0_d blocks (see Figure 1), and consists of one AND gate serving as "control" gate for each XOR gate in $\mathsf{G}^\oplus_d$ and each input gate. Each AND gate g in $G^\wedge_d$ receives its inputs from (i) the output of its companion XOR gate in $G^\oplus_d$ (resp. input gate) and (ii) the XOR gate in the last layer of $G^\oplus_d$ in (vertical) alignment with g (see Figure 1, formal definition below). As mentioned previously, intuitively, this block will act as an "error detection" mechanism for the $\mathsf{G}^\oplus_d$ block in the sense that it helps detect if (malformed) garblings of XOR gates evaluate wrongly.

More formally, for input size $n = 2^\kappa - 1$ with $\kappa \in \mathbb{N}$, and $d \leq n$, we describe the candidate circuit $\mathsf{G} = \mathsf{G}_d$ based on its underlying graph structure $G = G_d$ as follows, see Figure 1: G contains $\delta(d) := 2d + \lceil \log((d+1)n) \rceil + 2$ layers, each containing n nodes. The input gates (layer 0) have outdegree 2, the nodes on layers $[1, 2d+1]$ all have in- and outdegree 2, and the nodes in the last $\lceil \log(d \cdot n) \rceil$ layers have indegree 2 and outdegree 1. Since we will need to differentiate between left and right parents a node, we will define the edge sets of these graphs as the union of "right" and "left" edges. The first $d+1$ layers $[0, d]$ build a graph $G^\oplus$ of high pebbling complexity (see Section 4.3), defined as

$$
\begin{aligned}
G^\oplus :=& ([(d+1)\cdot n - 1]_0, \mathcal{E}^\oplus), \text{ where } \mathcal{E}^\oplus := \mathcal{E}^\oplus_L \cup \mathcal{E}^\oplus_R \text{ with} \\
\mathcal{E}^\oplus_L :=& \{((i-1)\cdot n + k, i \cdot n + k) \mid i \in [1, d]\,,\ k \in [0, n]\}, \\
\mathcal{E}^\oplus_R :=& \{((i-1)\cdot n + l, i \cdot n + k) \mid i \in [1, d]\,,\ k, l \in [0, n]\,,\ l = k + 1 \mod n\},
\end{aligned}
$$

where we number the input gates by $\{0, \ldots, n-1\}$. Graph $G^\oplus$ is followed by $d+1$ layers $[d+1, 2d+1]$ building $G^\wedge$, defined as

$$
\begin{aligned}
G^\wedge :=& ([(d+1)\cdot n, (2d+2)\cdot n - 1], \mathcal{E}^\wedge), \text{ where } \mathcal{E}^\wedge := \mathcal{E}^\wedge_L \cup \mathcal{E}^\wedge_{\mathsf{in},R} \cup \mathcal{E}^\wedge_R \text{ with} \\
\mathcal{E}^\wedge_L :=& \{(d\cdot n + k, (d+i)\cdot n + k) \mid i \in [1, d+1]\,,\ k \in [0, n]\}, \\
\mathcal{E}^\wedge_{\mathsf{in},R} :=& \{(d\cdot n + l, (d+1)\cdot n + k) \mid k, l \in [0, n]\,,\ l = k + 1 \mod n\}, \\
\mathcal{E}^\wedge_R :=& \{((i-2)\cdot n + k, (d+i)\cdot n + k) \mid i \in [2, d+1],\ k \in [0, n]\}.
\end{aligned}
$$

[8] In principle we could have used constant-0 gates in place of the AND gates, or simply a single constant-0 gate of high fan-in (which would considerably simplify the description). But we prefer to stick to the standard Boolean basis.

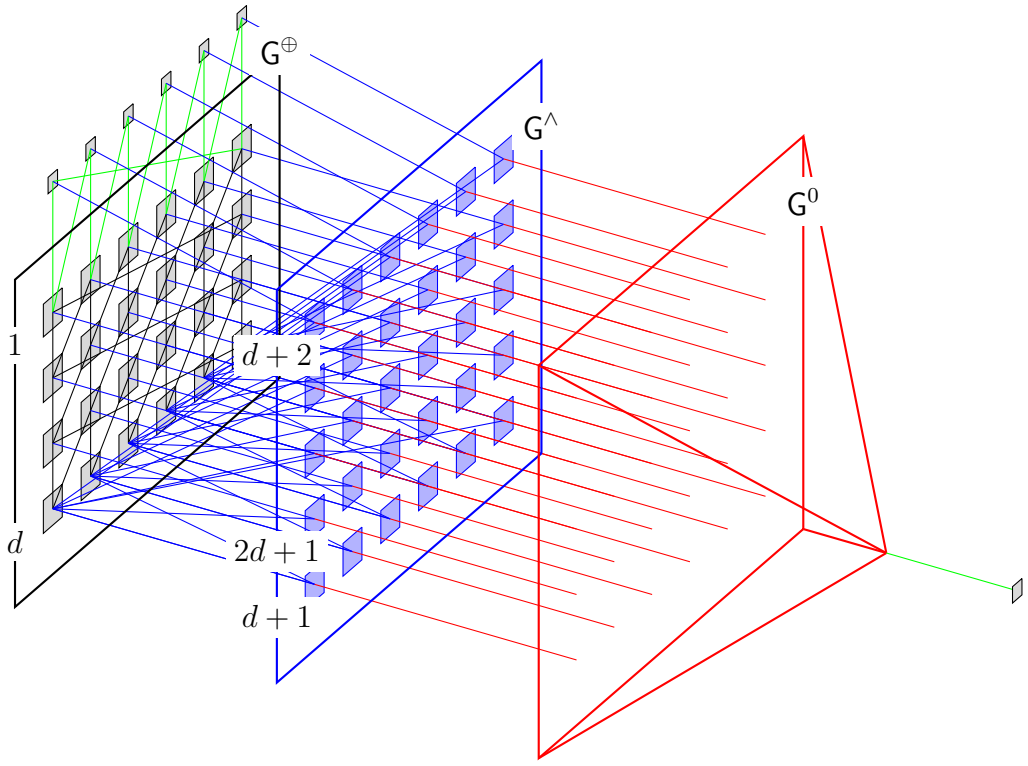


Figure 1: Schematic diagram for the candidate circuit of width 5 and depth 4. The input and output wires are coloured green. The layer number is indicated on the left. The first two blocks are the XOR and AND layers respectively; the final pyramid denotes the binary tree.

Finally, there is a binary tree structure G^0 on top of the $(d+1) \cdot n$ output gates of $G^{\wedge}$, to guarantee constant output 0 of the circuit.

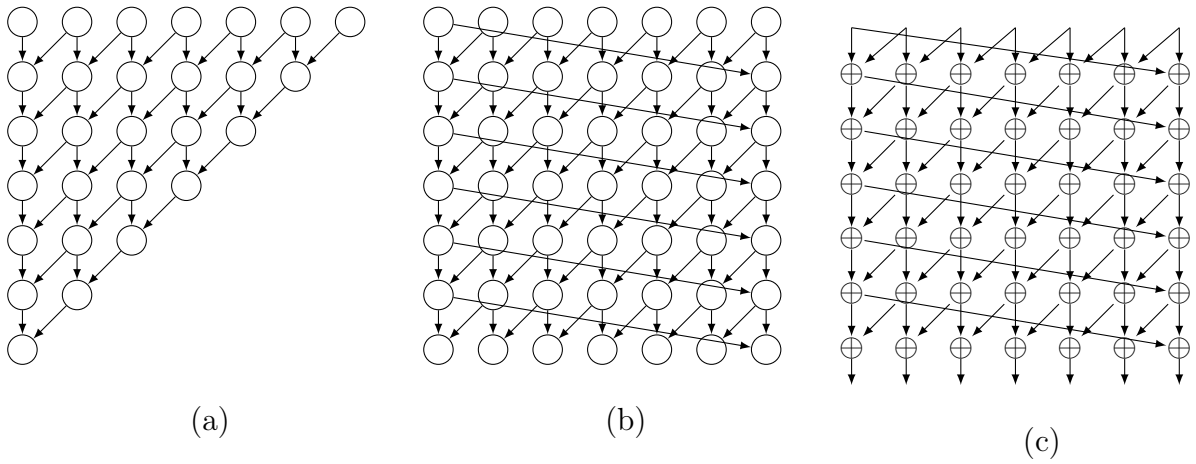
The candidate circuit G is now defined based on the graph structure G as follows:

- All gates on layers $[1, d]$ implement XOR gates.
- All other gates consist of AND gates.

In the following lemma we prove that G is indeed constant.

Lemma 1. $\mathsf{G}(x) = 0$ *for all* $x \in \{0,1\}^n$*, i.e.,* G *is constant.*

Proof. To see that this circuit has constant output 0, first note that G^0 outputs 1 only on the all-1 string 1^n. In Section 4.2 we will provide an explicit representation of the output of $\mathsf{G}^{\oplus}$ that in particular implies that the range of $\mathsf{G}^{\oplus}$ consists of strings $(y_1, \ldots, y_n) \in \{0,1\}^n$ that contain an even number of 1s (see Corollary 1). As we chose $n = 2^{\kappa} - 1$ odd, this implies that any output of $\mathsf{G}^{\oplus}$ must contain at least one 0. Hence, the input to $\mathsf{G}^{\wedge}$ contains at least one 0, and since $\mathsf{G}^{\wedge}$ only contains AND gates, the output of $\mathsf{G}^{\wedge}$ must contain a 0. This proves that G is constant. □



(a) (b) (c)

Figure 2: The graphs and the circuit for parameter $d = 6$: (a) A pyramid graph of depth d, (b) Extending the pyramid graph to get a tower graph $G_d^{\oplus}$ of depth d and (c) Circuit $\mathsf{G}_d^{\oplus}$ obtained replacing the vertices in $G_d^{\oplus}$ with XOR gates.

4.2 Vulnerability of the Circuit $\mathsf{G}^{\oplus}$

In Section 4.5 we will prove that any black-box reduction R that aims to use $\mathcal{A}$ to gain advantage in breaking the IND-CPA security of encryption scheme $\mathcal{F}$ has to simulate $(\tilde{\mathsf{G}}, \tilde{x})$ such that the extracted pebbling configuration on $\mathsf{G}^{\oplus}$ contains $d-1$ or d gray or black pebbles. Each of these pebbles implies that at least one of the ciphertexts associated to that gate must be malformed and modify the output of some input key pair. In the case that all AND gates are properly garbled, all keys can be mapped to bits and hence such a switch of the output can be detected (cf. Lemma 7). Thus, we consider the following game.

- On input a circuit C and a parameter d, R chooses a circuit C' of the same topology as C such that all except exactly d (non-input) gates coincide with the corresponding gates in C. R sends C' to $\mathcal{A}$.
- On receipt of C', $\mathcal{A}$ samples $x \leftarrow \{0,1\}^n$ uniformly at random.
- R wins if for all gates in C' the output during evaluation on input x coincides with the corresponding output bit when evaluating C.

We now prove that for $\mathsf{C} = \mathsf{G}^{\oplus}$, no algorithm R wins the above game with non-negligible (in d) probability.

Lemma 2. *Let $d \in [1, n]$. For $\mathsf{G} = \mathsf{G}^{\oplus}$ and any R, the probability that R wins the above game is at most $(\frac{3}{4})^{\sqrt{d}/4}$.*

First, note that all except d gates in G' are XOR gates, and in particular a linear function over $\mathbb{Z}_2$. For each of the remaining d malformed gates, on the other hand, at least one input pair is mapped to a different output bit than it would be in an XOR operation. We

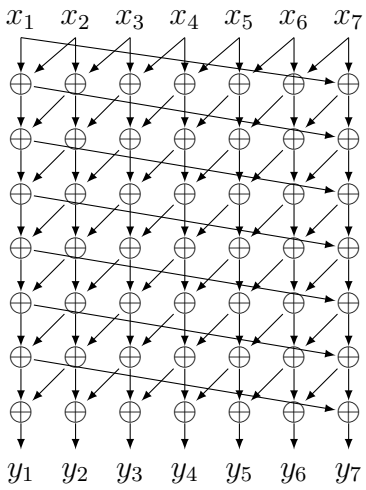


Figure 3: Circuit $\mathsf{G}^{\oplus}$ with $n = d = 7$.

call the corresponding gates in the original circuit $\mathsf{G}^{\oplus}$ *pebbled*. To prove Lemma 2, we will show that there exists a subset of at least $\sqrt{d}/4$ of those d pebbled gates such that their input is determined by independent linear functions. This implies that instead of choosing $x \leftarrow \{0,1\}^n$, $\mathcal{A}$ can equivalently choose the $\sqrt{d}/2$ input bits uniformly at random, and then choose x uniformly under the constraint that the values running over these wires during evaluation of $\mathsf{G}^{\oplus}$ must be consistent with the predetermined bits. Clearly, x chosen this way is still uniformly random in $\{0,1\}^n$. By definition of the game, R only wins the game if for all gates in G' the output during evaluation on input x coincides with the corresponding output bit when evaluating G, and this must in particular also hold for the pebbled gates. Since each of the malformed gates in G' flips the output of at least one of the four possible input pairs, and the input bits of $\sqrt{d}/4$ of the pebbled gates were chosen independently and uniformly at random, the probability that R wins is at most $(\frac{3}{4})^{\sqrt{d}/4}$.

Towards proving Lemma 2, let M denote the linear mapping corresponding to one layer of gates in the circuit $\mathsf{G}^{\oplus}$, i.e., written in matrix notation,

$$M = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

The output of the μth layer of $\mathsf{G}^{\oplus}$ on input $x \in \{0,1\}^n$ is given by $M^{\mu} \cdot x$, hence we denote the degree-1 polynomial in $\mathbb{Z}_2[x_1, \dots, x_n]$ which determines its ν-th bit by M^{μ}_{ν} (for $\mu \in [0,n]$ and $\nu \in [1,n]$). Denoting by $\overline{\nu+1}$ the representation of the residue class $\nu + 1 \mod n$ in $[n]$, we have e.g.,

$$M^0_{\nu} = x_{\nu}, \quad M^1_{\nu} = x_{\nu} \oplus x_{\overline{\nu+1}}, \quad M^2_{\nu} = x_{\nu} \oplus x_{\overline{\nu+2}}, \quad M^3_{\nu} = x_{\nu} \oplus x_{\overline{\nu+1}} \oplus x_{\overline{\nu+2}} \oplus x_{\overline{\nu+3}}$$

and in general it holds

$$M_\nu^\mu = M_\nu^{\mu-1} \oplus M_{\overline{\nu+1}}^{\mu-1} \tag{2}$$

for all $\mu \in \mathbb{N}, \nu \in [1,n]$. In the following we will associate gates with the corresponding polynomials that determine their outputs.

If the input length n is odd – for convenience we assume n to be one less than a power of 2 – then $\mathsf{G}^\oplus$ maintains high entropy; to prove this, we use the following explicit representation of the polynomials M_ν^μ.

Lemma 3 (explicit formula for the polynomials M_ν^μ)**.** *Let $n = 2^\kappa - 1$, $\kappa \in \mathbb{N}$, M defined above, $\mu \in \mathbb{N}$, and $\nu \in [1,n]$. For $\overline{\mu} \neq n$ and $\beta_k \in \{0,1\}$ its binary decomposition, i.e. $\overline{\mu} = \sum_{k\in[0,\kappa-1]} \beta_k 2^k$, it holds:*

$$M_\nu^\mu = \bigoplus_{i\in[1,n]} \alpha_i x_i, \text{ where } \alpha_i = \begin{cases} 1 & \text{if } i \in \nu + \sum_{k\in[0,\kappa-1]} \{0,\beta_k\} \cdot 2^k \mod n, \\ 0 & \text{else.} \end{cases} \tag{3}$$

Note, M_ν^μ only depends on $\overline{\mu}$, not on μ. For $\overline{\mu} = n = 2^\kappa - 1$, it holds:

$$M_\nu^\mu = \bigoplus_{i\in[1,n]} \alpha_i x_i, \text{ where } \alpha_i = \begin{cases} 1 & \text{if } i \neq \nu, \\ 0 & \text{else.} \end{cases} \tag{4}$$

Proof. We prove the claim via induction on $\mu \in \mathbb{N}$. For $\mu = 1$, we have $M_\nu^\mu = x_\nu \oplus x_{\overline{\nu+1}}$. On the other hand, for $\mu = 1$ we have $\beta_0 = 1$ and $\beta_k = 0$ for all $k \in [1,\kappa]$, which implies $\alpha_\nu = 1$, $\alpha_{\overline{\nu+1}}$ and $\alpha_i = 0$ for all $i \in [\ell] \setminus \{\nu, \overline{\nu+1}\}$. Hence, the claim is true for $\mu = 1$.

For $2 \leq \mu \leq n-1$, let $\mu - 1 = \sum_{k\in[0,\kappa-1]} \beta'_k 2^k$, hence for $\mu = \sum_{k\in[0,\kappa-1]} \beta_k 2^k = \sum_{k\in[0,\kappa-1]} \beta'_k 2^k + 1$ we obtain

$$\beta_k = \begin{cases} 1 - \beta'_k & \text{for } k \leq k' := \min\{k \in [0,\kappa-1] \mid \beta'_k = 0\}, \\ \beta'_k & \text{for } k > k'. \end{cases}$$

By induction hypothesis, we have

$$M_\nu^{\mu-1} = \bigoplus_{i\in[1,n]} \alpha_i^{(0)} x_i, \text{ where } \alpha_i^{(0)} = \begin{cases} 1 & \text{if } i \in \mathcal{I}^{(0)} := \{\nu + \sum_{k\in[0,\kappa-1]} [0,\beta'_k] \cdot 2^k \mod n\}, \\ 0 & \text{else.} \end{cases}$$

$$M_{\nu+1}^{\mu-1} = \bigoplus_{i\in[1,n]} \alpha_i^{(1)} x_i, \text{ where } \alpha_i^{(1)} = \begin{cases} 1 & \text{if } i \in \mathcal{I}^{(1)} := \{\nu + 1 + \sum_{k\in[0,\kappa-1]} [0,\beta'_k] \cdot 2^k \mod n\}, \\ 0 & \text{else.} \end{cases}$$

Let $\mathcal{I}^{(0)} \Delta \mathcal{I}^{(1)} := (\mathcal{I}^{(0)} \setminus \mathcal{I}^{(1)}) \cup (\mathcal{I}^{(1)} \setminus \mathcal{I}^{(0)})$ denote the symmetric difference of $\mathcal{I}^{(0)}$ and $\mathcal{I}^{(1)}$. Then, by Equation (2), we get

$$M_\nu^\mu = \bigoplus_{i\in[1,n]} \alpha_i x_i \quad \text{with } \alpha_i = \alpha_i^{(0)} \oplus \alpha_i^{(1)} = \begin{cases} 1 & \text{if } i \in \mathcal{I}^{(0)} \Delta \mathcal{I}^{(1)}, \\ 0 & \text{else.} \end{cases}$$

Since $\beta'_{k'} = 0$ and $\beta'_k = 1$ for $k < k'$, we have for $0 \le k < k'$:

$$\nu + 1 + \sum_{l=0}^{k-1} 2^l + \sum_{l=k+1}^{\kappa-1} [0, \beta'_l]\, 2^l \mod n = \nu + 2^k + \sum_{l=k+1}^{\kappa-1} [0, \beta'_l]\, 2^l \mod n \in \mathcal{I}^{(1)} \cap \mathcal{I}^{(0)},$$

and for $k = k'$:

$$\nu + 1 + \sum_{l=0}^{k'-1} 2^l + \sum_{l=k'+1}^{\kappa-1} [0, \beta'_l]\, 2^l \mod n = \nu + 2^{k'} + \sum_{l=k'+1}^{\kappa-1} [0, \beta'_l]\, 2^l \mod n \in \mathcal{I}^{(1)} \setminus \mathcal{I}^{(0)},$$

$$\text{and} \quad \nu + \sum_{l=k'+1}^{\kappa-1} [0, \beta'_l]\, 2^l \mod n \in \mathcal{I}^{(0)} \setminus \mathcal{I}^{(1)}.$$

Combining the above cases and using that $\beta_{k'} = 1$ and $\beta_k = 0$ for $k < k'$, proves Equation (3) for $\mu \in [1, n-1]$.

To prove Equation (4), note that for $\mu - 1 = n - 1 = 2^\kappa - 2 = \sum_{k \in [1,\kappa-1]} 2^k$, Equation (3) gives

$$M_\nu^{\mu-1} = \bigoplus_{i \in [1,n]} \alpha_i^{(0)} x_i, \text{ where } \alpha_i^{(0)} = \begin{cases} 1 & \text{if } i \in \mathcal{I}^{(0)} := \{\nu + 2 \cdot [0, (n-1)/2] \mod n\} \\ & \qquad = \{\nu, \nu+2, \ldots, \nu-1\}, \\ 0 & \text{else.} \end{cases}$$

$$M_{\nu+1}^{\mu-1} = \bigoplus_{i \in [1,n]} \alpha_i^{(1)} x_i, \text{ where } \alpha_i^{(1)} = \begin{cases} 1 & \text{if } i \in \mathcal{I}^{(1)} := \{\nu + 1 + 2 \cdot [0, (n-1)/2] \mod n\} \\ & \qquad = \{\nu+1, \nu+3, \ldots, \nu\}, \\ 0 & \text{else.} \end{cases}$$

Using $\mathcal{I}^{(0)} \Delta \mathcal{I}^{(1)} = \{\nu + [n-1] \mod n\} = [n] \setminus \{\nu\}$ now proves Equation (4).

Finally, for $\mu = 2^\kappa = n + 1$, Equation (4) implies

$$M_\nu^{2^\kappa} = M_\nu^n \oplus M_{\overline{\nu+1}}^n = \left(\bigoplus_{i \in [1,n] \setminus \{\nu\}} x_i \right) \oplus \left(\bigoplus_{i \in [1,n] \setminus \{\overline{\nu+1}\}} x_i \right) = x_\nu \oplus x_{\overline{\nu+1}} = M_\nu^1,$$

where we used the fact that $n = 2^\kappa - 1$ by definition. This proves the Lemma. □

Lemma 3 directly implies several useful properties, which we summarize in the following corollary.

Corollary 1 (Properties of M and $\mathsf{G}^\oplus$)**.** *For M defined as above, $n = 2^\kappa - 1$, $\kappa \in \mathbb{N}$, it holds*

1. *$M^{2^\kappa} = M$, which implies $\mathsf{rank}(M^k) = n - 1$ for all $k \ge 1$, i.e., $\mathsf{G}^\oplus = M^d$ is 2-to-1 for any d.*
2. *Any $n - 1$ output bits of M^k ($k \ge 1$) are determined by linearly independent degree-1 polynomials.*

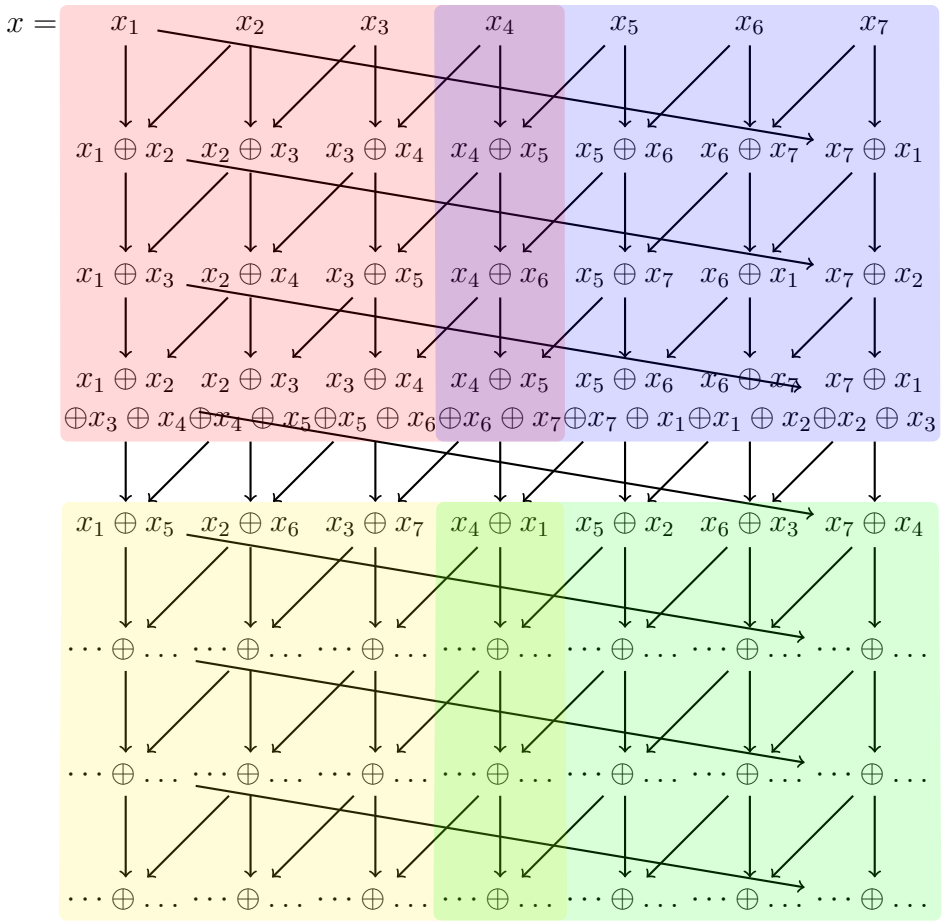


Figure 4: The circuit $\mathsf{G}^{\oplus}$ split into four equal-sized quarters.

3. $\mathsf{Image}(\mathsf{G}^{\oplus}) = \{x = (x_1, \ldots, x_n) \in \{0,1\}^n \mid \bigoplus_{i\in[1,n]} x_i = 0\}$, *i.e., all vectors in the image of* $\mathsf{G}^{\oplus}$ *contain an even number of 1s.*

The first property immediately follows from Lemma 3 since for $\mu = 2^\kappa$ we have $\overline{\mu} = 1$. The second property then follows from $\mathsf{rank}(M^k) = n - 1$. For the last property, note that the set $\nu + \sum_{k\in[0,\kappa-1]}\{0, \beta_k\} \cdot 2^k \mod n$ is even whenever a single bit β_k is nonzero (which is true for all $\overline{\mu} > 0$), and also the set $\{i \in [n] \mid i \neq \nu\}$ is even since n is odd.

The following Lemma now immediately implies Lemma 2.

Lemma 4. *Any subset* $\mathcal{S} \subset \{M^\mu_\nu\}_{\mu\in[0,n],\nu\in[1,n]}$ *of polynomials in* $\mathbb{Z}_2[x_1, \ldots, x_n]$ *with* $s := |\mathcal{S}|$ *contains a subset* $\mathcal{S}'$ *of size* $\sqrt{s}/4$ *such that* $|\mathsf{parents}(\mathcal{S}')| = \sqrt{s}/2$ *and* $\mathsf{parents}(\mathcal{S}')$ *is linearly independent, where* $\mathsf{parents}(M^\mu_\nu) := \{M^{\mu-1}_\nu, M^{\mu-1}_{\nu+1}\}$.

Proof. We split the $(n+1) \times n$ gates $\{M^\mu_\nu\}_{\mu\in[0,n],\nu\in[1,n]}$ into four equal-sized quarters $\mathcal{M}_i, i \in [1, 4]$, each containing a subset of $(n+1)/2 \times (n+1)/2$ gates, see Figure 4. Since $\mathcal{S}$ has size s, at least one of these quarters must contain $\geq s/4$ gates from $\mathcal{S}$. Furthermore, considering the $(n+1)/2$ vertical paths within such a quarter; then either 1) there is one vertical path

which contains $\geq \sqrt{s}/2$ gates from $\mathcal{S}$, or 2) there exist $\geq \sqrt{s}/2+1$ vertical paths which contain at least one pebble each.

For case 1), note that for all $\nu \in [1,n]$ the set of gates within any vertical path within $\mathcal{M}_i$, i.e., $\{M_\nu^\mu\}_{\mu\in[0,n]} \cap \mathcal{M}_i = \{M_\nu^{\mu_i+j}\}_{j\in[0,(n-1)/2]}$ with $\mu_1=\mu_2=0,\ \mu_3=\mu_4=(n+1)/2$, is linearly independent. To see this, first note that by Corollary 1 $\{M_\nu^{\mu_i},\dots,M_{\nu+(n-1)/2}^{\mu_i}\}$ is linearly independent for any $i\in[1,4]$ and generates the vertical path $\{M_\nu^{\mu_i+j}\}_{j\in[0,(n-1)/2]}$ in $\mathcal{M}_i$. Now, consider the explicit representation of $M_\nu^{\mu_i+j}$ with $j=\sum_{k\in[\kappa-2]_0}\beta_k 2^k$ in the basis $\{M_\nu^{\mu_i},\dots,M_{\nu+(n-1)/2}^{\mu_i}\}$ (which follows from Equation (3)):

$$M_\nu^{\mu_i+j} = \bigoplus_{l\in[0,(n-1)/2]} \alpha_l M_{\nu+l}^{\mu_i},\ \text{where } \alpha_l = \begin{cases} 1 & \text{if } \ l \in \nu + \sum_{k\in[\kappa-2]_0}[0,\beta_k]\cdot 2^k \mod n, \\ 0 & \text{else.}\end{cases}$$

In particular, it follows that $M_\nu^{\mu_i+j} = \bigoplus_{l\in[j]_0}\alpha_l M_{\nu+l}^{\mu_i}$ with $\alpha_j=1$ and $\alpha_l=0$ for all $l\in[j+1,(n-1)/2]$. This implies that for $b_j\in\{0,1\}^*$

$$\bigoplus_{j\in[0,(n-1)/2]} b_j\cdot M_\nu^{\mu_i+j}=0 \quad\Rightarrow\quad b_j=0 \quad \forall j\in[0,(n-1)/2]\,.$$

Thus, any subset of gates $\{M_\nu^{\mu_i+j}\}_{j\in[0,(n-1)/2]}$ along the ν-th vertical path in $\mathcal{M}_i$ is linearly independent over $\mathbb{Z}_2$. This implies that the set of $\geq\sqrt{s}/2$ gates in $\mathcal{S}$ which lie on one vertical path is linearly independant. It immediately follows that the left parents of this set are linearly independent as well. By basic linear algebra, replacing an element from a set of linearly independent equations by a linear combination of this element with other elements of the set preserves linear independence. Using Equation (2) and removing at most half of the left parents, we obtain a set $\mathcal{S}'$ of $\geq\sqrt{s}/4$ gates whose parents are distinct and form a linearly independent set.

For case 2), by assumption there exists a set $\mathcal{S}'$ consisting of $\sqrt{s}/4$ gates in $\mathcal{S}$ such that *their parents* lie on distinct vertical paths in $\mathcal{M}_i$.[9] Furthermore, since $\geq\sqrt{s}/2+1$ vertical paths contain gates from $\mathcal{S}$, we can choose $\mathcal{S}'$ such that $\mathsf{parents}(\mathcal{S}')$ does not contain the bottom right gate $M_{\nu_i+(n-1)/2}^{\mu_i+(n-1)/2}$ with $\nu_1=\nu_3=1$, $\nu_2=\nu_4=(n-1)/2$ (which is not necessary but more convenient for the analysis below). We will now argue that the set of parents of $\mathcal{S}'$ is linearly independent. Similar to above, we can uniquely represent the elements of $\mathsf{parents}(\mathcal{S}') := \{M_{\nu_i+\nu_j}^{\mu_i+\mu_j}\}_{j\in[\sqrt{s}/2]}$ with $\mu_j,\nu_j\in[0,(n-1)/2]$, and $\nu_j<\nu_{j+1}$ for all $j\in[\sqrt{s}/2-1]$ as a linear combination of the linearly independent set $\{M_{\nu_i}^{\mu_i},\dots,M_{\nu_i+n-1}^{\mu_i}\}$:

$$M_{\nu_i+\nu_j}^{\mu_i+\mu_j} = \bigoplus_{l\in[\mu_j]_0}\alpha_l M_{\nu_i+\nu_j+l}^{\mu_i} \quad \text{with } \alpha_0=1 \text{ and } \mu_j+\nu_j\leq n-1.$$

This implies that for $b_j\in\{0,1\}^*$

$$\bigoplus_{j\in[\sqrt{s}/2]} b_j\cdot M_{\nu_i+\nu_j}^{\mu_i+\mu_j}=0 \quad\Rightarrow\quad b_j=0 \quad \forall j\in[\sqrt{s}/2].$$

[9] Technically, for $i\in\{3,4\}$ we have to shift the window $\mathcal{M}_i$ by setting $\mu_i\leftarrow\mu_i-1$.

Hence, $\mathcal{S}'$ is indeed linearly independent over $\mathbb{Z}_2$. This proves the claim. □

Lemma 2 now follows, since for any set of d pebbled gates, by Lemma 4 there exists a subset $\mathcal{S}'$ of $\sqrt{d}/4$ pebbled gates such that their parents are distinct and form a linearly independent set.

4.3 Pebbling Game and Threshold

Recall that in Yao's garbling scheme, each gate g is associated with a (honest) garbling table $\tilde{\boldsymbol{g}}$, which consists of four double encryptions that encode g's gate table. However, a reduction is free to alter the contents of the honest garbling table in any way. In fact, the upper bounds in [LP09, JW16] crucially rely on the ability to do this in an indistinguishable manner: in the real game the garbling tables are all honest, whereas in the simulated game the garbling tables all encode the constant-0 gate, and the hybrids involve replacing the honest garbling tables one by one with that of the constant-0 gate.[10] We introduce a pebble game to precisely model such different simulations of the garbled circuit $\tilde{\mathsf{G}}$ (by the reduction). Loosely speaking, the extracted pebble configuration is an abstract representation of the simulation $(\tilde{\mathsf{G}}, \tilde{x}_b)$, and the pebbling rules model the reduction's ability to maul garbling tables in $\tilde{\mathsf{G}}$ without being noticed (indistinguishability).

The pebbles. Intuitively, the pebble on a gate g encodes how "different" the garbling table $\tilde{\boldsymbol{g}}'$ which $\mathcal{A}$ receives is from an honest garbling $\tilde{\boldsymbol{g}}$. To this end, we employ three different pebbles: white, gray and black.

- A white pebble on g indicates that $\tilde{\boldsymbol{g}}'$ and $\tilde{\boldsymbol{g}}$ are at "distance" 0 (defined below), i.e., $\tilde{\boldsymbol{g}}$ is (distributed identically to) an honest garbling table of g.
- A gray or black pebble on g indicates that $\tilde{\boldsymbol{g}}'$ is malformed. What differentiates gray from black is the degree of malformation: loosely speaking, a gray pebble indicates that $\tilde{\boldsymbol{g}}'$ is at a distance 1 from $\tilde{\boldsymbol{g}}$, whereas a black pebble indicates that $\tilde{\boldsymbol{g}}'$ is at a distance 2 (or more).

To understand what we mean by distance, we need to take a closer look at the structure of a garbling table. An honest garbling table $\tilde{\boldsymbol{g}}$ consists of the four double encryptions shown in Table 1.(a). We assign a gray pebble to a gate g if the garbling table of g in $\tilde{\mathsf{G}}$ can be proven indistinguishable from $\tilde{\boldsymbol{g}}$ by embedding a *single* IND-CPA challenge key (among k_u^0, k_u^1, k_v^0 and k_v^1). For example, let's consider an AND gate and its honest garbling table (Table 1.(b)): a malformed table that is at distance one (via the key k_u^1 or k_v^1) from it is, e.g., a garbling table that encodes the constant-0 gate (Table 1.(d)). A garbling of an XOR gate, in contrast, is at distance 2 from a garbling of a constant gate: If k_u^a and k_v^b are the keys

[10] Note, this simulation crucially relies on the fact that keys can be *equivocated*: While the output keys are all associated to 0, when altering the output mapping accordingly evaluation will still succeed. Note that in the selective setting for Yao's original scheme as well as in the adaptive setting for the modified scheme [JW16] the input is known before the output mapping is sent.

$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^{g(0,0)}))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^0))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^0))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^0))$
$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^{g(1,0)}))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^0))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^1))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^0))$
$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^{g(0,1)}))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^0))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^1))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^0))$
$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^{g(1,1)}))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^1))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^0))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^0))$
(a)	(b)	(c)	(d)

Table 1: Garbling tables for (a) general gate g, (b) AND gate, (c) XOR gate, and (d) constant-0 gate. u and v denote the two input wires, whereas w denotes the output wire.

revealed during evaluation, then the garbling of an XOR gate can be proven indistinguishable from the constant-$(a \oplus b)$ gate only by first embedding a challenge key at k_u^{1-a} and then a second challenge key at k_v^{1-b}, or vice versa; i.e. the reduction needs to embed challenges at each input wire.

Pebbling rules. To complete the description of a pebble game, we need to describe the pebbling rules. These rules essentially capture the following observation: a reduction (with overwhelming probability) cannot possess encryptions of its (IND-CPA) challenge key. Therefore, whenever the garbling table $\tilde{\boldsymbol{g}}$ of a gate g has been switched to a malformed garbling $\tilde{\boldsymbol{g}}'$ (say) at distance one, (at least) one of the garbling tables associated to its predecessor gates, say g_u, *must* have been first switched to a garbling that encodes only one of g_u's output keys. This is required to "free up" one of g_u's output keys (so that it can now be set as the challenge key). Looking ahead, we will be interested in pebbling the circuit $\mathsf{G}^{\oplus}$ which consists of XOR gates only. Hence, the pebbling rules are designed to capture the structure of XOR gates. Recall that an XOR gate is at distance 2 from a constant gate, thus, we end up with the following rules (where g_u and g_v denote the two predecessors of g):

1. a gray pebble can be placed on or removed from a gate g only if (at least) one of its predecessor gates (say g_u) carries a black pebble; and

2. a gray pebble on a gate g can be swapped with a black pebble if the *other* predecessor gate (i.e., g_v) carries a black pebble.

The actual game. The above white-gray-black ($\mathtt{WGB}$) pebble game is a simplified version of the ($\mathtt{WG^3B}$) pebble game we end up using, but it is sufficient to convey the essential ideas that we use. The actual game, defined in Definition 7 (Section 4.3), is more fine-grained: in order to keep track of the inner and outer encryptions, we introduce three types of gray pebbles (gray-left, gray-right and gray-free), and the pebbling rules are also modified accordingly.

Definition 7 (Reversible $\mathtt{WG^3B}$ pebbling game for indegree-2 graphs)**.** Consider a directed acyclic graph $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = [1, S]$ and let $\mathcal{X} = \{\mathtt{W}, \mathtt{G}_*, \mathtt{G}_L, \mathtt{G}_R, \mathtt{B}\}$ denote the set of colours of the pebbles. Consider a sequence $\boldsymbol{\mathcal{P}} := (\mathcal{P}_0, \ldots, \mathcal{P}_\tau)$ of pebbling configurations for G, where $\mathcal{P}_i \in \mathcal{X}^{\mathcal{V}}$ for all $i \in [0, \tau]$. We call such a sequence *a $\mathtt{WG^3B}$ pebbling strategy* for G if the following two criteria are satisfied:

1. In the initial configuration all the vertices are pebbled white (i.e., $\mathcal{P}_0 = (\mathtt{W}, \dots, \mathtt{W})$) and in the final configuration *at least one sink* of $\mathtt{G}$ is pebbled gray (i.e., $\mathcal{P}_\tau = (\dots, \mathtt{G}_\cdot, \dots)$), where $\mathtt{G}_\cdot$ denotes an arbitrary type of gray, i.e. $\mathtt{G}_\cdot \in \{\mathtt{G}_*, \mathtt{G}_L, \mathtt{G}_R\}$.

2. Two subsequent configurations differ only in one vertex and the following rules are respected in each move:

   (a) $\mathtt{W} \leftrightarrow \mathtt{G}_*$: a white pebble can be replaced by a $\mathtt{G}_*$ pebble (and vice versa) if one of its parents is black-pebbled

   (b) $\mathtt{W}/\mathtt{G}_* \leftrightarrow \mathtt{G}_L$: a white or $\mathtt{G}_*$ pebble can be replaced by a $\mathtt{G}_L$ pebble (and vice versa) if its *left* parent is black-pebbled

   (c) $\mathtt{W}/\mathtt{G}_* \leftrightarrow \mathtt{G}_R$: a white or $\mathtt{G}_*$ pebble can be replaced by a $\mathtt{G}_R$ pebble (and vice versa) if its *right* parent is black-pebbled

   (d) $\mathtt{G}_L \leftrightarrow \mathtt{B}$: a $\mathtt{G}_L$ pebble can be replaced by a black pebble (and vice versa) if its *right* parent is black-pebbled

   (e) $\mathtt{G}_R \leftrightarrow \mathtt{B}$: a $\mathtt{G}_R$ pebble can be replaced by a black pebble (and vice versa) if its *left* parent is black-pebbled

The *space-complexity* of a $\mathtt{WG^3B}$ pebbling strategy $\boldsymbol{\mathcal{P}} = (\mathcal{P}_0, \dots, \mathcal{P}_\tau)$ for a DAG G is defined as

$$\sigma_G(\boldsymbol{\mathcal{P}}) := \max_{i \in [0,\tau]} |\{j \in [1, S] : \mathcal{P}_i(j) \in \{\mathtt{G}_*, \mathtt{G}_L, \mathtt{G}_R, \mathtt{B}\}\}|.$$

For a subgraph G' induced on vertex set $\mathcal{V}' \subset \mathcal{V}$, the space-complexity of $\boldsymbol{\mathcal{P}}$ restricted to G' is defined as

$$\sigma_{|G'}(\boldsymbol{\mathcal{P}}) := \max_{i \in [0,\tau]} |\{j \in \mathcal{V}' : \mathcal{P}_i(j) \in \{\mathtt{G}_*, \mathtt{G}_L, \mathtt{G}_R, \mathtt{B}\}\}|.$$

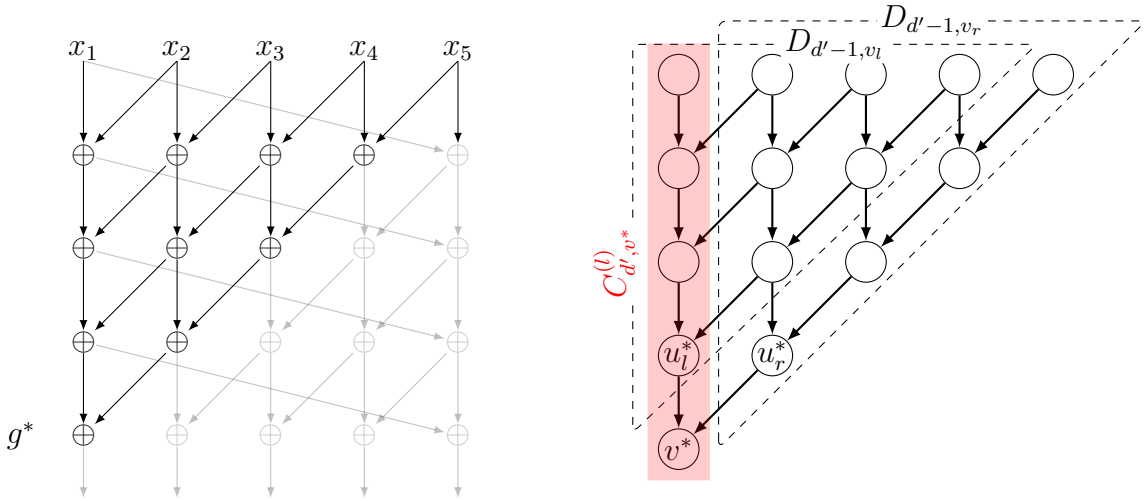
The space-complexity of a DAG G is the minimum space-complexity over all of its strategies $\boldsymbol{\mathcal{P}}^G$:

$$\sigma(G) := \min_{\boldsymbol{\mathcal{P}} \in \boldsymbol{\mathcal{P}}^G} \sigma_G(\boldsymbol{\mathcal{P}}). \tag{5}$$

A strategy matching the space-complexity of a DAG is deemed *space-optimal* for that DAG.

Remark 1. Note that for upper bounds the $\mathtt{G}_*$ pebbles would be redundant in the following sense: any $\mathtt{WG^3B}$ pebbling sequence including $\mathtt{G}_*$ pebbles can be replaced by a valid $\mathtt{WG^3B}$ sequence (potentially including redundant steps) that does not contain $\mathtt{G}_*$ pebbles and has a smaller or equal space-complexity. The reader familiar with pebbling games might notice that – ignoring the $\mathtt{G}_*$ pebbles – our $\mathtt{WG^3B}$ pebbling rules exactly correspond to *reversible edge-pebbling* [JKK+17]: In this game, pebbles are placed on edges instead of nodes, and a pebble can be placed on/removed from an edge $(u, v) \in \mathcal{E}$ if and only if all edges incident on u are pebbled.

The following lemma gives a lower bound on the $\mathtt{WG^3B}$ pebbling complexity of the graph $G \setminus G^0$ underlying the first two blocks $\mathsf{G}^\wedge \circ \mathsf{G}^\oplus$ of our candidate circuit G.



(a) (b)

Figure 5: Pyramid graph used in the proof of Lemma 5. (a) The gates in $\mathsf{G}^{\oplus}$ from which the gate g^* can be reached are highlighted (b) The correponding pyramid graph $D_{d'}$. The subgraphs of interest are highlighted.

Lemma 5 (Pebbling lower bound on $G \setminus G^0$)**.** *Let $G \setminus G^0$ be the graph underlying the circuit $\mathsf{G}^{\wedge} \circ \mathsf{G}^{\oplus}$. To gray-pebble a gate on layer $d' \in [1, d+1]$ following the reversible $\mathtt{WG^3B}$ pebbling rules from Definition 7, one requires space-complexity at least $d'-1$. Furthermore, to $\mathtt{G}_L$- or $\mathtt{B}$-pebble a gate on layer $d' \geq d+1$, one requires at least d gray or black pebbles simultaneously on the first d layers.*

Proof. We rely on the crucial observation that the ancestor graph of any vertex v in layer $d' \in [1, d+1]$ of G forms a so-called pyramid graph of depth d', with v as the unique sink. Let's denote this graph by $D_{d'}$. To prove the lemma, first note, that pebbling any node on layer d' requires to *black*-pebble one of its parents (see Figure 5). Thus, to prove the Lemma, it suffices to argue that it takes space-complexity $\geq d'-1$ to place a black pebble on the sink of $D_{d'-1}$. We will prove the following slightly stronger claim:

Claim 1. *Any $\mathtt{WG^3B}$ pebbling sequence $\boldsymbol{\mathcal{P}} = (\mathcal{P}_0, \ldots, \mathcal{P}_L)$ on $D_{d'}$ with unique sink v^*, where $\mathcal{P}_0 = \{\mathtt{W}, \ldots, \mathtt{W}\}$ is the all-white configuration and $\mathcal{P}_L = \{\ldots, \mathtt{B}\}$ is a configuration where the sink of $D_{d'}$ is black-pebbled, contains a pebbling configuration $\mathcal{P}^*$ such that $\sigma(\mathcal{P}^*) \geq d'$ and each path from a source to v^* contains at least one grey or black pebble.*

We prove the claim via induction on d'. For $d' = 1$ the claim is obviously true. For the induction step, note that, to black-pebble node v^*, one needs to black-pebble its parents v_l and v_r before, not necessarily at the same time though. Let $D_{d'-1,v_l}$, $D_{d'-1,v_r}$ denote the ancestor graphs of v_l and v_r, respectively, each being pyramid graphs of depth $d'-1$. By induction hypothesis, there must exist configurations $\mathcal{P}_l$, $\mathcal{P}_r$ in $\boldsymbol{\mathcal{P}}$ which contain $d'-1$ grey or

black pebbles in $D_{d'-1,v_l}$ and $D_{d'-1,v_r}$, respectively, such that each path from a source to v_l/v_r contains at least one grey or black pebble. Let $\mathcal{P}_l$, $\mathcal{P}_r$ denote the last such configurations. Let $C^{(l)}_{d',v^*}$ denote the left-most path in $D_{d'}$, which passes through v_l.

Let's first consider the case that $\mathcal{P}_l = \mathcal{P}_r$. Since $C^{(l)}_{d',v^*}$ and $D_{d'-1,v_r}$ are disjoint, the properties of $\mathcal{P}_l$ and $\mathcal{P}_r$ imply that there must be at least d' grey or black pebbles in $\mathcal{P}_l = \mathcal{P}_r$ and since all paths to v^* either go through v_l or through v_r, also the second property is true and the claim follows with $\mathcal{P}^* := \mathcal{P}_l = \mathcal{P}_r$.

Now, w.l.o.g., assume $\mathcal{P}_l$ occurs before $\mathcal{P}_r$ in $\boldsymbol{\mathcal{P}}$. Thus, either 1) there are less than $d'-1$ pebbles on $D_{d'-1,v_l}$ in configuration $\mathcal{P}_r$ or 2) there must exist a path $C' \in D_{d'-1,v_l}$ from a source to v_l which does not carry any grey or black pebbles, in particular also one of v_l's parents is white-pebbled; and this is true for all configurations in $(\mathcal{P}_r, \dots, \mathcal{P}_L)$. If case 2) does not occur, then similar to the case $\mathcal{P}_l = \mathcal{P}_r$ we can argue that there must be a grey or black pebble on $C^{(l)}_{d',v^*}$ and the claim follows. The same is true if node v^* is grey or black pebbled.

Finally, let's assume 2) is true for $\mathcal{P}_r$ and v^* is white-pebbled. Since v_l and v^* are W-pebbled, there must exist a configuration $\mathcal{P}' \in (\mathcal{P}_r, \dots, \mathcal{P}_L)$ such that v_l is black-pebbled in $\mathcal{P}'$. Let $\mathcal{P}'$ be the first such configuration in $(\mathcal{P}_r, \dots, \mathcal{P}_L)$. We will now construct a pebbling sequence on $D_{d'-1,v_l}$ which does *not* contain any configuration with at least $d'-1$ grey or black pebbles such that all paths to v_l carry at least one pebble – a contradiction to the induction hypothesis. Note that it suffices to show that the pebbling configuration on $D_{d'-1,v_l}$ induced by $\mathcal{P}_r$ can be reached by such a sequence, and then append this sequence by the pebbling strategy induced on $D_{d'-1,v_l}$ by $(\mathcal{P}_r, \dots, \mathcal{P}')$. To define a pebbling strategy on $D_{d'-1,v_l}$ from the all-white configuration to the one induced by $\mathcal{P}_r$, which always keeps one path all-white pebbled, we introduce some further notation: Let $C' = (u_1, \dots, u_{d'-1})$ with $u_{d'-1} = v_l$ be represented as $(b_2, \dots, b_{d-1}) \in \{0,1\}^{d'-2}$, where $b_i = 0/1$ indicates that u_{i-1} is the left/right parent of v_i. Furthermore, let $C^{(0)}_{i,u}/C^{(1)}_{i,u}$ denote the leftmost/rightmost path to node u on layer i. To define our pebbling strategy, we make the following simple observation: For any $i \in [2, d'-1]$, one can reach any configuration on $C^{(1-b_i)}_{i,u_i}$ with u_i white-pebbled while keeping the path $C^{(b_i)}_{i,u_i}$ all-`W` pebbled; this can be done by greedily black-pebbling all ancestors of grey or black pebbled nodes in $C^{(1-b_i)}_{i,u_i}$ and then reversibly switching all ancestors outside this set back to white (note, there are no ancestors in $C^{(b_i)}_{i,u_i}$, so this path remains white-pebbled). Thus, we define our pebbling strategy as follows

- For $i = d'-1, \dots, 2$: Greedily black-pebble all ancestors of nodes in $C^{(1-b_i)}_{i,u_i}$ which are grey or black pebbled, and reach the pebbling configuration $\mathcal{P}_r$ induces on $C^{(1-b_i)}_{i,u_i}$, then reversibly white-pebble all ancestors of $C^{(1-b_i)}_{i,u_i}$.

Note, throughout the i-th step the path $C^{(b_i)}_{i,u_i} \cup (u_i, \dots, u_{d'-1})$ remains white-pebbled and the pebbling configuration reached after the i-th step coincides with $\mathcal{P}_r$ on $\bigcup_{j \in [i,d'-1]} C^{(1-b_i)}_{i,u_i}$, and since u_i is white-pebbled in $\mathcal{P}_r$ the algorithm indeed terminates at the configuration which is induced on $D_{d'-1,v_l}$ by $\mathcal{P}_r$. This proves the claim. □

The following definition now gives a *cut in the configuration graph*; our adversary $\mathcal{A}$ will be a *threshold* adversary with respect to this cut.

Definition 8 (Good pebbling configurations)**.** A pebbling configuration $\mathcal{P}$ on DAG $G \setminus G^0$ is called *good* if it is *reachable* by reversible $\mathtt{WG^3B}$ pebbling moves using less than d gray or black pebbles on the first d layers simultaneously, i.e., there exists a $\mathtt{WG^3B}$ pebbling strategy $\boldsymbol{\mathcal{P}} := (\mathcal{P}_0, \ldots, \mathcal{P})$ for G such that $\sigma_{|G^{\oplus}}(\boldsymbol{\mathcal{P}}) \leq d - 1$.

In particular, by Lemma 5, any pebbling configuration $\mathcal{P}$ with a $\mathtt{G}_L$ or $\mathtt{B}$ pebble on a gate in $\mathsf{G}^{\wedge}$ is bad.

4.4 Extraction of Pebbling Configuration on $G \setminus G^0$

Given the garbled circuit $\tilde{\mathsf{G}}$ and input $\tilde{x}$, our adversary $\mathcal{A}$ maps $(\tilde{\mathsf{G}}, \tilde{x})$ to a *pebbling configuration* on the subgraph $G \setminus G^0$ of the DAG G underlying G. It's output behaviour then depends on whether this pebbling configuration lies in the cut defined by Definition 8. In this section we will discuss how to extract such a pebbling configuration. Note, that $\mathcal{A}$ is computationally unbounded, hence can extract messages and keys from ciphertexts by brute-force search.

1. First, check whether $(\tilde{\mathsf{G}}, \tilde{x})$ *evaluates correctly*, i.e., $\mathsf{GEval}(\tilde{\mathsf{G}}, \tilde{x}) = \mathsf{G}(x_0)$.
   If the evaluation check passes, check whether $\tilde{\mathsf{G}}, \tilde{x}$ have the *correct syntax*: Check whether $\tilde{\mathsf{G}}$ consists of four ciphertexts for each gate, which have the following form

$$\begin{aligned} c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(m_2)), \\ \{c_3, c_4\} = \{\mathsf{Enc}_{k_2}(m_3), \mathsf{Enc}_{k_2}(m_4)\}, \end{aligned} \tag{6}$$

   for distinct keys k_1, k_2, k_3, k_4, k_5 and arbitrary (not necessarily distinct) messages m_2, m_3, m_4, where keys k_1 and k_3 are revealed during evaluation $\mathsf{GEval}(\tilde{\mathsf{G}}, \tilde{x})$. I.e., two of the four ciphertexts are encryptions under the same left secret keys k_1 and k_2, respectively, one of them is a double encryption $\mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5))$ under left key k_1 and some right key k_3 of an output key k_5 (all these being revealed throughout evaluation), and the second encryption under k_1 encrypts an encryption under a second right key k_4 (of an arbitrary message m_2).
   Finally, check *consistency of keys*: For each gate, extract key pairs (k_1, k_2) and (k_3, k_4) corresponding to left and right input wires, and check whether they are consistent with the keys extracted from sibling gates: If gate g is the left sibling of g', then g's right input key pair must coincide with the left key pair extracted from g', i.e., $(k_3, k_4) = (k'_1, k'_2)$. Note, if this check passes, then all wires in the circuit can be uniquely associated with a key pair. Finally, check that all extracted keys are distinct. If any of these checks fails, map $(\tilde{\mathsf{G}}, \tilde{x})$ to a bad pebbling configuration, e.g., to the

pebbling configuration on $\mathtt{G}$ where all gates at levels $[d+1, 2d+1]$ are black pebbled[11] and quit.

Remark 2. Note, syntax and consistency checks allow a reduction to distinguish

- a ciphertext from a non-ciphertext,
- a ciphertext under key k from a ciphertext under key $k' \neq k$.

We will argue in Section 4.5 that this is of no help to the reduction for breaking IND-CPA security of the information-theoretic encryption scheme $\mathcal{F}$.

For all garblings $(\tilde{\mathsf{G}}, \tilde{x})$ that pass correctness, syntax, and consistency checks, $\mathcal{A}$ will extract a pebbling configuration on $G \setminus G^0$ by mapping each gate to a color in $\{\mathtt{W}, \mathtt{G}_*, \mathtt{G}_L, \mathtt{G}_R, \mathtt{B}\}$.

2. For each XOR gate g_j ($j \in [1, d] \cdot n + [0, n]$): Check whether g_j is garbled correctly with respect to input x_0. To this aim, let b_l, b_r, and $b_o = g_j(b_l, b_r) = b_l \oplus b_r$ denote the left/right input and the output bit of g_j, respectively, when evaluating G on x_0. We use the same notation as in Equation 6 above; furthermore, let k_6 be the second key associated with the output wire (which was extracted from the garbling tables of the successor gates).
   - If g_j is garbled similar to the case of an honest garbling of (G, x_0), i.e., $m_2 = k_6$, $m_3 = \mathsf{Enc}_{k_3}(k_6)$, and $m_4 = \mathsf{Enc}_{k_4}(k_5)$ (or the roles of m_3, m_4 permuted), then associate g_j with a $\mathtt{W}$ pebble.
   - If m_2 and m_3 are as in the previous case, but $m_4 = \mathsf{Enc}_{k_4}(m)$ for some message $m \neq k_5$, then associate g_j with a $\mathtt{G}_*$ pebble. Similarly for the case where the roles of m_3, m_4 are permuted.
   - If m_3 is as in the first case, $m_4 = \mathsf{Enc}_{k_4}(m)$ for an arbitrary message m, but $m_2 \neq k_6$, then associate g_j with a $\mathtt{G}_R$ pebble. Similarly for the case where the roles of m_3, m_4 are permuted.
   - If $m_2 = k_6$ is as in the first case, but $\{m_3, m_4\}$ differs from the previous cases, then associate g_j with a $\mathtt{G}_L$ pebble.
   - For all other cases, associate g_j with a $\mathtt{B}$ pebble.

Remark 3. Due to symmetry of the XOR operation, whether a gate is considered properly garbled (i.e. mapped to a white pebble) or not (i.e. mapped to gray or black) does *not* depend on the input keys. Thus, the set of black and gray pebbles on $G^{\oplus}$ can be extracted *independently* of x_0 and $\tilde{x}$.

3. For each AND gate g_j ($j \in [d+1, 2d+1] \cdot n + [0, n]$): Similar to the case of XOR gates, check whether the gate is correctly garbled with respect to x_0. Using the same notation as above, associate g_j with a pebble as follows:

[11] This choice was made for convenience (see Lemmas 7 to 9), but in principle could be an arbitrary bad configuration, and should simply guarantee that no reduction can gain any advantage by departing from the protocol in an obvious way.

- If g_j is garbled similar to the case of an honest garbling of (G, x_0), i.e., for

  $(b_l, b_r) = (0,0)$, we have $m_2 = k_5$, $m_3 = \mathsf{Enc}_{k_3}(k_5)$, and $m_4 = \mathsf{Enc}_{k_4}(k_6)$,
  $(b_l, b_r) = (0,1)$, we have $m_2 = k_5$, $m_3 = \mathsf{Enc}_{k_3}(k_6)$, and $m_4 = \mathsf{Enc}_{k_4}(k_5)$,
  $(b_l, b_r) = (1,0)$, we have $m_2 = k_6$, $m_3 = \mathsf{Enc}_{k_3}(k_5)$, and $m_4 = \mathsf{Enc}_{k_4}(k_5)$,
  $(b_l, b_r) = (1,1)$, we have $m_2 = k_6$, $m_3 = \mathsf{Enc}_{k_3}(k_6)$, and $m_4 = \mathsf{Enc}_{k_4}(k_6)$,

  (or the roles of m_3, m_4 permuted) then associate g_j with a $\mathtt{W}$ pebble.
- If m_2 and m_3 are as in the previous case, but $m_4 = \mathsf{Enc}_{k_4}(m)$ for some message m that differs from above, then associate g_j with a $\mathtt{G}_*$ pebble. (Similarly for the case where the roles of m_3, m_4 are permuted.)
- If m_3 is as in the first case, $m_4 = \mathsf{Enc}_{k_4}(m)$ for an arbitrary message m, but m_2 differs from the previous case, then associate g_j with a $\mathtt{G}_R$ pebble. (Similarly for the case where the roles of m_3, m_4 are permuted.)
- If m_2 is as in the first case, but $\{m_3, m_4\}$ differs from the previous cases, then associate g_j with a $\mathtt{G}_L$ pebble.
- For all other cases, associate g_j with a B pebble.

Remark 4. At first sight, it might seem counterintuitive that the mapping from gates to colours not only depends on the associated ciphertexts, but also on the input x_0. This however is unavoidable since the adversary $\mathcal{A}$ cannot simply map keys to bits, but can only *relate* them to the keys it learned from $\tilde{x}$, which might be properly garbled or not.

In the following lemma, we prove that the adversary $\mathcal{A}$ using the above pebbling extraction indeed breaks indistinguishability of Yao's garbling scheme.

Lemma 6. *$\mathcal{A}$ breaks indistinguishability of the garbling scheme with probability $1 - 1/2^{n-1}$.*

Proof. We defined our adversary $\mathcal{A}$ to output $b' = 0$ whenever the extracted pebbling configuration is *good*, and $b' = 1$ else. In particular, Definition 8 guarantees that $\mathcal{A}$ outputs $b' = 0$ if there are only white pebbles on the subgraph $G \setminus G^0$ of the topology graph G of G, i.e., when $(\tilde{\mathsf{G}}_b, \tilde{x}_b)$ is distributed identically to $(\tilde{\mathsf{G}}, \tilde{x}_0)$. On the other hand, when x_1 was garbled, then – as we will show below – there will be at least one gray or black pebble on layer $d+1$. Hence, since by Lemma 5 switching a pebble on layer $d' = d+1$ from $\mathtt{W}$ to $\mathtt{G}_*$, $\mathtt{G}_R$, $\mathtt{G}_L$ or $\mathtt{B}$ requires at least d gray or black pebles simultanously on the first d layers, $\mathcal{A}$ outputs $b' = 1$ in this case.

It remains to show that (with all but negligible probability) a proper garbling $(\tilde{\mathsf{G}}, \tilde{x}_1)$ will be mapped to a pebbling configuration which has at least one gray or black pebble on layer $d+1$. To this aim, we use the following properties of the circuit $\mathsf{G}^{\oplus}$, which were established in Section 4.2, Corollary 1: First, $\mathsf{G}^{\oplus}$ is 2-to-1, with x_0 and $x_0 \oplus 1^n$ being the two preimages of $\mathsf{G}^{\oplus}(x_0)$. Second, the image of $\mathsf{G}^{\oplus}$ only consists of strings containing an even number of 1s.

Now, assume $x_1 \notin \{x_0, x_0 \oplus 1^n\}$ (which happens with probability $1 - 1/2^{n-1}$), and $\mathsf{G}^{\oplus}(x_0)$ and $\mathsf{G}^{\oplus}(x_1)$ differ in the i-th bit and coincide in the $i+1$-th bit; note that such an i must

exist, since $\mathsf{G}^{\oplus}(x_0) \oplus 1^n$ is not in the image of $\mathsf{G}^{\oplus}$ (by the second property of $\mathsf{G}^{\oplus}$). Assume the i-th and $i+1$-th bits of $\mathsf{G}^{\oplus}(x_0)$ are 0, i.e., using the same notation as above, we analyze the case $(b_l, b_r) = (0, 0)$; the other cases work similarly. Consider the garbling of the i-th AND gate $\mathtt{G}$ on layer $d+1$ w.r.t. the keys k_1, k_3, k_5 revealed through evaluation of $\tilde{\mathsf{G}}$ on input $\tilde{x}_1$. Then this coincides with the case $(b_l, b_r) = (1, 0)$, in particular, $m_2 = k_6$ and $m_4 = \mathsf{Enc}_{k_4}(k_5)$ differ from the garbling for input x_0, while m_3 is similar. Hence, $\mathcal{A}$ associates this gate with a $\mathtt{G}_R$ pebble. Similarly, we can see that if the left input coincides but the right differs, $\mathcal{A}$ associates $\mathtt{G}$ with a $\mathtt{G}_L$ pebble. Finally, if both inputs differ, this implies that in the garbling m_2 and m_3 differ, hence $\mathcal{A}$ maps $\mathtt{G}$ to a $\mathtt{B}$ pebble. This proves the claim. □

Since $\mathcal{A}$ extracts the pebble mode of a gate with regard to the garbled input (i.e., the keys it learns through evaluation), the reduction can still change the mode of a gate *after* it output $\tilde{\mathsf{G}}$ by choosing different input keys for $\tilde{x}$. In the following lemmas we prove that this flexibility of choosing the input keys is of not much help to a reduction aiming at a good pebbling configuration, where in particular all gates at layers $[d+1, 2d+1]$ are mapped to $\mathtt{W}$, $\mathtt{G}_*$, or $\mathtt{G}_R$ pebbles.

First, we consider the case of a properly garbled AND gates. In this case, due to the asymmetry of the AND operation, input keys can be associated with bits and hence a properly garbled layer of AND gates has a similar function as an output mapping.

Lemma 7. *For any garbling of an AND gate on layer $[d+1, 2d+1]$, and any input bits b_l, b_r, there exists at most one input key pair (k_1, k_3) such that the gate will be mapped to a $\mathtt{W}$ pebble.*

Proof. For the claim on gates mapped to $\mathtt{W}$ pebbles, we only consider the case $(b_l, b_r) = (0, 0)$, the others work similarly. Let $g_\wedge$ be an AND gate that is mapped to a $\mathtt{W}$ pebble. Hence, the four associated ciphertexts must have the following form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(k_5)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_5)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(k_6)).$$

Since $g_\wedge$ is mapped to a $\mathtt{W}$ pebble, in particular evaluation, syntax, and consistency checks must pass. Hence, all keys are distinct, k_1, k_2 are associated to the left input wire, k_3, k_4 are associated to the right input wire, and during evaluation two input keys $k_l \in \{k_1, k_2\}$ and $k_r \in \{k_3, k_4\}$ are reveiled. We will now show that it must hold $(k_l, k_r) = (k_1, k_3)$. Assume, for contradiction, $(k_l, k_r) = (k_2, k_3)$. Then $g_\wedge$ will be mapped to a $\mathtt{G}_R$ pebble, because the inner encryptions (under key k_4) of ciphertexts c_2 and c_4 are malformed. Similarly, if $(k_l, k_r) = (k_1, k_4)$, then c_3 and c_4 are considered malformed and $g_\wedge$ is mapped to a $\mathtt{G}_L$ pebble. Finally, if $(k_l, k_r) = (k_2, k_4)$, then c_2, and c_3 are considered malformed and $g_\wedge$ is mapped to a $\mathtt{B}$ pebble. This implies that $g_\wedge$ is mapped to a $\mathtt{W}$ pebble only if $(k_l, k_r) = (k_1, k_3)$. □

The situation becomes a bit more involved if AND gates are not properly garbled, since in this case asymmetry might be broken. However, if the left input keys can be mapped to bits, then we can still obtain some meaningful guarantees. We first consider the case that an

AND gate is garbled in $\mathtt{G}_*$ mode, i.e. one ciphertext is malformed and there exist some input bits (b_l, b_r) such that it will be mapped to a $\mathtt{G}_*$ pebble. In the following Lemma we prove that for a different right input bit $1 - b_r$ the gate will be mapped to a $\mathtt{G}_L$ pebble instead.

Lemma 8. *For any garbling of an AND gate, any left input bit b_l, and fixed left input key, there exists at most one $b_r \in \{0,1\}$ such that there exists a (not necessarily unique) right input key such that the gate will be mapped to a $\mathtt{G}_*$ pebble. If such a right input bit b_r exists, then for right input bit $1 - b_r$ the gate will be mapped to a $\mathtt{G}_L$ pebble.*

Proof. Consider the case that an AND gate $g_\wedge$ is mapped to a $\mathtt{G}_*$ pebble for $(b_l, b_r) = (0,0)$. In this case, the four ciphertexts associated to $g_\wedge$ must have the form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(k_5)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_5)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(m)),$$

for some message $m \neq k_6$ and left input key $k_l = k_1$. Now, for $(b_l, b_r) = (0,1)$ and $k_l = k_1$, a gate garbled as above will be mapped to a $\mathtt{G}_L$ pebble, no matter whether the right input key is k_3 or k_4.
Next, consider the case that an AND gate $g_\wedge$ is mapped to a $\mathtt{G}_*$ pebble for $(b_l, b_r) = (1,0)$. In this case, the four ciphertexts associated to $g_\wedge$ must have the form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(k_6)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_5)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(m)),$$

for some message $m \neq k_5$ and left input key $k_l = k_1$. Then, for $(b_l, b_r) = (1,1)$ and $k_l = k_1$, a gate garbled as above will be mapped to a $\mathtt{G}_L$ pebble, no matter whether the right input key is k_3 or k_4. □

Next we consider the case of an AND gate that is garbled in $\mathtt{G}_R$ mode w.r.t. some input bits (b_l, b_r). In this case we have to distinguish two different ways to garble a gate such that it will be mapped to a $\mathtt{G}_R$ pebble. For one type of $\mathtt{G}_R$ pebble we can map keys to bits, just as in the case of properly garbled gates. For the second type of $\mathtt{G}_R$ pebble we obtain a similar guarantee as for $\mathtt{G}_*$ pebbles.

Lemma 9. *For any garbling of an AND gate on layer $[d+1, 2d+1]$, any left input bit b_l, and fixed left input key, one of the following is true:*

1. *For any right input bit $b_r \in \{0,1\}$ there exists at most one right input key such that the gate will be mapped to a $\mathtt{G}_R$ pebble. If such a key exists, then for any other right input key the gate will be mapped to a $\mathtt{B}$ pebble.*

2. *There exists at most one input bit $b_r \in \{0,1\}$ such that there exists a right input key k_r such that the gate will be mapped to a $\mathtt{G}_R$ pebble. If such a bit exists, then for right input bit $1 - b_r$ and any right input key the gate will be mapped to a $\mathtt{B}$ pebble.*

These two cases characterize two different types of $\mathtt{G}_R$ pebbled gates, where we denote a gate as $\mathtt{G}_R$-type-1 if case 1 is true, and $\mathtt{G}_R$-type-2 if only *case 2 is true.*

Proof. First, consider the case that an AND gate $g_\wedge$ is mapped to a $\mathtt{G}_R$ pebble for $(b_l, b_r) = (0, 0)$. In this case, the four ciphertexts associated to $g_\wedge$ must have the form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(m)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_5)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(m')),$$

for some message $m \neq k_5$, an arbitrary message m', and left input key $k_l = k_1$. We first consider the case $m = k_6$, $m' = k_5$. Then, for $b_r = 0$ and right input key $k_r = k_4$, a gate garbled as above will be mapped to a $\mathtt{B}$ pebble; i.e. case 1 happens. Similarly, for $b_r = 1$ and right input key $k_r = k_3$, a gate garbled as above will be mapped to a $\mathtt{B}$ pebble; i.e. case 1 happens. Next consider the case $m = k_6$, $m' = k_6$. Then, for $b_r = 1$ and any right input key k_r, a gate garbled as above will be mapped to a $\mathtt{B}$ pebble, i.e. case 2 happens. Finally, for $m, m' \notin \{k_5, k_6\}$, evaluation fails for $k_r = k_4$, hence the gate will be mapped to a $\mathtt{B}$ pebble. However, also for $b_r = 1$ the gate will be mapped to a $\mathtt{B}$ pebble; hence both cases 1 and 2 are true.

Next, consider the case that an AND gate $g_\wedge$ is mapped to a $\mathtt{G}_R$ pebble for $(b_l, b_r) = (1, 0)$. In this case, the four ciphertexts associated to $g_\wedge$ must have the form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(m)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_5)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(m')),$$

for some message $m \neq k_6$, an arbitrary message m', and left input key $k_l = k_1$. Then, for $m = k_5$, $m' = k_5$, and $b_r = 1$, the gate will be mapped to a $\mathtt{B}$ pebble, independently of the right input key, i.e. case 2 happens. For $m = k_5$, $m' = k_6$, on the other hand, if $b_r = 0$ and the right input key is $k_r = k_4$, then the gate will be mapped to a $\mathtt{B}$ pebble; and analogously, if $b_r = 1$ and the right input key is $k_r = k_3$, then the gate will be mapped to a $\mathtt{B}$ pebble, i.e. case 1 happens. For $m, m' \notin \{k_5, k_6\}$ the gate will be mapped to a $\mathtt{B}$ pebble and both cases are true.

Next, consider the case that an AND gate $g_\wedge$ is mapped to a $\mathtt{G}_R$ pebble for $(b_l, b_r) = (0, 1)$. In this case, the four ciphertexts associated to $g_\wedge$ must have the form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(m)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_6)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(m')),$$

for some message $m \neq k_5$, an arbitrary message m', and left input key $k_l = k_1$. Then, for $m = k_6$, $m' = k_5$, and $b_r = 0$, the gate will be mapped to a $\mathtt{B}$ pebble, independently of the right input key, i.e. case 2 happens. For $m = k_6$, $m' = k_6$, on the other hand, if $b_r = 0$ and the right input key is $k_r = k_3$, then the gate will be mapped to a $\mathtt{B}$ pebble; and analogously, if $b_r = 1$ and the right input key is $k_r = k_4$, then the gate will be mapped to a $\mathtt{B}$ pebble, i.e. case 1 happens. For $m, m' \notin \{k_5, k_6\}$ the gate will be mapped to a $\mathtt{B}$ pebble and both cases are true.

Finally, consider the case that an AND gate $g_\wedge$ is mapped to a $\mathtt{G}_R$ pebble for $(b_l, b_r) = (1,1)$. In this case, the four ciphertexts associated to $g_\wedge$ must have the form

$$c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_6)),\ c_2 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_4}(m)),$$

$$c_3 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_3}(k_5)),\ c_4 = \mathsf{Enc}_{k_2}(\mathsf{Enc}_{k_4}(m')),$$

for some message $m \neq k_5$, an arbitrary message m', and left input key $k_l = k_1$. Then, for $m = k_6$, $m' = k_5$, and $b_r = 0$, the gate will be mapped to a $\mathtt{B}$ pebble, independently of the right input key, i.e. case 2 happens. For $m = k_6$, $m' = k_6$, on the other hand, if $b_r = 0$ and the right input key is $k_r = k_3$, then the gate will be mapped to a $\mathtt{B}$ pebble; and analogously, if $b_r = 1$ and the right input key is $k_r = k_4$, then the gate will be mapped to a $\mathtt{B}$ pebble, i.e. case 1 happens. For $m, m' \notin \{k_5, k_6\}$ the gate will be mapped to a $\mathtt{B}$ pebble and both cases are true.

□

4.5 Lower Bound on Security Loss for any Reduction

In this section we will combine all previous results to prove a lower bound on adaptive security of Yao's garbling scheme. More precisely, we will prove that any black-box reduction which aims to exploit $\mathcal{A}$'s distinguishing advantage to break IND-CPA security of the underlying encryption scheme loses a factor subexponential in the depth of the circuit.

Let R be an arbitrary PPT reduction which has black-box access to an adversary $\mathcal{A}$ that breaks indistinguishability of Yao's garbling scheme, and attempts to solve an IND-CPA challenge with respect to an encryption scheme $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$. Following the approach of Kamath et al. [KKPW21], we define an *information-theoretically* secure encryption scheme $\mathcal{F} = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ as follows: For $l \in \{1, 6\}$, let $\mathsf{E}_l : \{0,1\}^{(l+2)\lambda} \to \{0,1\}^{2(l+2)\lambda}$ be a random expanding function (which is injective with overwhelming probability).

- Key generation $\mathsf{Gen}(1^\lambda)$: On input a security parameter λ in unary, output a key $k \leftarrow \{0,1\}^*$ uniformly at random.
- Encryption $\mathsf{Enc}(k, m)$: On input a key $k \in \{0,1\}^\lambda$ and a message $m \in \{0,1\}^{l \cdot \lambda}$ with $l \in \{1, 6\}$, sample randomness $r \leftarrow \{0,1\}^\lambda$, and output $\mathsf{E}_l(k, m; r)$.
- Decryption $\mathsf{Dec}(k, c)$ is simulated to be consistent with Enc: On input a key $k \in \{0,1\}^\lambda$ and a ciphertext $c \in \{0,1\}^{2(l+2)\lambda}$ with $l \in \{1, 6\}$, check whether c lies in the image of $\mathsf{E}_l(k, \cdot; \cdot)$, if so extract $m \in \{0,1\}^{l \cdot \lambda}$, $r \in \{0,1\}^\lambda$ such that $c = \mathsf{E}_l(k, m; r)$ and output m, otherwise output $\perp$.

Choosing E_l ($l \in \{1, 6\}$) to be random functions implies that $\mathcal{F}$ is information-theoretically IND-CCA secure. Thus, since R only makes polynomially many queries, the only non-negligible advantage R has in breaking the IND-CPA security of $\mathcal{F}$ must stem from its interaction with $\mathcal{A}$. Furthermore, with all but negligible (in λ) probability $\mathcal{F}$ satisfies the special property (Definition 6), hence can be used in Yao's garbling scheme.

We first argue that neither checking correctness, syntax, nor consistency (cf. Section 4.4) is of any help to R. Obviously, this is true for the correctness check, since R can efficiently evaluate $\mathsf{GEval}(\tilde{\mathsf{G}}, \tilde{x})$. However, we have to argue a bit more to prove that also syntax and consistency checks are of no help to R. To this aim, we construct an oracle $\mathcal{O}$ that allows to distinguish

- a ciphertext from an arbitrary string in $\{0,1\}^{2(l+2)\lambda}$ for $l \in \{1,6\}$,
- a ciphertext under key $k \in \{0,1\}^\lambda$ from a ciphertext under key $k' \neq k$.

More precisely, $\mathcal{O}$ takes as input two strings $s \in \{0,1\}^{2(l+2)\lambda}$ and $s' \in \{0,1\}^{2(l'+2)\lambda}$ ($l, l' \in \{1,6\}$) and checks whether s, s' lie in the image of $\mathsf{E}_l, \mathsf{E}_{l'}$, respectively. If this check fails for one of the strings, then $\mathcal{O}$ outputs $\perp$. Otherwise, it extracts preimages $(k, m, r) \in \{0,1\}^{(l+2)\lambda}$ under E_l and $(k', m', r') \in \{0,1\}^{(l'+2)\lambda}$ under $\mathsf{E}_{l'}$. If $k = k'$, $\mathcal{O}$ outputs 1, otherwise 0.

We will first prove that access to oracle $\mathcal{O}$ allows R to efficiently carry out syntax and consistency checks. Then we will prove that $\mathcal{F}$ remains information-theoretically IND-CPA secure even against adversaries that have access to $\mathcal{O}$.

Lemma 10. *There exists an algorithm $\mathsf{B}^{\mathcal{F},\mathcal{O}}$ with oracle access to $\mathcal{O}$ that given a garbled circuit and input pair $(\tilde{\mathsf{G}}, \tilde{x})$ such that $\mathsf{GEval}(\tilde{\mathsf{G}}, \tilde{x}) = \mathsf{G}(x_0)$ efficiently checks whether $(\tilde{\mathsf{G}}, \tilde{x})$ satisfies syntax and consistency (as defined in Section 4.4).*

Proof. We first describe how $\mathsf{B}^{\mathcal{F},\mathcal{O}}$ check the syntax of the four strings $s_1, s_2, s_3, s_4 \in \{0,1\}^{16\lambda}$ associated to a gate g. Since evaluation succeeds, one of these strings must have the form $c_1 = \mathsf{Enc}_{k_1}(\mathsf{Enc}_{k_3}(k_5))$ for three keys $k_1, k_3, k_5 \in \{0,1\}^\lambda$ that are revealed during evaluation; w.l.o.g., assume $s_1 = c_1$. Given k_1, B can easily check if the strings s_2, s_3, s_4 can be decrypted under k_1 by querying the decryption oracle Dec on (k_1, s_i) for all $i \in \{2,3,4\}$. The algorithm B aborts except if exactly on of the strings s_2, s_3, s_4 can be decrypted under k_1, w.l.o.g., assume this string is s_2. Now given the decryption $\mathsf{Dec}(k_1, s_2)$, B now checks whether this is a valid ciphertext by querying $\mathcal{O}(\mathsf{Dec}(k_1, s_1), \mathsf{Dec}(k_1, s_2))$. If the output of $\mathcal{O}$ is $\perp$, then $\mathsf{Dec}(k_1, s_2)$ is not a valid ciphertext and B aborts. If the output is 1, then both s_1 and s_2 are double encryptions under the same key pair $(k_1, k_4) = (k_1, k_3)$, hence B aborts. Otherwise, B continues and queries $\mathcal{O}(s_3, s_4)$. B aborts, whenever the output to this query is not 1, i.e., s_3 and s_4 are not encryptions under the same key k_2. Finally, B has to check that all keys used to generate s_1, s_2, s_3, s_4 are distinct. It already learned by previous queries that $k_1 \neq k_2$ and $k_3 \neq k_4$, hence it suffices to query $\mathcal{O}(\mathsf{Dec}(k_1, s_i), s_j)$ for $i = 1, 2,\ j = 1, 3$ and abort if any output is not 0. Otherwise accept the syntax of $(\tilde{\mathsf{G}}, \tilde{x})$.

Consistency of keys can be verified in a similar way using th oracle $\mathcal{O}$: Assume (k_1, k_2), (k_3, k_4) are the (partially unknown) distinct keys involved in the encryptions c_1, c_2, c_3, c_4 (as guaranteed by the syntax check) associated to gate g; and similarly $(k'_1, k'_2), (k'_3, k'_4)$ and c'_1, c'_2, c'_3, c'_4 are the keys and ciphertexts associated with its right sibling g'. Since k_3 and k'_1 are revealed through evaluation, it is trivially true that $k_3 = k'_1$. For the unknown keys k_4 and k'_2, on the other hand, B can check equality using $\mathcal{O}$ on $(\mathsf{Dec}(k_1, c_2), c'_3)$.

After checking that each wire in the circuit can uniquely be associated to a key pair, finally, B checks that all these keys are distinct: This works in a similar way as before by querying the oracle $\mathcal{O}$ on appropriate ciphertexts. □

Lemma 11. *The encryption scheme $\mathcal{F}$ is information-theoretically IND-CPA secure against adversaries with oracle access to $\mathcal{O}$.*

Proof. To see this, we show how the oracle $\mathcal{O}$ can be simulated by an IND-CPA challenger with oracle access to $\mathcal{F}$ that sees all the adversary's queries to $\mathcal{F}$. First, note that the probability of sampling a ciphertext from $\{0,1\}^{2(l+2)\lambda}$ ($l \in \{1,6\}$) *without* either calling Enc on a triple $(k,m,r) \in \{0,1\}^{(l+2)\lambda}$ or querying the IND-CPA oracle is negligible in λ. Thus, an oracle that allows to distinguish a ciphertext from an arbitrary string can be implemented simply by checking whether the queried string was output of a previous query; this is indistinguishable from the real functionality for any computationally bounded reduction. Furthermore, for all ciphertexts the reduction sees, it actually knows the corresponding keys, except for those derived from the IND-CPA challenger. Hence, distinguishing whether two given ciphertexts were derived under the same key is easy: In the case that both ciphertexts were derived from the IND-CPA challenger, obviously both ciphertexts were derived under the same key. In the case that at least one of the associated keys is known, the reduction can query the Dec oracle on this key and the other ciphertext. Since Enc is injective with all-but-negligible probability, the answer to this query will be $\perp$ if the keys do not coincide. □

Now, to prove that any black-box reduction from indistinguishability of Yao's garbling scheme to IND-CPA security of the underlying encryption scheme suffers from a loss that is subexponential in the depth δ of the circuit, we construct an adversary $\mathcal{A}[c^*]$ that behaves just like $\mathcal{A}$ but doesn't decrypt challenge ciphertext c^*. More precisely, $\mathcal{A}[c^*]$ with input a ciphertext c^*, has oracle access to $\mathcal{O}$, $\mathcal{F}$, as well as an IND-CCA decryption oracle Dec_{k^*} that it can query on any ciphertext $c \neq c^*$. We construct $\mathcal{A}[c^*]$ such that it never decrypts c^* unless it already knows the encryption key k^* from other keys and ciphertexts in $\tilde{\mathsf{G}}, \tilde{x}$:

- First $\mathcal{A}[c^*]$ runs evaluation, syntax, and consistency checks using oracle $\mathcal{O}$. If these checks pass, similar to $\mathcal{A}$, the algorithm $\mathcal{A}[c^*]$ uses brute-force search to decrypt all ciphertexts *except* for those encrypted under k^* (to check whether a ciphertext is encrypted under k^* it uses $\mathcal{O}$ and c^*). Ciphertexts $c \neq c^*$ encrypted under k^* it decrypts using oracle Dec_{k^*}. For c^*, there are two cases:
  - If the key k^* was learned from previous decryptions (this can be checked by decrypting c^* under all known keys), $\mathcal{A}[c^*]$ simply decrypts c^* using k^*.
  - If the k^* is not known to $\mathcal{A}[c^*]$, then it simply *assumes* $c^* \in \{0,1\}^{2(l+2)\lambda}$ with $l \in \{1,6\}$ would decrypt to $0^{l\cdot\lambda}$.

  $\mathcal{A}[c^*]$ then continues analogous to $\mathcal{A}$ by mapping $(\tilde{\mathsf{G}}, \tilde{x})$ to a pebbling configuration and outputting 0 whenever the pebbling configuration is good per Definition 8, and 1 otherwise.

Clearly, since $\mathcal{A}[c^*]$ never decrypts c^* except if k^* is known, there is no chance for R to use $\mathcal{A}[c^*]$ to break IND-CPA security of $\mathcal{F}$.[12] It remains to bound the success probability of any PPT distinguisher D to distinguish $\mathcal{A}[c^*]$ from $\mathcal{A}$.[13] To this aim, we will first show how the $\mathtt{WG}^3\mathtt{B}$ pebbling game relates to this issue.

Lemma 12. *Let $c^* \leftarrow \mathsf{Enc}_{k^*}(m)$ be an arbitrary ciphertext and let $\mathcal{P}$, $\mathcal{P}^*$ be the two pebbling configurations extracted by $\mathcal{A}$ and $\mathcal{A}[c^*]$, respectively, in the same execution of the game, i.e. using the same randomness. Then $\mathcal{P}^*$ differs from $\mathcal{P}$ by at most one valid $\mathtt{WG}^3\mathtt{B}$ pebbling move.*

Proof. First, note that whenever c^* is not embedded into $\tilde{\mathsf{G}}$ then $\mathcal{A}[c^*]$ is trivially indistinguishable from $\mathcal{A}$, as $\mathcal{A}[c^*] \equiv \mathcal{A}$ in this case. Similarly, if $k^* \in \tilde{x}$ or there exists any encryption of k^* in $\tilde{\mathsf{G}}$, then also $\mathcal{A}[c^*] \equiv \mathcal{A}$; in particular, $\mathcal{P} = \mathcal{P}^*$. Now, assume that evaluation fails for $\mathcal{A}[c^*]$ but passes in $\mathcal{A}$. Then the key k^* must be revealed through $\mathsf{GEval}(\tilde{\mathsf{G}}, \tilde{x})$. But then $\mathcal{A}[c^*]$ properly decrypts c^*; hence this cannot happen and evaluation fails for $\mathcal{A}[c^*]$ if and only it fails for $\mathcal{A}$. Also syntax and consistency checks $\mathcal{A}[c^*]$ passes if and only if $\mathcal{A}$ passes. Thus, whenever such an initial check fails, then $\mathcal{P} = \mathcal{P}^*$.

In the following we will assume that c^* is embedded in $\tilde{\mathsf{G}}$, the key k^* is never encrypted or opened in $\tilde{x}$, and all initial checks pass. The second assumption implies that the key k^* must either be embedded at a non-opened input wire or at the output wire of a gate whose associated ciphertexts only encode one of the two output keys. We will now argue that such XOR gates will be mapped to $\mathtt{B}$ pebbles: Using the notation from Section 4.4, the key k_5 is learned during evaluation, while k_6 is the second key associated to the output wire. If the garbling of the XOR gate is independent of k_6, this implies that messages m_2 and m_3 differ from the case of an honest garbling. By definition, such gates are mapped to $\mathtt{B}$ pebbles. Hence, c^* can only be embedded in the garbling table of a gate g if at least one of g's parents is black pebbled.

Now, assume there exists a (XOR or AND) gate that is $\mathtt{W}$ in $\mathcal{P}$ and $\mathtt{B}$ in $\mathcal{P}^*$; the case opposite case works analogously. For ease of notation, we consider the case of an XOR gate here, the case of AND gates follows analogously. By definition, an XOR gate is only mapped to a $\mathtt{B}$ gate, if either (1) $m_2 \neq k_6$ and $m_3 \notin \mathsf{Enc}_{k_3}(k_6)$, or (2) $m_2 \neq k_6$ and $m_4 \notin \mathsf{Im}(\mathsf{Enc}_{k_4})$. For case (1), note that since k_3 is known, m_2 and m_3 can only be switched by embedding c^* *twice*, for right key k_4 *and* left key k_2. The same is true for case (2). But since by consistency $k_2 \neq k_4$, this implies that $\mathcal{P}$ and $\mathcal{P}^*$ cannot differ by a switch from $\mathtt{W}$ to B.

Next, consider the case that a gate is mapped to $\mathtt{W}$ in $\mathcal{P}$ and to $\mathtt{G}_L$ in $\mathcal{P}^*$. We will argue that in this case the key k^* must be embedded at the left input wire. Again, for ease of notation, we consider the case of an XOR gate g; the case of AND gates follows analogously. Since $\mathcal{A}$ maps g to $\mathtt{W}$ and $\mathcal{A}[c^*]$ maps it to $\mathtt{G}_L$, g must be properly garbled and the change arrises from $\mathcal{A}[c^*]$ "decrypting" c^* to 0. Hence, c^* must be embedded either at c_3 or c_4, in

[12] Recall that our ideal encryption scheme $\mathcal{F}$ is IND-CCA secure, hence access to the oracle Dec_{k^*} used by $\mathcal{A}[c^*]$ is of no help to R.

[13] Note, we assume that $\mathcal{A}[c^*]$ has *private* access to its oracles and D cannot observe its oracle queries to distinguish it from $\mathcal{A}$.

particular $k^* = k_2$ the *left* input key. Furthermore, since g is properly garbled, it must hold either $c^* \leftarrow \mathsf{Enc}_{k^*}(\mathsf{Enc}_{k_3}(k_6))$ or $c^* \leftarrow \mathsf{Enc}_{k^*}(\mathsf{Enc}_{k_4}(k_5))$, i.e. c^* can only be embedded *once* in this gate. If c^* was additionally embedded in another gate, then this must be the left sibling g' of g due to consistency and distinctness of keys. In g', however, k^* is employed as a right input key, hence c^* can only be embedded as an inner encryption and in particular constitutes a malformed encryption (note the difference in length between inner and outer encryptions). Thus, both $\mathcal{A}$ and $\mathcal{A}[c^*]$ will map g' to the same pebble.

In a similar way, one can prove that whenever a gate is mapped to $\mathtt{W}$ in $\mathcal{P}$ and to $\mathtt{G}_R$ in $\mathcal{P}^*$, then the key k^* must be embedded at the right input wire. Also in this case it follows that c^* can be embedded in at most one further gate – the right sibling – and there will be considered as a malformed ciphertext by both $\mathcal{A}$ and $\mathcal{A}[c^*]$.

For the case that a gate is mapped to $\mathtt{W}$ in $\mathcal{P}$ and to $\mathtt{G}_*$ in $\mathcal{P}^*$, the key k^* can be embedded either at the right or the left input wire – however (by consistency) not at both. Also in this case it follows that c^* can be embedded in at most one further gate, and for this other gate c^* will be considered as a malformed ciphertext by both $\mathcal{A}$ and $\mathcal{A}[c^*]$.

Analogously, one can verify the remaining $\mathtt{WG^3B}$ pebbling rules by analysing the cases of a gate being mapped to $\mathtt{G}_*$ ($\mathtt{G}_L/\mathtt{G}_R$) in $\mathcal{P}$ and to $\mathtt{G}_L/\mathtt{G}_R$ ($\mathtt{B}$) in $\mathcal{P}^*$. Also in these cases, embedding c^* at any further gate leads to $\mathcal{A}$ and $\mathcal{A}[c^*]$ extracting the same pebble for this further gate. □

We will now bound the distinguishing advantage of $\mathsf{D}^{\mathcal{F}}$. Recall that a pebbling configuration on $G \setminus G^0$ is good per Definition 8 if it can be reached by $\mathtt{WG^3B}$ pebbling moves using at most $d-1$ pebbles on the first d layers. Thus, by Lemma 12, any successful distinguisher D has to simulate $\tilde{\mathsf{G}}$ and $\tilde{x}$ such that the pebbling configurations $\mathcal{P}, \mathcal{P}^*$ on G extracted by $\mathcal{A}$ and $\mathcal{A}[c^*]$, respectively, contain exactly $d-1$ or d black and gray pebbles on the first d layers (depending on the IND-CPA challenge bit b^*), contain only $\mathtt{W}$, $\mathtt{G}_*$, and $\mathtt{G}_R$ pebbles on higher layers, and differ by a valid $\mathtt{WG^3B}$ pebbling move within layers $[1, d+1]$.

In the following we will first restrict our analysis to *non-rewinding* distinguishers and assume x_0, x_1 were chosen uniformly at random by $\mathcal{A}$ *after* it sees $\tilde{\mathsf{G}}$. Finally we will discuss how to slightly modify our adversary $\mathcal{A}$ to also cover the case that D chooses $\mathcal{A}$'s randomness and rewinds $\mathcal{A}$.

To bound the success probability of D, let r be arbitrary random coins and consider two cases:

(1) there exists s such that the output of $\mathcal{A}(s)$ and $\mathcal{A}[c^*](s)$ after interaction with $\mathsf{D}(r, c^*)$ differs and in $\mathcal{P}$ and $\mathcal{P}^*$ there are *more than* $\bar{d}$ $\mathtt{G}_*$ and $\mathtt{G}_R$-type-2 (as defined in Lemma 9) pebbles in layers $[d+2, 2d+1]$,

(2) there exists s such that the output of $\mathcal{A}(s)$ and $\mathcal{A}[c^*](s)$ after interaction with $\mathsf{D}(r, c^*)$ differs and in $\mathcal{P}$ and $\mathcal{P}^*$ there are *at most* $\bar{d}$ $\mathtt{G}_*$ and $\mathtt{G}_R$-type-2 pebbles in layers $[d+2, 2d+1]$.

We leave the parameter $\bar{d} < d/3$ undefined for now and optimze it later. In Lemmas 13 and 14, we will argue that, intuitively, in both cases the distinguisher D must have correctly guessed many of the input bits in x_0.

Lemma 13. *Let r be arbitrary coins such that case (1) is true. Then the probability (over uniformly random coins s) that the output of $\mathcal{A}(s)$ and $\mathcal{A}[c^*](s)$ differs after interaction with $\mathsf{D}(r, c^*)$ is at most $(3/4)^{\sqrt{\bar{d}}/7}$.*

Proof. To prove this lemma, we will use Lemmas 7 to 9. First, note that D can only succeed if at most one of the gates at layer $d+1$ is not mapped to a $\mathtt{W}$ pebble, since the adversary $\mathcal{A}$ outputs 1 whenever any gate at layer $d+1$ is not $\mathtt{W}$ pebbled. Now, by Lemma 7, there is at most one pair of input keys to an AND gate that leads to this gate being mapped to a $\mathtt{W}$ pebble. As the input to all but one gate at layer $d+1$ comprises *all* input to layer $d+1$, this implies that D can only succeed, if it properly garbles all gates at layer $d+1$ and the input keys which are revealed through $\mathsf{GEval}(\tilde{\mathsf{G}}, \tilde{x})$ are associated with the corresponding bits in $\mathsf{G}^{\oplus}(x_0)$.

Next, consider the AND gates at layers $[d+2, 2d+1]$. For D to succeed, these gates must *not* end up $\mathtt{G}_L$ or $\mathtt{B}$ pebbled. Since all these gates have their left input from layer d and by the previous argument all these keys are fixed, we can apply Lemmas 8 and 9: Let $\mathcal{S}$ denote the set of $\bar{d}$ gates in layers $[d+2, 2d+1]$ that are mapped to $\mathtt{G}_*$ or $\mathtt{G}_R$-type-2 pebbles (for some random coins s such that (1) is true). Then by Lemma 4 there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size $\sqrt{\bar{d}}/4$ such that the set of right parents $\mathcal{S}_R$ of $\mathcal{S}'$ is linearly independent over $\mathbb{Z}_2$; and for each gate $g \in \mathcal{S}'$ left and right parent are linearly independent. To see that the latter is true, note that any subset smaller than n of gates within one layer or within one column is linearly independent (cf. Lemma 3). It directly follows that left and right parents of any gate $g \in \mathcal{S}'$ since they lie in the same column. Furthermore, the set of left parents $\mathcal{S}_L$ to $\mathcal{S}'$ is linearly independent since it is a subset of $\leq \bar{d} < n$ gates at layer d.

To argue that D must have guessed many of the right input bits to $S^{\wedge}$ correctly, we use the following simple result from linear algebra.

Claim 2. *Let $m \in [1, n]$ and $\mathcal{S}_1 = \{u_i\}_{i\in[1,m]}$ a subset of $\{0,1\}^n$ that is linearly independent over $\mathbb{Z}_2$. Let $\mathcal{S}_2 = \{v_i\}_{i\in[1,m]}$ be a multiset of elements in $\{0,1\}^n$ such that $\mathcal{S}_2$ as a set is linearly independent over $\mathbb{Z}_2$. Furthermore, assume $\{u_i, v_i\}$ is linearly independent for all $i \in [1, m]$. Then there exists an index set $\mathcal{I} \subset [1, m]$ of size $|\mathcal{I}| = \lfloor m/4 \rfloor$ such that $\bigcup_{i\in\mathcal{I}}\{u_i\} \cup \{v_i\}$ is linearly independent.*

Proof of the claim. Let $i_1 \in [1, m]$ be arbitrary and set $\mathcal{I}_1 := \{i_1\}$. Then clearly $|\mathcal{I}_1| = 1$ and $\mathcal{U}_1 := \bigcup_{j\in\mathcal{I}_1}\{u_j\} \cup \{v_j\}$ is linearly independent. For $k > 1$, choose $i_k \in [1, m] \setminus \mathcal{I}_{k-1}$ such that $\mathcal{U}_k := \mathcal{U}_{k-1} \cup \{u_{i_k}\} \cup \{v_{i_k}\}$ is linearly independent, if exists, otherwise set $i_k := i_{k-1}$; set $\mathcal{I}_k := \mathcal{I}_{k-1} \cup \{i_k\}$. We show that such $i_k \in [1, m] \setminus \mathcal{I}_{k-1}$ must exist as long as $k \leq m/4$: Since $\dim(\mathcal{U}_{k-1}) \leq 2(k-1) \leq m/2 - 2$ and $\mathcal{S}_1$ is linearly independent, it must hold $|\mathcal{S}_1 \cap \langle\mathcal{U}_{k-1}\rangle| \leq m/2 - 2$; we denote the index set of $\mathcal{S}_1 \cap \langle\mathcal{U}_{k-1}\rangle$ by $\mathcal{I}_{\mathcal{S}_1,k-1}$. For $\mathcal{S}_2$, on the other hand, it also holds that $|\mathcal{S}_2 \cap \langle\mathcal{U}_{k-1}\rangle| \leq m/2 - 2$, but since $\mathcal{S}_2$ is a *multiset* we don't obtain a lower bound on $|\mathcal{S}_2 \setminus \langle\mathcal{U}_{k-1}\rangle|$. However, since for any linearly independent set $\mathcal{U}$ and $v \in \mathcal{U}$ the set $\mathcal{U} \cup \{v\} = U$ is linearly independent, we choose $\mathcal{I}_{\mathcal{S}_2,k-1} \subset [1, m]$ minimal such that $\bigcup_{j\in\mathcal{I}_{\mathcal{S}_2,k-1}}\{u_j\} = \mathcal{S}_2 \cap \langle\mathcal{U}_{k-1}\rangle$. Then again we have $|\mathcal{I}_{\mathcal{S}_2,k-1}| \leq m/2 - 2$. Thus, by pigeonhole principle, there must exist $i_k \in [1, m] \setminus (\mathcal{I}_{\mathcal{S}_1,k-1} \cup \mathcal{I}_{\mathcal{S}_2,k-1})$ such that $\mathcal{U}_k := \mathcal{U}_{k-1} \cup \{u_{i_k}\} \cup \{v_{i_k}\}$ is linearly independent and we have $|\mathcal{I}_k| = |\mathcal{I}_{k-1} \cup \{i_k\}| = k$. □

Since the multiset $\mathcal{S}_L$ and the set $\mathcal{S}_R$ of left and right parents of $\mathcal{S}'$ are linearly independent (as sets), respectively, and for any $g \in \mathcal{S}'$ left and right input to $\mathtt{G}$ are linearly independent, we can apply the claim to obtain a subset $\mathcal{S}'' \subset \mathcal{S}'$ of size $|\mathcal{S}'|/4$ such that the union of the parents of $\mathcal{S}''$ is linearly independent. For $\mathcal{S}''$, we can now use Lemmas 8 and 9 to see that any successful D must have correctly guessed all right input bits to $\mathcal{S}''$; i.e., for s sampled uniformly at random, the probability that D succeeds is at most $(1/2)^{|\mathcal{S}''|}$. As $|\mathcal{S}'| \geq \sqrt{\bar{d}}/4$, the probability that D succeeds can be upper-bounded by

$$\mathsf{Pr}[\mathsf{D} \text{ succeeds in case (1)}] \leq \left(\frac{1}{2}\right)^{\sqrt{\bar{d}}/16} < \left(\frac{3}{4}\right)^{\sqrt{\bar{d}}/7}.$$

□

Lemma 14. *Let r be arbitrary coins such that case (2) is true. Then the probability (over uniformly random coins s) that the output of $\mathcal{A}(s)$ and $\mathcal{A}[c^*](s)$ differs after interaction with $\mathsf{D}(r, c^*)$ is at most $(3/4)^{\sqrt{d-3\bar{d}}/4}$.*

Proof. Recall that whenever the consistency check passes, each wire in $\tilde{\mathsf{G}}$ can be uniquely associated with two keys. Now, in case (2), for all but $\bar{d}$ wires in $G \setminus G^0$ the following holds: By Lemmas 7 and 9, for each bit running over the wire w in G, there exists at most one key associated with w in $\tilde{\mathsf{G}}^{\oplus}$ such that the AND gates with right input wire w is mapped to a "good" (W or $\mathtt{G}_R$-type-1) pebble, while for the other key associated to w it would be mapped to a "bad" pebble ($\mathtt{G}_L$ or $\mathtt{B}$). Note that in the latter case D immediately fails.

This allows us to map keys associated with wires in $\tilde{\mathsf{G}}^{\oplus}$ to bits, hence implies a mapping from $(\tilde{\mathsf{G}}, \tilde{x})$ to a circuit $\hat{\mathsf{G}}$ and input $\hat{x}$, where $\hat{\mathsf{G}}$ contains at most $3\bar{d}$ "undefined" gates (note, each internal wire effects 3 gates in $\mathsf{G}^{\oplus}$). Now, for D to succeed, it has to simulate $(\tilde{\mathsf{G}}, \tilde{x})$ such that at least $d' := d - 3\bar{d}$ "well-defined" gates in the circuit $\hat{\mathsf{G}}$ differ from XOR gates and $\hat{x} = x_0$. At the same time, all input and output wires of the well-defined gates have to carry the correct bits during evaluation (for "evaluation" of $\hat{\mathsf{G}}$ on $\hat{x}$ we apply the mapping from keys to bits to $\mathsf{Eval}(\tilde{\mathsf{G}}, \tilde{x})$ to extract a bit for all wires connected to well-defined gates).

Ignoring the undefined gates in $\hat{\mathsf{G}}$, this exactly corresponds to the game introduced in Section 4.2: D simulates a circuit such that all but d' gates are garbled correctly as XOR gates, and D succeeds, if for all gates the (input and) output bits correspond to the respective bits during evaluation of $\mathsf{G}^{\oplus}$ on input x_0. Lemma 2 now implies an upper bound on D's success probability in case (2):

$$\mathsf{Pr}[\mathsf{D} \text{ succeeds in case (2)}] \leq \left(\frac{3}{4}\right)^{\sqrt{d'}/4} = \left(\frac{3}{4}\right)^{\sqrt{d-3\bar{d}}/4}.$$

□

Thus, Lemmas 13 and 14 imply the following bound on any non-rewinding PPT distinguisher D (choose $\bar{d} = d/4$):

Corollary 2. *No non-rewinding PPT distinguisher* $\mathsf{D}^{\mathcal{F}}$ *can distinguish* $\mathcal{A}[c^*]$ *from* $\mathcal{A}$ *with probability larger than* $(3/4)^{\sqrt{d}/14}$.

To handle arbitrary – potentially rewinding – distinguishers D, we modify $\mathcal{A}$ as follows: Instead of sampling x_0, x_1 using random coins s, we assume a pseudorandom function f_k with uniformly random key k was hardcoded in $\mathcal{A}$, which takes as input a garbled circuit $\tilde{\mathsf{G}}$ and coins s, and outputs a tuple (x_0, x_1). Since D only has *black-box* access to $\mathcal{A}/\mathcal{A}[c^*]$, the secret key k is hidden from D, thus for two different inputs $(\tilde{\mathsf{G}}, s), (\tilde{\mathsf{G}}', s')$ to $\mathcal{A}/\mathcal{A}[c^*]$ the input pairs $(x_0, x_1), (x_0', x_1')$ look like independently sampled uniformly random strings.

With this modification in place, we finally arrive at the following lower bound on the security loss of any *black-box reduction* R (where we used $\delta < 3d$, hence $\sqrt{d}/14 > \sqrt{\delta}/25$). Note that our bounds naturally only apply to $d \leq n$, hence we assume $\delta < 2n$ in our theorem statement.

Theorem 1. *Any black-box reduction from the indistinguishability of Yao's garbling scheme (or its variant from [JW16]) on the class of circuits with input length* n *and depth* $\delta \leq 2n$ *to the IND-CPA security of the underlying encryption scheme loses at least a factor* $\frac{1}{q} \cdot (\frac{4}{3})^{\sqrt{\delta}/25} > \frac{1}{q} \cdot 2^{\sqrt{\delta}/61}$, *where* q *denotes the number of times the reduction rewinds the adversary.*

5 Discussion and Open Problems

In this work we prove that any black-box reduction from indistinguishability of (the modification [JW16] of) Yao's garbling scheme to IND-CPA security of the underlying encryption scheme must involve a loss in security that is sub-exponential in the depth of the circuit. This clearly also implies limitations to the stronger and more common simulation-based security and shows that the approach of [JW16] is essentially optimal. However, we leave it to future work if our fine-grained separation can be turned into an actual attack against Yao's garbling scheme.

Beside this most exciting open problem, one can also consider if our approach can be optimized. It might be possible to push our lower bound to an exponential loss, which would exactly match the upper bound from [JW16]. Following our approach, this requires a more sophisticated pebbling lower bound. Another interesting question would be if an even stronger bound can be found for the original construction of Yao, where the output mapping is sent in the offline phase, and certain limitations are already known from [AIKW13].

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References

[AIKW13] Benny Applebaum, Yuval Ishai, Eyal Kushilevitz, and Brent Waters. Encoding functions with constant online rate or how to compress garbled circuits keys. In Ran Canetti and Juan A. Garay, editors, *CRYPTO 2013, Part II*, volume 8043 of *LNCS*, pages 166–184. Springer, Heidelberg, August 2013.

[AL18] Prabhanjan Ananth and Alex Lombardi. Succinct garbling schemes from functional encryption through a local simulation paradigm. In Amos Beimel and Stefan Dziembowski, editors, *TCC 2018, Part II*, volume 11240 of *LNCS*, pages 455–472. Springer, Heidelberg, November 2018.

[AS16] Prabhanjan Vijendra Ananth and Amit Sahai. Functional encryption for turing machines. In Eyal Kushilevitz and Tal Malkin, editors, *TCC 2016-A, Part I*, volume 9562 of *LNCS*, pages 125–153. Springer, Heidelberg, January 2016.

[BGG+14] Dan Boneh, Craig Gentry, Sergey Gorbunov, Shai Halevi, Valeria Nikolaenko, Gil Segev, Vinod Vaikuntanathan, and Dhinakaran Vinayagamurthy. Fully key-homomorphic encryption, arithmetic circuit ABE and compact garbled circuits. In Phong Q. Nguyen and Elisabeth Oswald, editors, *EUROCRYPT 2014*, volume 8441 of *LNCS*, pages 533–556. Springer, Heidelberg, May 2014.

[BHK13] Mihir Bellare, Viet Tung Hoang, and Sriram Keelveedhi. Instantiating random oracles via UCEs. In Ran Canetti and Juan A. Garay, editors, *CRYPTO 2013, Part II*, volume 8043 of *LNCS*, pages 398–415. Springer, Heidelberg, August 2013.

[BHR12a] Mihir Bellare, Viet Tung Hoang, and Phillip Rogaway. Adaptively secure garbling with applications to one-time programs and secure outsourcing. In Xiaoyun Wang and Kazue Sako, editors, *ASIACRYPT 2012*, volume 7658 of *LNCS*, pages 134–153. Springer, Heidelberg, December 2012.

[BHR12b] Mihir Bellare, Viet Tung Hoang, and Phillip Rogaway. Foundations of garbled circuits. In Ting Yu, George Danezis, and Virgil D. Gligor, editors, *ACM CCS 2012*, pages 784–796. ACM Press, October 2012.

[DKW11] Stefan Dziembowski, Tomasz Kazana, and Daniel Wichs. Key-evolution schemes resilient to space-bounded leakage. In Phillip Rogaway, editor, *CRYPTO 2011*, volume 6841 of *LNCS*, pages 335–353. Springer, Heidelberg, August 2011.

[GS18] Sanjam Garg and Akshayaram Srinivasan. Adaptively secure garbling with near optimal online complexity. In Jesper Buus Nielsen and Vincent Rijmen, editors, *EUROCRYPT 2018, Part II*, volume 10821 of *LNCS*, pages 535–565. Springer, Heidelberg, April / May 2018.

[HJO+16] Brett Hemenway, Zahra Jafargholi, Rafail Ostrovsky, Alessandra Scafuro, and Daniel Wichs. Adaptively secure garbled circuits from one-way functions. In Matthew Robshaw and Jonathan Katz, editors, *CRYPTO 2016, Part III*, volume 9816 of *LNCS*, pages 149–178. Springer, Heidelberg, August 2016.

[JKK+17] Zahra Jafargholi, Chethan Kamath, Karen Klein, Ilan Komargodski, Krzysztof Pietrzak, and Daniel Wichs. Be adaptive, avoid overcommitting. In Jonathan Katz and Hovav Shacham, editors, *CRYPTO 2017, Part I*, volume 10401 of *LNCS*, pages 133–163. Springer, Heidelberg, August 2017.

[JO20] Zahra Jafargholi and Sabine Oechsner. Adaptive security of practical garbling schemes. In Karthikeyan Bhargavan, Elisabeth Oswald, and Manoj Prabhakaran, editors, *Progress in Cryptology – INDOCRYPT 2020*, pages 741–762, Cham, 2020. Springer International Publishing.

[JSW17] Zahra Jafargholi, Alessandra Scafuro, and Daniel Wichs. Adaptively indistinguishable garbled circuits. In Yael Kalai and Leonid Reyzin, editors, *TCC 2017, Part II*, volume 10678 of *LNCS*, pages 40–71. Springer, Heidelberg, November 2017.

[JW16] Zahra Jafargholi and Daniel Wichs. Adaptive security of Yao's garbled circuits. In Martin Hirt and Adam D. Smith, editors, *TCC 2016-B, Part I*, volume 9985 of *LNCS*, pages 433–458. Springer, Heidelberg, October / November 2016.

[KKPW21] Chethan Kamath, Karen Klein, Krzysztof Pietrzak, and Michael Walter. On the cost of adaptivity in graph-based games. Cryptology ePrint Archive, Report 2021/059, 2021. https://eprint.iacr.org/2021/059.

[LP09] Yehuda Lindell and Benny Pinkas. A proof of security of Yao's protocol for two-party computation. *Journal of Cryptology*, 22(2):161–188, April 2009.

[Sav98] John E. Savage. *Models of computation - exploring the power of computing.* Addison-Wesley, 1998.

[Yao82] Andrew Chi-Chih Yao. Protocols for secure computations (extended abstract). In *23rd FOCS*, pages 160–164. IEEE Computer Society Press, November 1982.

[Yao86] Andrew Chi-Chih Yao. How to generate and exchange secrets (extended abstract). In *27th FOCS*, pages 162–167. IEEE Computer Society Press, October 1986.