Multi-Designated Receiver Signed Public Key Encryption

Ueli Maurer[1], Christopher Portmann[2], and Guilherme Rito[1]

[1] Department of Computer Science, ETH Zürich, Switzerland
{maurer,gteixeir}@inf.ethz.ch
[2] Concordium, Zürich, Switzerland
cp@concordium.com

Abstract. This paper introduces a new type of public-key encryption scheme, called Multi-Designated Receiver Signed Public Key Encryption (MDRS-PKE), which allows a sender to select a set of designated receivers and both encrypt and sign a message that only these receivers will be able to read and authenticate (*confidentiality* and *authenticity*). An MDRS-PKE scheme provides several additional security properties which allow for a fundamentally new type of communication not considered before. Namely, it satisfies *consistency*—a dishonest sender cannot make different receivers receive different messages—*off-the-record*—a dishonest receiver cannot convince a third party of what message was sent (e.g., by selling their secret key), because dishonest receivers have the ability to forge signatures—and *anonymity*—parties that are not in the set of designated receivers cannot identify who the sender and designated receivers are.

We give a construction of an MDRS-PKE scheme from standard assumptions. At the core of our construction lies yet another new type of public-key encryption scheme, which is of independent interest: Public Key Encryption for Broadcast (PKEBC) which provides all the security guarantees of MDRS-PKE schemes, except authenticity.

We note that MDRS-PKE schemes give strictly more guarantees than Multi-Designated Verifier Signature (MDVS) schemes with privacy of identities. This in particular means that our MDRS-PKE construction yields the first MDVS scheme with privacy of identities from standard assumptions. The only prior construction of such schemes was based on Verifiable Functional Encryption for general circuits (Damgård et al., TCC '20).

1 Introduction

1.1 Public Key Encryption security properties

The most common use case for cryptography is sending a message to a single receiver. Here one usually desires to have *confidentiality* (only the desired receiver can read the message) and *authenticity* (the receiver is convinced that the message is from the declared sender). Although one might be interested in signatures that can be publicly verified (e.g. for a judge to verify a contract), when trying to protect the privacy of personal communication one often wants the opposite: not only is the intended receiver the only one that can verify the signature, but even if this person sells their secret key, no third party will be convinced of the authenticity of the message. This latter property is called *off-the-record* in the Designated Verifier Signature (DVS) literature [13, 20, 23–25, 27, 33–35], and is achieved by designing the scheme so that the receiver's secret key can be used to forge signatures. One may take this a step further and require *anonymity*, i.e. third parties cannot even learn who the sender and receiver are (this is called *privacy of identities* in the (M)DVS literature) [13].[3]

Another setting of interest is where the message is sent to many recipients. Consider, for example, the case of sending an email to multiple receivers. Apart from all the security properties listed above, here one would additionally require *consistency*: all the (intended) receivers will get the same email when decrypting the same ciphertext, even if the sender is dishonest. We note that it is crucial that a receiver can decrypt ciphertexts using only their secret key, i.e. without having to use the public key of the sender and other receivers. It is common in the literature to assume that the receiver knows who the sender and other receivers are so that their public keys can be used for decryption [6, 26]. But in many contexts adding this information in plain to the ciphertext would violate crucial properties, e.g., in broadcast encryption the ciphertext size would not be small any longer and in MDVS schemes anonymity (privacy) would be violated [26].

Many different schemes have been introduced in the literature that satisfy some of the properties listed here, see Sect. 1.5. In this work we propose two new primitives, Public Key Encryption for Broadcast (PKEBC) and Multi-Designated Receiver Signed Public Key Encryption (MDRS-PKE), which we explain in the following two subsections.

1.2 Public Key Encryption for Broadcast

The first type of primitive that we introduce, PKEBC, can be seen as an extension of Broadcast Encryption (BE) [15] which additionally gives consistency guarantees in the case of a dishonest sender.[4] More specifically, we expect PKEBC schemes to provide the following guarantees:

Correctness If a ciphertext c is honestly generated as the encryption of a message m with respect to a vector of receivers, say $\vec{R} \coloneqq (\mathrm{Bob}, \mathrm{Charlie})$, then we want that if Bob is honest and decrypts c using its secret key, it obtains a pair $((\mathtt{pk}_{\mathrm{Bob}}, \mathtt{pk}_{\mathrm{Charlie}}), m)$, where $\mathtt{pk}_{\mathrm{Bob}}$ and $\mathtt{pk}_{\mathrm{Charlie}}$ are, respectively, Bob's and Charlie's public keys;

[3] With off-the-record, a third party will know that either the alleged sender or the receiver wrote the message, whereas anonymity completely hides who the sender and receiver are. However, anonymity only holds when the receiver is honest whereas off-the-record provides guarantees against a dishonest receiver.

[4] Though BE usually requires the ciphertext size to be sublinear in the number of receivers, which PKEBC does not.

Robustness Let c be the ciphertext from above. We do not want Dave, who is honest but yet not an intended receiver of c, to think c was meant for himself. In other words, we do not want Dave to successfully decrypt c.

Consistency Now consider a dishonest party Alice who wants to confuse Bob and Charlie, both of whom are honest. We do not want Alice to be capable of creating a ciphertext c such that when Bob decrypts c, it obtains some pair $((\texttt{pk}_{\text{Bob}}, \texttt{pk}_{\text{Charlie}}), m)$, but when Charlie decrypts c it obtains some different pair. Instead, we want that if Bob obtains a pair $((\texttt{pk}_{\text{Bob}}, \texttt{pk}_{\text{Charlie}}), m)$, then so will Charlie (and vice-versa).

Confidentiality Now, suppose that Alice is honest. If Alice encrypts a message m to Bob and Charlie (who are both honest), we do not want Eve, who is dishonest, to find out what m is.

Anonymity Finally, suppose there are two more honest receivers, say Frank and Grace, to whom Alice could also be sending a message to. If, again, Alice encrypts a message m to both Bob and Charlie, and letting c be the corresponding ciphertext, we do not want Eve to find out that the receivers of ciphertext c are Bob and Charlie; in fact, we do not want Eve to learn anything about the intended receivers of c, other than the number of receivers.

The formal definitions of PKEBC are given in Sect. 3. In Sect. 4 we show how to construct a PKEBC from standard assumptions. Our construction is a generalization of Naor-Yung's scheme [29] that enhances the security guarantees given by the original scheme. In particular, as we will see if the underlying PKE scheme is anonymous, then this anonymity is preserved by the PKEBC construction.

One important difference from other public key schemes for multiple parties is that to decrypt, a receiver only needs to know their own secret key; the decryption of a ciphertext yields not only the underlying plaintext but also the set of receivers for the ciphertext. This then allows the corresponding public keys to be used as needed.[5]

1.3 Multi-Designated Receiver Signed Public Key Encryption

Our main primitive has all of the properties listed in Sect. 1.1. Namely, a MDRS-PKE scheme is expected to provide the following guarantees:

Correctness If a ciphertext c is honestly generated as the encryption of a message m from a sender Alice to a vector of receivers $\vec{R} := (\text{Bob}, \text{Charlie})$ then we want that if Bob is honest and decrypts c using its secret key, it obtains a triple $(\texttt{spk}_{\text{Alice}}, (\texttt{rpk}_{\text{Bob}}, \texttt{rpk}_{\text{Charlie}}), m)$, where $\texttt{spk}_{\text{Alice}}$ is Alice's public sending key, and $\texttt{rpk}_{\text{Bob}}$ and $\texttt{rpk}_{\text{Charlie}}$ are, respectively, Bob's and Charlie's receiver public keys;

Consistency Now consider a dishonest party Donald who is a sender and wants to confuse Bob and Charlie, both of whom are honest. We do not want

[5] We note that this is only important since we want to achieve anonymity, otherwise once could send the public keys of the other parties together with the ciphertext.

Donald to be able to create a ciphertext c such that when Bob decrypts c, it obtains some triple $(\mathtt{spk}_{\mathrm{Donald}}, (\mathtt{pk}_{\mathrm{Bob}}, \mathtt{pk}_{\mathrm{Charlie}}), m)$, but when Charlie decrypts c it obtains some different triple (or does not even decrypt). Instead, we want that if Bob obtains a triple $(\mathtt{spk}_{\mathrm{Donald}}, (\mathtt{pk}_{\mathrm{Bob}}, \mathtt{pk}_{\mathrm{Charlie}}), m)$, then so will Charlie (and vice-versa).

Unforgeability We do not want that Eve can forge a ciphertext as if it were from an honest sender, say Alice, to a vector of receivers Bob and Charlie.

Confidentiality If an honest sender Alice encrypts a message m to Bob and Charlie (who are both honest), we do not want Eve, who is dishonest, to find out what m is.

Anonymity Suppose there is another honest sender, say Heidi. If Alice encrypts a message m to Bob, and letting c be the corresponding ciphertext, we do not want Eve to find out that Alice is the sender or that Bob is the receiver; Eve should at most learn that someone sent a message to a single receiver.

Off-The-Record Suppose Alice sends a message to Bob, Charlie and Donald. Donald, being dishonest, might be enticed to try convincing Eve that Alice sent some message. However, we do not want Donald to have this capability.

The formal definitions of MDRS-PKE are given in Sect. 5. In Sect. 6 we show how to construct a MDRS-PKE from standard assumptions. As we will see, our construction essentially consists of using the MDVS scheme to sign messages, and then using the PKEBC scheme to encrypt the signed messages, together with their MDVS signatures.

Since a MDRS-PKE scheme is an extension of an MDVS scheme with privacy of identities and confidentiality, for completeness, we show in Appendix E that any MDRS-PKE scheme yields an MDVS scheme with privacy of identities. Since we give an MDRS-PKE scheme which is secure under standard assumptions, this in particular implies that our construction is the first achieving privacy of identities from standard assumptions. The only previous construction of an MDVS scheme with privacy of identities relied on a Verifiable Functional Encryption scheme for general circuits [13].

1.4 Applications to Secure (Group) Messaging

As we now discuss, one main application of MDRS-PKE schemes is secure messaging, and in particular secure group messaging.

Suppose Alice and Bob are using a secure messaging application to chat with each other. Of course, they expect the messenger to provide basic guarantees such as *Correctness*—if Alice sends a message to Bob, Bob receives this message—*Confidentiality*—no one other than Alice and Bob should learn the contents of the messages—and *Authenticity*—if Alice reads a message m, then Bob must have sent m. Another desirable guarantee they could expect from the messenger is *Anonymity*: suppose that in parallel to Alice and Bob's chat, Charlie and Dave are also chatting; then, if a third party Eve intercepts a ciphertext c from Alice and Bob's chat and Eve cannot *a priori* tell that c came from and/or is addressed to Alice or Bob, then Eve should not gain any additional information about the identity of c's sender and/or receiver from inspecting the contents of ciphertext c itself (in other words, Eve cannot tell if the ciphertext is from Alice and Bob's chat, from Alice and Charlie's chat, from Bob and Charlie's chat, or from Charlie and Dave's chat). Finally, imagine that Bob, who wants to keep the history of his chat with Alice, outsources the storage of the chat's ciphertexts to an external storage service which reliably, but not authentically, stores these ciphertexts. An important additional guarantee Alice expects from the messaging application is *Off-The-Record Deniability* (*Off-The-Record*) [11,13]: if, somehow, Eve manages to access whatever is stored by Bob's storage service, Eve cannot tell by inspecting the stored ciphertexts, even if Bob chooses to cooperate with Eve[6], if these ciphertexts are authentic ones corresponding to real messages sent by Alice to Bob in their chat, or if they are fake ones generated by Bob (in case Bob is cooperating with Eve) or generated by anyone else (in case Bob is not cooperating with Eve) to incriminate Alice.

A related, yet very different property that secure messaging applications like Signal [12] provide is *Forward Secrecy* [21]. Informally, Forward Secrecy guarantees that even if Eve stores any ciphertexts received by Bob and later *hacks* into Bob's computer to learn his secret key, Eve cannot learn the decryptions (i.e. the plaintexts) of the ciphertexts she previously intercepted. Off-The-Record, on the other hand, does not give any guarantees about hiding the contents of previously exchanged messages. However, it hides from Eve whether Alice really sent a message m to Bob or if Bob faked receiving m. Furthermore, Forward Secrecy assumes Bob is honest: if Bob were dishonest, he could simply store the decryptions of the ciphertexts he receives to later disclose them to Eve. Off-The-Record does not make such assumption: even if Bob is dishonest, Eve cannot tell if it was Alice sending a message m, or if Bob faked receiving m from Alice (in case Bob is dishonest), or anyone else faked Alice sending m to Bob (in case Bob is honest). Finally, as one can deduce, Forward Secrecy is incompatible with parties keeping a history of their chats, whereas this is not the case for Off-The-Record. A different problem is Alice's computer getting *hacked* by Eve. In such scenario it would be desirable to still give the Off-The-Record guarantee to Alice: Eve should not be able to tell if Alice ever sent any message or not. However, current Off-The-Record notions [13], including the one given in this paper, do not capture this.

A natural generalization of two party secure messaging is secure group messaging [2,13]. Suppose Alice, Bob and Charlie now share a group chat. The key difference between Alice, Bob and Charlie sharing a group chat or having multiple two party chats with each other is *Consistency*: even if Charlie is dishonest, he cannot create confusion among Alice and Bob as to whether he sent a message to the group chat or not [13]. In other words, honest group members have a consistent view of the chat. Surprisingly, for the case of MDVS, this guarantee was only recently introduced by Damgård et al. in [13].

[6] By Bob collaborating with Eve we mean that Bob shares all his secrets (including secret keys) with Eve.

To achieve Off-The-Record in the group messaging case, one must consider that any subset of the parties participating in the group chat may be dishonest [13]. This property, also known as *Any-Subset Off-The-Record Deniability* (or more simply *Off-The-Record*) was first introduced by Damgård et al. in [13]. Returning to Alice, Bob and Charlie's group chat, this property essentially guarantees that regardless of who (among Bob and Charlie) cooperate with Eve in trying to convince her that Alice sent some message, Eve will not be convinced because any of them (or the two together) could have created a fake message to pretend that Alice sent it.

1.5 Related Work

A closely related type of encryption scheme are Broadcast Encryption (BE) schemes [10,15]. However, BE schemes do not give the consistency guarantee that PKEBC give; the main goal of BE schemes is actually making ciphertexts short—ideally the size of ciphertexts would be independent from the number recipients. Conversely, the size of the ciphertexts of the PKEBC scheme construction we give in this paper grows quadratically with the number of recipients. Diament et al. introduce a special type of BE scheme, called Dual-Receiver Encryption schemes, which allow a sender to send messages to two (and only two) receivers. By limiting the number of receivers to two receivers, these schemes allow for efficient constructions with relatively short ciphertexts and public keys from standard assumptions.

As already mentioned, PKEBC schemes allow receivers to decrypt a ciphertext meant for multiple receivers using their secret key only. This problem had been noticed before by Barth et al. in [6], and by Libert et al. in [26]. Barth et al. modify the definition of BE schemes in a way that allows receivers to decrypt ciphertexts without knowing who the other recipients are a priori [6]. Libert et al. strengthens this by guaranteeing that receivers do not learn who the other receivers are, even after decrypting ciphertexts.

Other closely related works are Multi-Designated Verifier Signature (MDVS) schemes [13]. They provide consistency, authenticity, and off-the record and sometimes also anonymity (called privacy). However, to the best of our knowledge, MDVS schemes require the public keys of the sender and other designated receivers to be used to verify signatures, and the existing literature does not discuss how the receiver gets that information, e.g. sending this information in plain would violate privacy. Thus, existing constructions of MDVS with privacy can only be used if the number of combinations of possible sender and receivers is small enough that all combinations can be tried by the verifier.

2 Preliminaries

We now introduce conventions and notation we use throughout the paper. We denote the arity of a vector $\vec{x}$ by $|\vec{x}|$ and its i-th element by x_i. We write $\alpha \in \vec{x}$ to denote $\exists i \in \{1, \ldots, |\vec{x}|\}$ with $\alpha = x_i$. We write $\mathrm{Set}(\vec{x})$ to denote the set induced by vector $\vec{x}$, i.e. $\mathrm{Set}(\vec{x}) \coloneqq \{x_i \mid x_i \in \vec{x}\}$.

Throughout the paper we frequently use vectors. We use upper case letters to denote vectors of parties, and lower case letters to denote vectors of artifacts such as public keys, messages, sequences of random coins, and so on. Moreover, we use the convention that if $\vec{V}$ is a vector of parties, then $\vec{v}$ denotes $\vec{V}$'s corresponding vector of public keys. For example, for a vector of parties $\vec{V} \coloneqq (\mathrm{Bob}, \mathrm{Charlie})$, $\vec{v} \coloneqq (\mathtt{pk}_{\mathrm{Bob}}, \mathtt{pk}_{\mathrm{Charlie}})$ is $\vec{V}$'s corresponding vector of public keys. In particular, V_1 is Bob and v_1 is Bob's public key $\mathtt{pk}_{\mathrm{Bob}}$, and V_2 is Charlie and v_2 is Charlie's public key $\mathtt{pk}_{\mathrm{Charlie}}$. More generally, for a vector of parties $\vec{V}$ with corresponding vector of public keys $\vec{v}$, V_i's public key is v_i, for $i \in \{1, \ldots, |\vec{V}|\}$.

3 Public Key Encryption for Broadcast Schemes

We now introduce the first new type of scheme we give in this paper, namely Public Key Encryption for Broadcast (PKEBC). A PKEBC scheme Π with message space $\mathcal{M}$ is a quadruple $\Pi = (S, G, E, D)$ of Probabilistic Polynomial Time Algorithms (PPTs), where:

- S: on input 1^k, generates public parameters $\mathtt{pp}$;
- G: on input $\mathtt{pp}$, generates a receiver key-pair;
- E: on input $(\mathtt{pp}, \vec{v}, m)$, where $\vec{v}$ is a vector of public keys of the intended receivers and m is the message, generates a ciphertext c;
- D: on input $(\mathtt{pp}, \mathtt{sk}, c)$, where $\mathtt{sk}$ is the receiver's secret key, D decrypts c using $\mathtt{sk}$, and outputs the decrypted receiver-vector/message pair $(\vec{v}, m)$ (or $\perp$ if the ciphertext did not decrypt correctly).

3.1 The Security of PKEBC Schemes

We now state the definitions of Correctness, Robustness, Consistency, and IND-CCA-2 and IK-CCA-2 security for PKEBC schemes. Before proceeding to the actual definitions, we first introduce some oracles the game systems from Definitions 1, 2 and 3 use. In the following, consider a PKEBC scheme $\Pi = (S, G, E, D)$ with message space $\mathcal{M}$. The oracles below are defined for a game-system with (an implicitly defined) security parameter k:

Public Parameters Oracle: $\mathcal{O}_{PP}$

1. On the first call, compute and store $\mathtt{pp} \leftarrow S(1^k)$; output $\mathtt{pp}$;
2. On subsequent calls, output the previously generated $\mathtt{pp}$.

Secret Key Generation Oracle: $\mathcal{O}_{SK}(B_j)$

1. If $\mathcal{O}_{SK}$ was queried on B_j before, simply look up and return the previously generated key for B_j;
2. Otherwise, store $(\mathtt{pk}_j, \mathtt{sk}_j) \leftarrow G(\mathtt{pp})$ as B_j's key-pair, and output $(\mathtt{pk}_j, \mathtt{sk}_j)$.

Public Key Generation Oracle: $\mathcal{O}_{PK}(B_j)$

1. $(\mathtt{pk}_j, \mathtt{sk}_j) \leftarrow \mathcal{O}_{SK}(B_j)$;
2. Output $\mathtt{pk}_j$.

Encryption Oracle: $\mathcal{O}_E(\vec{V}, m)$

1. $\vec{v} \leftarrow (\mathcal{O}_{PK}(V_1), \ldots, \mathcal{O}_{PK}(V_{|\vec{V}|}))$;
2. Create and output a fresh encryption $c \leftarrow E_{\mathsf{pp},\vec{v}}(m)$.

In addition to the oracles above, the game systems from Definitions 1 and 2 further provide adversaries with access to the following oracles:

Decryption Oracle: $\mathcal{O}_D(B_j, c)$

1. Query $\mathcal{O}_{SK}(B_j)$ to obtain the corresponding secret-key $\mathtt{sk}_j$;
2. Decrypt c using $\mathtt{sk}_j$, $(\vec{v}, m) \leftarrow D_{\mathsf{pp},\mathtt{sk}_j}(c)$, and then output the resulting receivers-message pair $(\vec{v}, m)$, or $\perp$ (if $(\vec{v}, m) = \perp$, i.e. the ciphertext is not valid with respect to B_j's secret key).

Definition 1 (Correctness). *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Corr}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{PK}, \mathcal{O}_{SK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if there are two queries* q_E *and* q_D *to* $\mathcal{O}_E$ *and* $\mathcal{O}_D$*, respectively, where* q_E *has input* $(\vec{V}, m)$ *and* q_D *has input* (B_j, c)*, satisfying* $B_j \in \vec{V}$*, the input* c *in* q_D *is the output of* q_E*, the output of* q_D *is either* $\perp$ *or* $(\vec{v}', m')$ *with* $(\vec{v}, m) \neq (\vec{v}', m')$*, and* $\mathbf{A}$ *did not query* $\mathcal{O}_{SK}$ *on input* B_j*.*

The advantage of $\mathbf{A}$ *in winning the Correctness game, denoted* $Adv^{\mathsf{Corr}}(\mathbf{A})$*, is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Corr}}$ *as described above.*

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Corr}}, t)$-breaks the (n, d_E, q_E, q_D)-Correctness of a PKEBC scheme Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{PK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n different parties[7], makes at most q_E and q_D queries to $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, with the sum of lengths of the party vectors input to $\mathcal{O}_E$ being at most d_E, and satisfies $Adv^{\mathsf{Corr}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Corr}}$.

The following notion captures the guarantee that if a ciphertext c is an honestly generated ciphertext for a vector of receivers $\vec{R}$ (for some message), then no honest receiver B who is not one of the intended receivers of c can successfully decrypt c (i.e. if $B \notin \vec{R}$ then the decryption of c with B's secret key outputs $\perp$). As one might note, this notion is a variant of the Weak Robustness notion introduced in [1], but adapted to PKEBC schemes.

Definition 2 (Robustness). *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Rob}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{PK}, \mathcal{O}_{SK}, \mathcal{O}_E, \mathcal{O}_D}$

[7] Here, querying on most n parties means that the number of different parties in all queries is at most n. In particular, the number of different parties in a query $\mathcal{O}_E((B_1, B_2, B_3), (\ldots))$ is 3, assuming $B_1 \neq B_2 \neq B_3 \neq B_1$; the number of different parties in a query $\mathcal{O}_D(B_j, \cdot)$ is 1.

$\mathbf{A}$ *wins the game if there are two queries* q_E *and* q_D *to* $\mathcal{O}_E$ *and* $\mathcal{O}_D$*, respectively, where* q_E *has input* $(\vec{V}, m)$ *and* q_D *has input* (B_j, c)*, satisfying* $B_j \notin \vec{V}$*, the input* c *in* q_D *is the output of* q_E*, the output of* q_D *is* $(\vec{v}', m')$ *with* $(\vec{v}', m') \neq \bot$*, and* $\mathbf{A}$ *did not query* $\mathcal{O}_{SK}$ *on input* B_j*.*

The advantage of $\mathbf{A}$ *in winning the Robustness game is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Rob}}$ *as described above, and is denoted* $Adv^{\mathsf{Rob}}(\mathbf{A})$*.*

An adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Rob}}, t)$-breaks the Robustness of a PKEBC scheme Π if $\mathbf{A}$ runs in time at most t and satisfies $Adv^{\mathsf{Rob}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Rob}}$.

Remark 1. Correctness and Robustness are properties only relevant to honest parties. It is common in the literature to either define such security notions without any adversary or to consider a stronger adversary that is unbounded or has access to the honest parties' secret keys. We choose the weaker definitions above for two main reasons: first, it has been proven that analogous Correctness and Robustness notions [1, 5] for PKE schemes—also defined with respect to computationally bounded adversaries who are not given access to the secret keys of honest parties—imply (corresponding) composable security notions (see [5] and [22]); second, since the remaining PKEBC security notions (e.g. IND-CCA-2 security) are defined with respect to computationally bounded adversaries that cannot obtain the secret keys of honest parties, there is no advantage in considering strengthened Correctness and Robustness security notions. Nevertheless, as we will see, if the PKE scheme underlying our PKEBC scheme's construction satisfies Correctness against unbounded adversaries, then the PKEBC scheme's construction can be proven to satisfy such stronger Correctness and Robustness security notions.

We now introduce the notion of Consistency. Essentially, this notion captures the guarantee that a dishonest sender cannot create confusion between any pair of honest receivers as to whether they received some message m with respect to a vector of receivers $\vec{R}$ that includes both parties.

Definition 3 (Consistency). *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Cons}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{PK}, \mathcal{O}_{SK}, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if there is a ciphertext* c *such that* $\mathcal{O}_D$ *is queried on inputs* (B_i, c) *and* (B_j, c) *for some* B_i *and* B_j *(possibly with* $B_i = B_j$*), there is no prior query on either* B_i *or* B_j *to* $\mathcal{O}_{SK}$*, query* $\mathcal{O}_D(B_i, c)$ *outputs some* $(\vec{v}, m)$ *satisfying* $(\vec{v}, m) \neq \bot$ *with* $\mathtt{pk}_j \in \vec{v}$ *(where* $\mathtt{pk}_j$ *is* B_j*'s public key), and query* $\mathcal{O}_D(B_j, c)$ *does not output* $(\vec{v}, m)$*.*

The advantage of $\mathbf{A}$ *in winning the Consistency game is denoted* $Adv^{\mathsf{Cons}}(\mathbf{A})$ *and corresponds to the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Cons}}$ *as described above.*

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Cons}}, t)$-breaks the (n, q_D)-Consistency of Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{SK}$, $\mathcal{O}_{PK}$ and $\mathcal{O}_D$ on at most n different parties, makes at most q_D queries to $\mathcal{O}_D$ and satisfies $Adv^{\mathsf{Cons}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Cons}}$.

Remark 2. Similarly to Remark 1, Consistency is a security property only relevant to honest receivers, for which reason Definition 3 disallows adversaries from querying for the secret keys of honest receivers. It was proven in [28] that an analogous Consistency notion for MDVS schemes (introduced in [13]) implies composable security. Yet, as we will see, if the PKE scheme underlying our PKEBC scheme's construction satisfies Correctness against unbounded adversaries, then our PKEBC scheme can be proven to satisfy a stronger Consistency notion in which the adversary can query for any party's secret key.

The two following security notions are the multi-receiver variants of $\mathsf{IND\text{-}CCA\text{-}2}$ security (introduced in [30]) and $\mathsf{IK\text{-}CCA\text{-}2}$ security (introduced in [7]). The games defined by these notions provide adversaries with access to the oracles $\mathcal{O}_{PP}$ and $\mathcal{O}_{PK}$ defined above as well as to oracles $\mathcal{O}_E$ and $\mathcal{O}_D$. For both notions, $\mathcal{O}_D$ is defined as follows:

Decryption Oracle: $\mathcal{O}_D(B_j, c)$

1. If c was the output of some query to $\mathcal{O}_E$, output $\texttt{test}$;
2. Otherwise, compute and output $(\vec{v}, m) \leftarrow D_{\texttt{pp},\texttt{sk}_j}(c)$, where $\texttt{sk}_j$ is B_j's secret key.

The $\mathcal{O}_E$ oracle provided by the $\mathsf{IND\text{-}CCA\text{-}2}$ games differs from the one provided by the $\mathsf{IK\text{-}CCA\text{-}2}$ games; for $\mathsf{IND\text{-}CCA\text{-}2}$, $\mathcal{O}_E$ is as follows:

Encryption Oracle: $\mathcal{O}_E(\vec{V}, m_0, m_1)$

1. For game system $\mathbf{G}_\mathbf{b}^{\mathsf{IND\text{-}CCA\text{-}2}}$, encrypt $m_\mathbf{b}$ under $\vec{v}$ (the vector of public keys corresponding to $\vec{V}$); output c.

Adversaries do not have access to $\mathcal{O}_{SK}$ in either notion.

Definition 4 ($\mathsf{IND\text{-}CCA\text{-}2}$ Security). *Consider the following game played between an adversary* $\mathbf{A}$ *and a game system* $\mathbf{G}_\mathbf{b}^{\mathsf{IND\text{-}CCA\text{-}2}}$*, with* $\mathbf{b} \in \{0, 1\}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{PK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and every query* $\mathcal{O}_E(\vec{V}, m_0, m_1)$ *satisfies* $|m_0| = |m_1|$.
We define the advantage of $\mathbf{A}$ *in winning the* $\mathsf{IND\text{-}CCA\text{-}2}$ *game as*

$$Adv^{\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_\mathbf{0}^{\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{A}\mathbf{G}_\mathbf{1}^{\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] - 1 \right|.$$

For the $\mathsf{IK\text{-}CCA\text{-}2}$ security notion, $\mathcal{O}_E$ behaves as follows:

Encryption Oracle: $\mathcal{O}_E(\vec{V}_0, \vec{V}_1, m)$

1. For game system $\mathbf{G}_\mathbf{b}^{\mathsf{IK\text{-}CCA\text{-}2}}$, encrypt m under $\vec{v}_\mathbf{b}$, the vector of public keys corresponding to $\vec{V}_\mathbf{b}$, creating a fresh ciphertext c; output c.

Definition 5 ($\mathsf{IK\text{-}CCA\text{-}2}$ Security). *Consider the following game played between an adversary* $\mathbf{A}$ *and a game system* $\mathbf{G}_\mathbf{b}^{\mathsf{IK\text{-}CCA\text{-}2}}$*, with* $\mathbf{b} \in \{0, 1\}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{PK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and every query* $\mathcal{O}_E(\vec{V}_0, \vec{V}_1, m)$ *satisfies* $|\vec{V}_0| = |\vec{V}_1|$.

We define the advantage of $\mathbf{A}$ *in winning the* $\mathsf{IK\text{-}CCA\text{-}2}$ *security game as*

$$Adv^{\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \Big| \Pr[\mathbf{AG}_{\mathbf{0}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1 \Big|.$$

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}, t)$-breaks (resp. $(\varepsilon_{\mathsf{IK\text{-}CCA\text{-}2}}, t)$-breaks) the (n, d_E, q_E, q_D)-$\mathsf{IND\text{-}CCA\text{-}2}$ (resp. (n, d_E, q_E, q_D)-$\mathsf{IK\text{-}CCA\text{-}2}$) security of Π if $\mathbf{A}$ runs in time at most t, queries the oracles it has access to on at most n different parties, makes at most q_E and q_D queries to oracles $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, with the sum of lengths of all the party vectors input to $\mathcal{O}_E$ being at most d_E, and satisfies $Adv^{\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \geq \varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}$ (resp. $Adv^{\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \geq \varepsilon_{\mathsf{IK\text{-}CCA\text{-}2}}$).

Finally, we say that Π is

$$(\varepsilon_{\mathsf{Corr}}, \varepsilon_{\mathsf{Rob}}, \varepsilon_{\mathsf{Cons}}, \varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}, \varepsilon_{\mathsf{IK\text{-}CCA\text{-}2}}, t, n, d_E, q_E, q_D)\text{-secure},$$

if no adversary $\mathbf{A}$:

- $(\varepsilon_{\mathsf{Corr}}, t)$-breaks the (n, d_E, q_E, q_D)-Correctness of Π;
- $(\varepsilon_{\mathsf{Rob}}, t)$-breaks the Robustness of Π;
- $(\varepsilon_{\mathsf{Cons}}, t)$-breaks the (n, q_D)-Consistency of Π;
- $(\varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}, t)$-breaks the (n, d_E, q_E, q_D)-$\mathsf{IND\text{-}CCA\text{-}2}$ security of Π; or
- $(\varepsilon_{\mathsf{IK\text{-}CCA\text{-}2}}, t)$-breaks the (n, d_E, q_E, q_D)-$\mathsf{IK\text{-}CCA\text{-}2}$ security of Π.

4 A PKEBC Scheme from Standard Assumptions

We now present our construction of a PKEBC scheme. The construction is a generalization of Naor-Yung's scheme [29] that enhances the security guarantees given by the original scheme. In particular, if the underlying PKE scheme is anonymous, then this anonymity is preserved by the PKEBC construction. First, while the scheme should preserve the anonymity of the underlying PKE scheme, parties should still be able to obtain the vector of receivers from ciphertexts, using only their own secret key. For this reason, the underlying PKE scheme is used to encrypt not only the messages to be sent, but also the vector of receivers to which each message is being sent to. As one might note, however, to preserve the anonymity of the underlying PKE scheme, the NIZK proof that proves the consistency of the ciphertexts for the various receivers can no longer be a proof for a statement in which the public keys are part of the statement. This introduces an extra complication since for some PKE schemes such as ElGamal, for every ciphertext c and message m, there is a public key $\mathtt{pk}$ and a sequence of random coins r such that c is an encryption of m under $\mathtt{pk}$, using r as the sequence of random coins for encrypting m. In particular, this means that the NIZK proof is not actually proving the consistency of the ciphertexts. To solve this issue, we further add a (binding) commitment to the vector of receiver public keys used to encrypt each ciphertext, and then use the NIZK proof to show that each ciphertext is an encryption of this same message under the public keys of the vector to which the commitment is bound. Note, however, that this is still not sufficient: despite now having the guarantee that if the NIZK proof verifies then all ciphertexts are encryptions of the same plaintext with respect a vector of public keys, since a party can still decrypt ciphertexts not meant for itself without realizing it, it could happen that a receiver decrypts the wrong ciphertext, thus getting the wrong vector of receivers-plaintext pair. To avoid this, the commitment additionally commits to the message being sent, and the sequence of random coins used to create the commitment are now encrypted along with the vector of public keys of the parties and the message being sent. This then allows a receiver to recompute the commitment from the vector of parties and message it decrypted. Given the commitment is binding, this implies that if the recomputed commitment matches the one in the ciphertext then decryption worked correctly (as otherwise the recomputed commitment would not match the one in the ciphertext).

We note that our security reductions are tight, and that there are tightly secure instantiations of each of the schemes we use as building blocks for our construction. For instance, ElGamal could be used as the underlying IND-CPA secure encryption scheme, as it is tightly multi-user multi-challenge IND-CPA secure [8]. For completeness, we show in the appendix that ElGamal is also tightly multi-user multi-challenge IK-CPA secure under the DDH assumption (see Appendix F). Furthermore, we could use any perfectly correct PKE scheme as the statistically binding commitment scheme needed by our scheme (in particular ElGamal), and the tightly unbounded simulation sound NIZK scheme from [16].

Algorithm 1 gives a construction of a Public Key Encryption for Broadcast scheme $\mathit{\Pi} = (S, G, E, D)$ from a Public Key Encryption scheme $\mathit{\Pi}_{\mathrm{PKE}} = (G, E, D)$, a Commitment Scheme $\mathit{\Pi}_{\mathrm{CS}} = (G_{CRS}, \mathit{Commit}, \mathit{Verify})$ and a Non Interactive Zero Knowledge scheme $\mathit{\Pi}_{\mathrm{NIZK}} = (G_{CRS}, \mathit{Prove}, \mathit{Verify}, S := (S_{CRS}, S_{Sim}))$. Consider relation R_{Cons} defined as

$$\begin{aligned} R_{\mathrm{Cons}} := \Big\{ \big((\mathtt{crs}_{\mathrm{CS}}, \mathtt{comm}, \vec{c}), (\rho, \vec{v}, m, \vec{r}) \big) \mid \\ |\vec{c}| = |\vec{v}| \\ \wedge\ \mathtt{comm} = \mathit{\Pi}_{\mathrm{CS}}.\mathit{Commit}_{\mathtt{crs}}(\vec{v}, m; \rho) \\ \wedge\ \big(\forall j \in \{1, \ldots, |\vec{c}|\}, \forall b \in \{0,1\}, \\ c_{j,b} = \mathit{\Pi}_{\mathrm{PKE}}.E_{v_{j,b}}(\rho, \vec{v}, m; r_{j,b}) \big) \Big\}. \end{aligned} \tag{4.1}$$

In Algorithm 1, we consider the language induced by R_{Cons}, which is defined as

$$\begin{aligned} L_{\mathrm{Cons}} := \{ (\mathtt{crs}_{\mathrm{CS}}, \mathtt{comm}, \vec{c}) \mid \\ \exists (\rho, \vec{v}, m, \vec{r}) \\ \big((\mathtt{crs}_{\mathrm{CS}}, \mathtt{comm}, \vec{c}), (\rho, \vec{v}, m, \vec{r}) \big) \in R_{\mathrm{Cons}} \}. \end{aligned} \tag{4.2}$$

4.1 Security Analysis of PKEBC Construction

Due to space constraints, the full proofs of the following results are in the appendix (see Appendix G).

Algorithm 1 Construction of a PKEBC scheme $\mathit{\Pi} = (S, G, E, D)$.

```
$S(1^k)$
    return $(1^k, \mathit{\Pi}_{\mathrm{NIZK}}.G_{CRS}(1^k), \mathit{\Pi}_{\mathrm{CS}}.G_{CRS}(1^k))$

$G(\mathtt{pp} := (1^k, \mathtt{crs}_{\mathrm{NIZK}}, \mathtt{crs}_{\mathrm{CS}}))$
    $(\mathtt{pk}_0, \mathtt{sk}_0) \leftarrow \mathit{\Pi}_{\mathrm{PKE}}.G(1^k)$
    $(\mathtt{pk}_1, \mathtt{sk}_1) \leftarrow \mathit{\Pi}_{\mathrm{PKE}}.G(1^k)$
    return $\big(\mathtt{pk} := (\mathtt{pk}_0, \mathtt{pk}_1), \mathtt{sk} := ((\mathtt{pk}_0, \mathtt{sk}_0), (\mathtt{pk}_1, \mathtt{sk}_1))\big)$

$E(\mathtt{pp} := (1^k, \mathtt{crs}_{\mathrm{NIZK}}, \mathtt{crs}_{\mathrm{CS}}), \vec{v} := \big((\mathtt{pk}_{1,0}, \mathtt{pk}_{1,1}), \ldots, (\mathtt{pk}_{|\vec{v}|,0}, \mathtt{pk}_{|\vec{v}|,1})\big), m \in \mathcal{M})$
    $\rho \leftarrow RandomCoins$
    $\mathtt{comm} \leftarrow \mathit{\Pi}_{\mathrm{CS}}.Commit_{\mathtt{crs}_{\mathrm{CS}}}(\vec{v}, m; \rho)$
    for $(\mathtt{pk}_{j,0}{'}, \mathtt{pk}_{j,1}{'}) \in \vec{v}$ do
        $(r_{j,0}, r_{j,1}) \leftarrow (RandomCoins, RandomCoins)$
        $(c_{j,0}, c_{j,1}) \leftarrow (\mathit{\Pi}_{\mathrm{PKE}}.E_{\mathtt{pk}_{j,0}}(\rho, \vec{v}, m; r_{j,0}), \mathit{\Pi}_{\mathrm{PKE}}.E_{\mathtt{pk}_{j,1}}(\rho, \vec{v}, m; r_{j,1}))$
    $\vec{r} := \big((r_{1,0}, r_{1,1}), \ldots, (r_{|\vec{v}|,0}, r_{|\vec{v}|,1})\big)$
    $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{v}|,0}, c_{|\vec{v}|,1})\big)$
    $p \leftarrow \mathit{\Pi}_{\mathrm{NIZK}}.Prove_{\mathtt{crs}_{\mathrm{NIZK}}}\big((\mathtt{crs}_{\mathrm{CS}}, \mathtt{comm}, \vec{c}) \in L_{\mathrm{Cons}}, (\vec{v}, m, \rho, \vec{r})\big)$
    return $(p, \mathtt{comm}, \vec{c})$

$D(\mathtt{pp} := (1^k, \mathtt{crs}_{\mathrm{NIZK}}, \mathtt{crs}_{\mathrm{CS}}), \mathtt{sk}_j := \big((\mathtt{pk}_{j,0}, \mathtt{sk}_{j,0}), (\mathtt{pk}_{j,1}, \mathtt{sk}_{j,1})\big), c := (p, \mathtt{comm}, \vec{c}))$
    if $\mathit{\Pi}_{\mathrm{NIZK}}.Verify_{\mathtt{crs}_{\mathrm{NIZK}}}\big((\mathtt{crs}_{\mathrm{CS}}, \mathtt{comm}, \vec{c}) \in L_{\mathrm{Cons}}, p\big) = \mathtt{valid}$ then
        for $i \in \{1, \ldots, |\vec{c}|\}$ do
            $\big(\rho, \vec{v} := \big((\mathtt{pk}_{1,0}{'}, \mathtt{pk}_{1,1}{'}), \ldots, (\mathtt{pk}_{|\vec{v}|,0}{'}, \mathtt{pk}_{|\vec{v}|,1}{'})\big), m\big) \leftarrow \mathit{\Pi}_{\mathrm{PKE}}.D_{\mathtt{sk}_{j,0}{'}}(c_{i,0})$
            if $(\rho, \vec{v}, m) \neq \bot \wedge (\mathtt{pk}_{j,0}, \mathtt{pk}_{j,1}) = (\mathtt{pk}_{i,0}{'}, \mathtt{pk}_{i,1}{'})$ then
                if $\mathtt{comm} = \mathit{\Pi}_{\mathrm{CS}}.Commit_{\mathtt{crs}_{\mathrm{CS}}}(\vec{v}, m; \rho)$ then
                    return $(\vec{v}, m)$
    return $\bot$
```

Theorem 1. *If $\mathit{\Pi}_{\mathrm{PKE}}$ is*

$$\begin{aligned}(&\varepsilon_{\text{PKE-Corr}}, \varepsilon_{\text{PKE-IND-CPA}}, \varepsilon_{\text{PKE-IK-CPA}},\\ &t_{\mathrm{PKE}}, n_{\mathrm{PKE}}, q_{E\,\mathrm{PKE}}, q_{D\,\mathrm{PKE}}, \mathsf{Corr})\text{-}\mathit{secure},\end{aligned} \tag{4.3}$$

$\mathit{\Pi}_{\mathrm{NIZK}}$ is

$$\begin{aligned}(&\varepsilon_{\text{NIZK-Complete}}, \varepsilon_{\text{NIZK-Sound}}, \varepsilon_{\text{NIZK-ZK}}, \varepsilon_{\text{NIZK-SS}},\\ &t_{\mathrm{NIZK}}, q_{P\,\mathrm{NIZK}}, q_{V\,\mathrm{NIZK}})\text{-}\mathit{secure},\end{aligned} \tag{4.4}$$

and $\mathit{\Pi}_{\mathrm{CS}}$ is

$$(\varepsilon_{\text{CS-Hiding}}, \varepsilon_{\text{CS-Binding}}, t_{\mathrm{CS}}, q_{\mathrm{CS}}, \mathsf{Binding})\text{-}\mathit{secure}, \tag{4.5}$$

then no adversary $\mathbf{A}$ (ε, t)-breaks $\mathit{\Pi}$'s

$$\begin{aligned}(&n := n_{\mathrm{PKE}}, d_E := q_{E\,\mathrm{PKE}}, q_E := q_{P\,\mathrm{NIZK}},\\ &q_D := \min(q_{V\,\mathrm{NIZK}}, q_{D\,\mathrm{PKE}}))\text{-}\mathit{Correctness},\end{aligned}$$

with $\varepsilon > \varepsilon_{\text{CS-Binding}} + \varepsilon_{\text{PKE-Corr}} + \varepsilon_{\text{NIZK-Complete}}$, and $t_{\mathrm{CS}}, t_{\mathrm{PKE}}, t_{\mathrm{NIZK}} \approx t + t_{\mathsf{Corr}}$, where t_{Corr} is the time to run $\mathit{\Pi}$'s $\mathbf{G}^{\mathsf{Corr}}$ game.

Remark 3. Theorem 1 states that $\mathit{\Pi}$'s Correctness holds against computationally bounded adversaries who do not have access to the secret keys of honest parties.

However, since we use an underlying PKE with correctness against unbounded adversaries, the proof of Theorem 1 implies something stronger, namely that Π is Correct according to a stronger Correctness notion wherein adversaries are allowed to query for the secret key of any honest receiver.

Theorem 2. *If Π_{CS} is*

$$(\varepsilon_{\mathrm{CS}\text{-}\mathsf{Hiding}},\varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}},t_{\mathrm{CS}},q_{\mathrm{CS}},\mathsf{Binding})\text{-}secure, \tag{4.6}$$

then no adversary $\mathbf{A}$ (ε)-breaks Π's Robustness, with $\varepsilon > \varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}}$.

Remark 4. Note that Theorem 2 states that Π's Robustness holds against computationally unbounded adversaries; such adversaries can compute the private key of any party from its public key.

In the following we assume, without loss of generality for any practical purpose, that the NIZK proof verification algorithm is deterministic. For instance, the NIZK scheme given in [16] has deterministic proof verification and is tightly unbounded simulation sound. The reason for this assumptions is that an adversary could potentially come up with a NIZK proof for a valid statement which would only be considered as valid by the NIZK verification algorithm sometimes.

Theorem 3. *If Π_{PKE} is*

$$\begin{aligned}(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}},\varepsilon_{\mathrm{PKE}\text{-}\mathsf{IND\text{-}CPA}},\varepsilon_{\mathrm{PKE}\text{-}\mathsf{IK\text{-}CPA}},&\\ t_{\mathrm{PKE}},n_{\mathrm{PKE}},q_{E\,\mathrm{PKE}},q_{D\,\mathrm{PKE}},\mathsf{Corr})\text{-}secure,&\end{aligned} \tag{4.7}$$

Π_{NIZK} is

$$\begin{aligned}(\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Complete}},\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Sound}},\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{ZK}},\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{SS}},&\\ t_{\mathrm{NIZK}},q_{P\,\mathrm{NIZK}},q_{V\,\mathrm{NIZK}})\text{-}secure,&\end{aligned} \tag{4.8}$$

Π_{CS} is

$$(\varepsilon_{\mathrm{CS}\text{-}\mathsf{Hiding}},\varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}},t_{\mathrm{CS}},q_{\mathrm{CS}},\mathsf{Binding})\text{-}secure, \tag{4.9}$$

and $\Pi_{\mathrm{NIZK}}.V$ is a deterministic algorithm, then no adversary $\mathbf{A}$ (ε,t)-breaks Π's

$$(n \coloneqq n_{\mathrm{PKE}}, q_D \coloneqq q_{V\,\mathrm{NIZK}})\text{-}Consistency,$$

with $\varepsilon > \varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}} + \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Sound}} + \varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}$ and with $t_{\mathrm{PKE}}, t_{\mathrm{CS}}, t_{\mathrm{NIZK}} \approx t + t_{\mathsf{Cons}}$, where t_{Cons} is the time to run Π's $\mathbf{G}^{\mathsf{Cons}}$ game.

Remark 5. Theorem 3 states that Π's Consistency holds against computationally bounded adversaries who do not have access to the secret keys of honest parties. However, similarly to Remark 3, its proof implies something stronger, namely that Π is Consistent with respect to a stronger Consistency notion which allows adversaries to query for the secret key of any honest receiver.

Theorem 4. *If $\mathit{\Pi}_{\mathrm{PKE}}$ is*

$$(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}, \varepsilon_{\mathrm{PKE}\text{-}\mathsf{IND}\text{-}\mathsf{CPA}}, \varepsilon_{\mathrm{PKE}\text{-}\mathsf{IK}\text{-}\mathsf{CPA}}, \\ t_{\mathrm{PKE}}, n_{\mathrm{PKE}}, q_{E_{\mathrm{PKE}}}, q_{D_{\mathrm{PKE}}}, \mathsf{Corr})\text{-}\textit{secure,} \tag{4.10}$$

$\mathit{\Pi}_{\mathrm{NIZK}}$ *is*

$$(\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Complete}}, \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Sound}}, \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{ZK}}, \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{SS}}, \\ t_{\mathrm{NIZK}}, q_{P_{\mathrm{NIZK}}}, q_{V_{\mathrm{NIZK}}})\text{-}\textit{secure,} \tag{4.11}$$

and $\mathit{\Pi}_{\mathrm{CS}}$ *is*

$$(\varepsilon_{\mathrm{CS}\text{-}\mathsf{Hiding}}, \varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}}, t_{\mathrm{CS}}, q_{\mathrm{CS}}, \mathsf{Binding})\text{-}\textit{secure,} \tag{4.12}$$

then no adversary $\mathbf{A}$ (ε, t)*-breaks* $\mathit{\Pi}$*'s*

$$(n \coloneqq n_{\mathrm{PKE}}, d_E \coloneqq q_{E_{\mathrm{PKE}}}, \\ q_E \coloneqq \min(q_{P_{\mathrm{NIZK}}}, q_{\mathrm{CS}}), q_D \coloneqq q_{V_{\mathrm{NIZK}}})\text{-}\mathsf{IK}\text{-}\mathsf{CCA}\text{-}\mathsf{2}\ \textit{security,}$$

with

$$\begin{aligned} \varepsilon &> 4 \cdot (\varepsilon_{\mathrm{PKE}\text{-}\mathsf{IND}\text{-}\mathsf{CPA}} + \varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}) \\ &\quad + 2 \cdot (\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{ZK}} + \varepsilon_{\mathrm{PKE}\text{-}\mathsf{IK}\text{-}\mathsf{CPA}} + \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{SS}}) \\ &\quad + \varepsilon_{\mathrm{CS}\text{-}\mathsf{Hiding}}, \\ t_{\mathrm{PKE}}, t_{\mathrm{CS}} &\approx t + t_{\mathsf{IK}\text{-}\mathsf{CCA}\text{-}\mathsf{2}} + q_E \cdot t_{S_{Sim}} + t_{S_{CRS}}, \\ t_{\mathrm{NIZK}} &\approx t + t_{\mathsf{IK}\text{-}\mathsf{CCA}\text{-}\mathsf{2}}, \end{aligned}$$

where $t_{\mathsf{IK}\text{-}\mathsf{CCA}\text{-}\mathsf{2}}$ *is the time to run* $\mathit{\Pi}$*'s* $\mathbf{G}_{\mathbf{b}}^{\mathsf{IK}\text{-}\mathsf{CCA}\text{-}\mathsf{2}}$ *game experiment,* $t_{S_{Sim}}$ *is the runtime of* S_{Sim}*, and* $t_{S_{CRS}}$ *is the runtime of* S_{CRS}*.*

Theorem 5. *If $\mathit{\Pi}_{\mathrm{PKE}}$ is*

$$(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}, \varepsilon_{\mathrm{PKE}\text{-}\mathsf{IND}\text{-}\mathsf{CPA}}, \varepsilon_{\mathrm{PKE}\text{-}\mathsf{IK}\text{-}\mathsf{CPA}}, \\ t_{\mathrm{PKE}}, n_{\mathrm{PKE}}, q_{E_{\mathrm{PKE}}}, q_{D_{\mathrm{PKE}}}, \mathsf{Corr})\text{-}\textit{secure,} \tag{4.13}$$

$\mathit{\Pi}_{\mathrm{NIZK}}$ *is*

$$(\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Complete}}, \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Sound}}, \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{ZK}}, \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{SS}}, \\ t_{\mathrm{NIZK}}, q_{P_{\mathrm{NIZK}}}, q_{V_{\mathrm{NIZK}}})\text{-}\textit{secure,} \tag{4.14}$$

and $\mathit{\Pi}_{\mathrm{CS}}$ *is*

$$(\varepsilon_{\mathrm{CS}\text{-}\mathsf{Hiding}}, \varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}}, t_{\mathrm{CS}}, q_{\mathrm{CS}}, \mathsf{Binding})\text{-}\textit{secure,} \tag{4.15}$$

then no adversary $\mathbf{A}$ (ε, t)*-breaks* $\mathit{\Pi}$*'s*

$$(n \coloneqq n_{\mathrm{PKE}}, d_E \coloneqq q_{E_{\mathrm{PKE}}}, \\ q_E \coloneqq \min(q_{P_{\mathrm{NIZK}}}, q_{\mathrm{CS}}), q_D \coloneqq q_{V_{\mathrm{NIZK}}})\text{-}\mathsf{IND}\text{-}\mathsf{CCA}\text{-}\mathsf{2}\ \textit{security,}$$

with

$$\begin{aligned}\varepsilon &> 4 \cdot (\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + \varepsilon_{\text{PKE-}\mathsf{Corr}}) \\ &\quad + 2 \cdot (\varepsilon_{\text{NIZK-}\mathsf{ZK}} + \varepsilon_{\text{NIZK-}\mathsf{SS}}) \\ &\quad + \varepsilon_{\text{CS-}\mathsf{Hiding}} \\ t_{\text{PKE}} &\approx t + t_{\mathsf{IND\text{-}CCA\text{-}2}} + q_E \cdot t_{S_{Sim}} + t_{S_{CRS}}, \\ t_{\text{NIZK}}, t_{\text{CS}} &\approx t + t_{\mathsf{IND\text{-}CCA\text{-}2}},\end{aligned}$$

where $t_{\mathsf{IND\text{-}CCA\text{-}2}}$ *is the time to run* Π*'s* $\mathbf{G}_{\mathbf{b}}^{\mathsf{IND\text{-}CCA\text{-}2}}$ *game,* $t_{S_{Sim}}$ *is the runtime of* S_{Sim}*, and* $t_{S_{CRS}}$ *is the runtime of* S_{CRS}*.*

5 Multi-Designated Receiver Signed Public Key Encryption Schemes

We now introduce the second new type of scheme we give in this paper: Multi-Designated Receiver Signed Public Key Encryption (MDRS-PKE). An MDRS-PKE scheme $\Pi = (S, G_S, G_V, E, D)$ with message space $\mathcal{M}$ is a five-tuple of PPTs, where:

- S: on input 1^k, generates public parameters $\mathtt{pp}$;
- G_S: on input $\mathtt{pp}$, generates a sender key-pair;
- G_V: on input $\mathtt{pp}$, generates a receiver key-pair;
- E: on input $(\mathtt{pp}, \mathtt{ssk}, \vec{v}, m)$, where $\mathtt{ssk}$ is the secret sending key, $\vec{v}$ is a vector of public keys of the intended receivers, and m is the message, generates a ciphertext c;
- D: on input $(\mathtt{pp}, \mathtt{rsk}, c)$, where $\mathtt{rsk}$ is the receiver's secret key, D decrypts c using $\mathtt{rsk}$, obtaining a triple sender/receiver-vector/message $(\mathtt{spk}, \vec{v}, m)$ (or $\perp$ if decryption fails) which it then outputs.

5.1 The Security of MDRS-PKE Schemes

Below we state the definitions of Correctness, Consistency, Unforgeability, IND-CCA-2 security, IK-CCA-2 security, and Off-The-Record for MDRS-PKE schemes. Before proceeding to the actual definitions, we first introduce some oracles the game systems for MDRS-PKE use. In the following, consider an MDRS-PKE scheme $\Pi = (S, G_S, G_V, E, D)$ with message space $\mathcal{M}$. The oracles below are defined for a game-system with (an implicitly defined) security parameter k:

Public Parameter Generation Oracle: $\mathcal{O}_{PP}$

1. On the first call, compute $\mathtt{pp} \leftarrow S(1^k)$; output $\mathtt{pp}$;
2. On subsequent calls, simply output $\mathtt{pp}$.

Sender Key-Pair Oracle: $\mathcal{O}_{SK}(A_i)$

1. On the first call on input A_i, compute and store $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow G_S(\mathtt{pp})$; output $(\mathtt{spk}_i, \mathtt{ssk}_i)$;
2. On subsequent calls, simply output $(\mathtt{spk}_i, \mathtt{ssk}_i)$.

Receiver Key-Pair Oracle: $\mathcal{O}_{RK}(B_j)$

1. Analogous to the Sender Key-Pair Oracle.

Sender Public-Key Oracle: $\mathcal{O}_{SPK}(A_i)$

1. $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow \mathcal{O}_{SK}(A_i)$; output $\mathtt{spk}_i$.

Receiver Public-Key Oracle: $\mathcal{O}_{RPK}(B_j)$

1. Analogous to the Sender Public-Key Oracle.

Encryption Oracle: $\mathcal{O}_E(A_i, \vec{V}, m)$

1. $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow \mathcal{O}_{SK}(A_i)$;
2. $\vec{v} \leftarrow (\mathcal{O}_{RPK}(V_1), \ldots, \mathcal{O}_{RPK}(V_{|\vec{V}|}))$;
3. Output $c \leftarrow E_{\mathsf{pp}}(\mathtt{ssk}_i, \vec{v}, m)$.

Decryption Oracle: $\mathcal{O}_D(B_j, c)$

1. $(\mathtt{vpk}_j, \mathtt{vsk}_j) \leftarrow \mathcal{O}_{RK}(B_j)$;
2. Output $(\mathtt{spk}, \vec{v} \coloneqq (\mathtt{rpk}_1, \ldots, \mathtt{rpk}_{|\vec{v}|}), m) \leftarrow D_{\mathsf{pp}}(\mathtt{vsk}_j, c)$.

We now introduce the game-based notions. Let $\Pi = (S, G_S, G_V, E, D)$ be an MDRS-PKE.

Definition 6 (Correctness). *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Corr}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SPK}, \mathcal{O}_{SK}, \mathcal{O}_{RPK}, \mathcal{O}_{RK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if there are two queries* q_E *and* q_D *to* $\mathcal{O}_E$ *and* $\mathcal{O}_D$*, respectively, where* q_E *has input* $(A_i, \vec{V}, m)$ *and* q_D *has input* (B_j, c)*, satisfying* $B_j \in \vec{V}$*, the input* c *in* q_D *is the output of* q_E*, the output of* q_D *is* $(\mathtt{spk}_i{}', \vec{v}', m')$ *with* $(\mathtt{spk}_i{}', \vec{v}', m') = \bot$ *or* $(\mathtt{spk}_i{}', \vec{v}', m') \neq (\mathtt{spk}_i, \vec{v}, m)$*—where* $\mathtt{spk}_i$ *is* A_i*'s public key and* $\vec{v}$ *is the corresponding vector of public keys of the parties of* $\vec{V}$*— and* $\mathbf{A}$ *did not query* $\mathcal{O}_{SK}$ *on* A_i *nor* $\mathcal{O}_{RK}$ *on* B_j*.*

The advantage of $\mathbf{A}$ *in winning the Correctness game, denoted* $Adv^{\mathsf{Corr}}(\mathbf{A})$*, is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Corr}}$ *as described above.*

As already noted in Remark 1, Correctness is a property only relevant to honest parties. As these parties are not corrupted, their keys do not leak to the adversary. Definition 6 hence disallows adversaries from querying for the secret keys of honest parties. Note that the analogous Correctness notion for MDVS schemes introduced in [28]—which also does not allow adversaries to query for the secret keys of honest parties—is known to imply the composable security of MDVS schemes (see [28]). As noted in Remark 9, the MDRS-PKE construction we give actually satisfies a stronger Correctness notion analogous to the one mentioned in Remark 1, as long as both of the underlying (PKEBC and MDVS) schemes satisfy analogous Correctness notions.

The following notion captures Consistency for MDRS-PKE schemes, and is analogous to the PKEBC Consistency notion.

Definition 7 (Consistency). *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Cons}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SPK}, \mathcal{O}_{SK}, \mathcal{O}_{RPK}, \mathcal{O}_{RK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if there is a ciphertext* c *such that* $\mathcal{O}_D$ *is queried on inputs* (B_i, c) *and* (B_j, c) *for some* B_i *and* B_j *(possibly with* $B_i = B_j$*), there is no prior query on either* B_i *or* B_j *to* $\mathcal{O}_{RK}$*, query* $\mathcal{O}_D(B_i, c)$ *outputs some* $(\mathtt{spk}_l, \vec{v}, m)$ *satisfying* $(\mathtt{spk}_l, \vec{v}, m) \neq \bot$*,* $\mathtt{spk}_l$ *is some party* A_l*'s public sender key (i.e.* $\mathcal{O}_{SPK}(A_l) = \mathtt{spk}_l$*) and* $\mathtt{rpk}_j \in \vec{v}$ *(where* $\mathtt{rpk}_j$ *is* B_j*'s public key), and query* $\mathcal{O}_D(B_j, c)$ *does not output the same triple* $(\mathtt{spk}_l, \vec{v}, m)$*.*

The advantage of $\mathbf{A}$ *in winning the Consistency game is denoted* $Adv^{\mathsf{Cons}}(\mathbf{A})$ *and corresponds to the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Cons}}$ *as described above.*

The following security notion is analogous to the EUF-CMA security notion for Digital Signature Schemes. For the case of a single receiver, it informally states that if a sender A is honest, then no dishonest party can forge a ciphertext that fools an honest receiver into believing A sent it some message that A actually did not send.

Definition 8 (Unforgeability). *Consider the following game played between adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Unforg}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SPK}, \mathcal{O}_{SK}, \mathcal{O}_{RPK}, \mathcal{O}_{RK}, \mathcal{O}_E, \mathcal{O}_D}$

We say that $\mathbf{A}$ *wins the game if there is a query* q *to* $\mathcal{O}_D$ *on an input* (B_j, c) *that outputs* $(\mathtt{spk}_i, \vec{v}, m) \neq \bot$ *with* $\mathtt{spk}_i$ *being some party* A_i*'s sender public key (i.e.* $\mathcal{O}_{SPK}(A_i) = \mathtt{spk}_i$*), there was no query* $\mathcal{O}_E(A_i, \vec{V}, m)$ *where* $\vec{V}$ *is the vector of parties with corresponding public keys* $\vec{v}$*,* $\mathcal{O}_{SK}$ *was not queried on input* A_i*, and* $\mathcal{O}_{RK}$ *was not queried on input* B_j*.*

The advantage of $\mathbf{A}$ *in winning the Unforgeability game is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Unforg}}$ *as described above, and is denoted* $Adv^{\mathsf{Unforg}}(\mathbf{A})$*.*

We say that an adversary $\mathbf{A}$ (ε, t)-breaks the $(n_S, n_R, d_E, q_E, q_D)$-Correctness, Consistency, or Unforgeability of Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{SPK}$, $\mathcal{O}_{SK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n_S different senders, queries $\mathcal{O}_{RPK}$, $\mathcal{O}_{RK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n_R different receivers, makes at most q_E and q_D queries to $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, with the sum of lengths of the party vectors input to $\mathcal{O}_E$ being at most d_E, and $\mathbf{A}$'s advantage in winning the (corresponding) security game is at least ε.

The following security notions are the MDRS-PKE variants of Definitions 4 and 5. The games defined by these notions provide adversaries with access to the oracles $\mathcal{O}_{PP}$, $\mathcal{O}_{SPK}$, $\mathcal{O}_{SK}$ and $\mathcal{O}_{RPK}$ defined above as well as to oracles $\mathcal{O}_E$ and $\mathcal{O}_D$. For both notions, $\mathcal{O}_D$ is defined as follows:

Decryption Oracle: $\mathcal{O}_D(B_j, c)$

1. If c was the output of some query to $\mathcal{O}_E$, output `test`;
2. Otherwise, compute $(\mathtt{spk}_i, \vec{v}, m) \leftarrow D_{\mathsf{pp},\mathsf{sk}_j}(c)$, where $\mathtt{sk}_j$ is B_j's secret key; output $(\mathtt{spk}_i, \vec{v}, m)$.

The $\mathcal{O}_E$ oracle provided by the IND-CCA-2 games differs from the one provided by the IK-CCA-2 games; for IND-CCA-2, $\mathcal{O}_E$ is as follows:

Encryption Oracle: $\mathcal{O}_E(A_i, \vec{V}, m_0, m_1)$

1. For game system $\mathbf{G}_{\mathbf{b}}^{\mathsf{IND\text{-}CCA\text{-}2}}$, encrypt $m_{\mathbf{b}}$ under $\mathtt{ssk}_i$ (A_i's sender secret key) and $\vec{v}$ ($\vec{V}$'s corresponding vector of receiver public keys); output c.

Definition 9 (IND-CCA-2 Security). *Consider the following game played between an adversary* $\mathbf{A}$ *and a game system* $\mathbf{G}_{\mathbf{b}}^{\mathsf{IND\text{-}CCA\text{-}2}}$*, with* $\mathbf{b} \in \{0,1\}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SPK}, \mathcal{O}_{SK}, \mathcal{O}_{RPK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query* $\mathcal{O}_E(A_i, \vec{V}, m_0, m_1)$*:*

- $|m_0| = |m_1|$*; and*
- *there is no query on* A_i *to* $\mathcal{O}_{SK}$*.*

We define the advantage of $\mathbf{A}$ *in winning the* IND-CCA-2 *game as*

$$Adv^{\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \left| \Pr[\mathbf{AG}_{\mathbf{0}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1 \right|.$$

For the IK-CCA-2 security notion, $\mathcal{O}_E$ behaves as follows:

Encryption Oracle: $\mathcal{O}_E((A_{i,0}, \vec{V}_0), (A_{i,1}, \vec{V}_1), m)$

1. For game system $\mathbf{G}_{\mathbf{b}}^{\mathsf{IK\text{-}CCA\text{-}2}}$, encrypt m under $\mathtt{ssk}_{i,\mathbf{b}}$ ($A_{i,\mathbf{b}}$'s secret key) and $\vec{v}_{\mathbf{b}}$ (the vector of public keys corresponding to $\vec{V}_{\mathbf{b}}$), creating a fresh ciphertext c; output c.

Definition 10 (IK-CCA-2 Security). *Consider the following game played between an adversary* $\mathbf{A}$ *and a game system* $\mathbf{G}_{\mathbf{b}}^{\mathsf{IK\text{-}CCA\text{-}2}}$*, with* $\mathbf{b} \in \{0,1\}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SPK}, \mathcal{O}_{SK}, \mathcal{O}_{RPK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query* $((A_{i,0}, \vec{V}_0), (A_{i,1}, \vec{V}_1), m)$ *to* $\mathcal{O}_E$*:*

- $|\vec{V}_0| = |\vec{V}_1|$*; and*
- $\mathcal{O}_{SK}$ *is not queried on neither* $A_{i,0}$ *and* $A_{i,1}$*.*

We define the advantage of $\mathbf{A}$ *in winning the* IK-CCA-2 *security game as*

$$Adv^{\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \left| \Pr[\mathbf{AG}_{\mathbf{0}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1 \right|.$$

We say that an adversary $\mathbf{A}$ (ε, t)-breaks the (n_R, d_E, q_E, q_D)-IND-CCA-2 security or IK-CCA-2 security of Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{RPK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n_R different receivers, makes at most q_E and q_D queries to $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, with the sum of lengths of the party vectors input to $\mathcal{O}_E$ being at most d_E, and has at least ε advantage in winning the corresponding security game.

Remark 6. The IND-CCA-2 and IK-CCA-2 security notions for MDRS-PKE schemes capture, respectively, confidentiality and anonymity. Even though one could define stronger variants of these notions wherein the adversary is allowed to query for the secret key of any sender, we chose these definitions because they are weaker, but yet strong enough to imply composable security (see [3, 4, 17] for the analogous case of the Outsider Security Model for Signcryption). Nonetheless, our MDRS-PKE construction satisfies the stronger IND-CCA-2 and IK-CCA-2 security notions in which the adversary is allowed to query for the secret key of every sender.

The following notion captures the *Off-The-Record* property of MDRS-PKE schemes, and resembles the (Any-Subset) Off-The-Record security notion introduced in [13] for MDVS schemes. This notion defines two game systems, $\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$ and $\mathbf{G}_{\mathbf{1}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$, which are parameterized by an algorithm *Forge*. The game systems also provide adversaries with access to an oracle $\mathcal{O}_E$, whose behavior varies depending on the underlying game system, i.e. depending on $\mathbf{b} \in \{0,1\}$. $\mathcal{O}_E$ behaves as follows:

Encryption Oracle: $\mathcal{O}_E(\mathtt{type} \in \{\mathtt{sign}, \mathtt{forge}\}, A_i, \vec{V}, m, \mathcal{D})$

For game system $\mathbf{G}_{\mathbf{b}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$, the oracle behaves as follows:

1. $c_{\mathbf{0}} \leftarrow E_{\mathsf{pp}}(\mathtt{ssk}_i, \vec{v}, m)$;
2. $c_{\mathbf{1}} \leftarrow \mathit{Forge}_{\mathsf{pp}}(\mathtt{spk}_i, \vec{v}, m, \{\mathtt{rsk}_j\}_{B_j \in \mathcal{D}})$;
3. If $\mathbf{b} = 0$, output $c_{\mathbf{0}}$ if $\mathtt{type} = \mathtt{sign}$ and $c_{\mathbf{1}}$ if $\mathtt{type} = \mathtt{forge}$;
4. Otherwise, if $\mathbf{b} = 1$, output $c_{\mathbf{1}}$.

Definition 11 (Off-The-Record). *Let Forge be a PPT algorithm that on input* pp, $\mathtt{spk}_{i^*}$, $\vec{v}$, m^* *and* $\{\mathtt{rsk}_j\}_{B_j \in \mathcal{D}^*}$*, outputs a forged ciphertext* c'*. For* $\mathbf{b} \in \{0,1\}$*, consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}_{\mathbf{b}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SPK}, \mathcal{O}_{SK}, \mathcal{O}_{RPK}, \mathcal{O}_{RK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query* $(\mathtt{type}, A_i, \vec{V}, m, \mathcal{D})$ *to* $\mathcal{O}_E$*, and letting* c *be the output of* $\mathcal{O}_E$*, all of the following hold:*

1. $\mathcal{D} \subseteq \mathit{Set}(\vec{V})$*;*
2. *for every query* B_j *to* $\mathcal{O}_{VK}$*,* $B_j \notin \mathit{Set}(\vec{V}) \setminus \mathcal{D}$*;*
3. *for every query* A_l *to* $\mathcal{O}_{SK}$*,* $A_l \neq A_i$*; and*
4. *for all queries* $\mathcal{O}_D(A_l, B_j, \vec{V}', m', c')$ *with* $A_l = A_i$ *and* $\vec{V}' = \vec{V}$*,* $c' \neq c$*.*

$\mathbf{A}$*'s advantage in winning the Off-The-Record security game with respect to Forge is defined as*

$$\mathit{Adv}^{\mathsf{OTR}\text{-}\mathit{Forge}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}\mathit{Forge}} = \mathtt{win}] + \Pr[\mathbf{A}\mathbf{G}_{\mathbf{1}}^{\mathsf{OTR}\text{-}\mathit{Forge}} = \mathtt{win}] - 1 \right|.$$

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{OTR}}, t)$-breaks the $(n_S, n_R, d_E, q_E, q_D)$-Off-The-Record security of $\mathit{\Pi}$ with respect to algorithm *Forge* if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{SPK}$, $\mathcal{O}_{SK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n_S different senders, queries $\mathcal{O}_{RPK}$, $\mathcal{O}_{RK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n_R different receivers, makes at most q_E and q_D queries to $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, with the sum of lengths of the party vectors input to $\mathcal{O}_E$ being at most d_E, and satisfies $\mathit{Adv}^{\mathsf{OTR}\text{-}\mathit{Forge}}(\mathbf{A}) \geq \varepsilon_{\mathsf{OTR}}$.

Finally, we say that $\mathit{\Pi}$ is

$$(\varepsilon_{\mathsf{Corr}}, \varepsilon_{\mathsf{Cons}}, \varepsilon_{\mathsf{Unforg}}, \varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}, \varepsilon_{\mathsf{IK\text{-}CCA\text{-}2}}, \varepsilon_{\mathsf{OTR}},$$
$$t, n_S, n_R, d_E, q_E, q_D, \mathit{Forge})\text{-secure},$$

if no adversary $\mathbf{A}$:

- $(\varepsilon_{\mathsf{Corr}}, t)$-breaks the $(n_S, n_R, d_E, q_E, q_D)$-Correctness of Π;
- $(\varepsilon_{\mathsf{Cons}}, t)$-breaks the $(n_S, n_R, d_E, q_E, q_D)$-Consistency of Π;
- $(\varepsilon_{\mathsf{Unforg}}, t)$-breaks the $(n_S, n_R, d_E, q_E, q_D)$-Unforgeability of Π;
- $(\varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}, t)$-breaks the (n_R, d_E, q_E, q_D)-$\mathsf{IND\text{-}CCA\text{-}2}$ security of Π;
- $(\varepsilon_{\mathsf{IK\text{-}CCA\text{-}2}}, t)$-breaks the (n_R, d_E, q_E, q_D)-$\mathsf{IK\text{-}CCA\text{-}2}$ security of Π; or
- $(\varepsilon_{\mathsf{OTR}}, t)$-breaks the $(n_S, n_R, d_E, q_E, q_D)$-Off-The-Record security of Π with respect to *Forge*.

Remark 7. As one may note, due to the Off-The-Record property of MDRS-PKE schemes (see Definition 11), any receiver B_j can generate a ciphertext that decrypts correctly under B_j's own receiver secret key using only its own secret key and the public keys of the sender and any other receivers. It is thus crucial that, when defining ciphertext Unforgeability (see Definition 8), the adversary is not allowed to query for the secret key of any receiver with respect to which it is trying forge a signature.

It is equally important that the adversary is not allowed to query for the secret keys of honest receivers in the Off-The-Record security notion (Definition 11): as honest receivers do not participate in the ciphertext forgery, due to the Unforgeability of ciphertexts (Definition 8)—which in particular guarantees that if a receiver is honest, then it only decrypts ciphertexts generated by the actual sender, assuming the sender is honest—if an adversary could query for the secret key of an honest receiver B_j, it would be able to distinguish real ciphertexts generated by the sender—which B_j would decrypt successfully using its secret key—from fake ciphertexts generated by dishonest receivers—which, by the Unforgeability of ciphertexts, B_j would not decrypt successfully.

Finally, the adversary can also not be given access to the secret key of any honest receiver B_j in the Consistency game of Definition 7, as otherwise, by the Off-The-Record guarantee (Definition 11), it would be able to use B_j's receiver secret key to forge a ciphertext c that B_j would decrypt successfully (as if it really had been sent by the actual sender), whereas any other honest (designated) receiver's decryption of c would fail.

6 A Multi-Designated Receiver Signed Public Key Encryption Scheme from Standard Assumptions

In this section we give a construction of an MDRS-PKE scheme from a PKEBC scheme and an MDVS scheme (see Algorithm 2). The construction essentially consists of using the MDVS scheme to sign both the messages and the vectors of public PKEBC keys of the receivers, and then using the PKEBC scheme to encrypt the signed message, together with its MDVS signature, the public MDVS signer key of the sender and the vector of public MDVS verifier keys of the receivers.

Remark 8. Even though our MDRS-PKE construction allows parties to locally generate their keys, to achieve the Off-The-Record guarantee it is required that dishonest receivers know their secret keys. This is only so as otherwise one could

Algorithm 2 Construction of an MDRS-PKE scheme $\Pi = (S, G_S, G_V, E, D)$ from a PKEBC scheme $\Pi_{\text{PKEBC}} = (G, S, E, D)$, and an MDVS scheme $\Pi_{\text{MDVS}} = (\mathit{Setup}, G_S, G_V, \mathit{Sign}, \mathit{Vfy})$.

```
$\mathit{Setup}(1^k)$
    $\mathtt{pp}_{\text{MDVS}} \leftarrow \Pi_{\text{MDVS}}.\mathit{Setup}(1^k)$
    $\mathtt{pp}_{\text{PKEBC}} \leftarrow \Pi_{\text{PKEBC}}.S(1^k)$
    $\mathtt{pp} \coloneqq (\mathtt{pp}_{\text{MDVS}}, \mathtt{pp}_{\text{PKEBC}})$
    return $\mathtt{pp}$

$G_S(\mathtt{pp} \coloneqq (\mathtt{pp}_{\text{MDVS}}, \mathtt{pp}_{\text{PKEBC}}))$
    $(\mathtt{spk}_{\text{MDVS}}, \mathtt{ssk}_{\text{MDVS}}) \leftarrow \Pi_{\text{MDVS}}.G_S(\mathtt{pp}_{\text{MDVS}})$
    $\mathtt{spk} \coloneqq \mathtt{spk}_{\text{MDVS}}$
    $\mathtt{ssk} \coloneqq (\mathtt{spk}, \mathtt{ssk}_{\text{MDVS}})$
    return $(\mathtt{spk}, \mathtt{ssk})$

$G_V(\mathtt{pp} \coloneqq (\mathtt{pp}_{\text{MDVS}}, \mathtt{pp}_{\text{PKEBC}}))$
    $(\mathtt{vpk}_{\text{MDVS}}, \mathtt{vsk}_{\text{MDVS}}) \leftarrow \Pi_{\text{MDVS}}.G_V(\mathtt{pp}_{\text{MDVS}})$
    $(\mathtt{pk}_{\text{PKEBC}}, \mathtt{sk}_{\text{PKEBC}}) \leftarrow \Pi_{\text{PKEBC}}.G(\mathtt{pp}_{\text{PKEBC}})$
    $\mathtt{rpk} \coloneqq (\mathtt{vpk}_{\text{MDVS}}, \mathtt{pk}_{\text{PKEBC}})$
    $\mathtt{rsk} \coloneqq \big(\mathtt{rpk}, (\mathtt{vsk}_{\text{MDVS}}, \mathtt{sk}_{\text{PKEBC}})\big)$
    return $(\mathtt{rpk}, \mathtt{rsk})$

$E_{\mathtt{pp}}(\mathtt{ssk}_i, \vec{v}, m)$
    With
        $\mathtt{pp} \coloneqq (\mathtt{pp}_{\text{MDVS}}, \mathtt{pp}_{\text{PKEBC}})$
        $\mathtt{ssk}_i \coloneqq (\mathtt{spk}_i, \mathtt{ssk}_{\text{MDVS}\,i})$
        $\vec{v} \coloneqq \big(\mathtt{rpk}_1, \ldots, \mathtt{rpk}_{|\vec{v}|}\big)$
        for each $i \in \{1, \ldots, |\vec{v}|\}$
            $\mathtt{rpk}_i \coloneqq (\mathtt{vpk}_{\text{MDVS}\,i}, \mathtt{pk}_{\text{PKEBC}\,i})$

    $\vec{v}_{\text{PKEBC}} \leftarrow (\mathtt{pk}_{\text{PKEBC}\,1}, \ldots, \mathtt{pk}_{\text{PKEBC}\,|\vec{v}|})$
    $\vec{v}_{\text{MDVS}} \leftarrow (\mathtt{vpk}_{\text{MDVS}\,1}, \ldots, \mathtt{vpk}_{\text{MDVS}\,|\vec{v}|})$
    $\sigma \leftarrow \Pi_{\text{MDVS}}.\mathit{Sign}_{\mathtt{pp}_{\text{MDVS}}}(\mathtt{ssk}_{\text{MDVS}\,i}, \text{Set}(\vec{v}_{\text{MDVS}}), (\vec{v}_{\text{PKEBC}}, m))$
    return $\Pi_{\text{PKEBC}}.E_{\mathtt{pp}_{\text{PKEBC}}}\big(\vec{v}_{\text{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\text{MDVS}}, m, \sigma)\big)$

$D_{\mathtt{pp}}(\mathtt{rsk}_j, c)$
    With
        $\mathtt{pp} \coloneqq (\mathtt{pp}_{\text{MDVS}}, \mathtt{pp}_{\text{PKEBC}})$
        $\mathtt{rsk}_j \coloneqq \big(\mathtt{rpk}_j, (\mathtt{vsk}_{\text{MDVS}\,j}, \mathtt{sk}_{\text{PKEBC}\,j})\big)$
        $\mathtt{rpk}_j \coloneqq (\mathtt{vpk}_{\text{MDVS}\,j}, \mathtt{pk}_{\text{PKEBC}\,j})$

    $\big(\vec{v}_{\text{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\text{MDVS}}, m, \sigma)\big) \leftarrow \Pi_{\text{PKEBC}}.D_{\mathtt{pp}_{\text{PKEBC}}}(\mathtt{sk}_{\text{PKEBC}\,j}, c)$
    if $\big(\vec{v}_{\text{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\text{MDVS}}, m, \sigma)\big) = \bot \;\vee\; |\vec{v}_{\text{PKEBC}}| \neq |\vec{v}_{\text{MDVS}}|$ then
        return $\bot$
    $\vec{v} \coloneqq \big((v_{\text{MDVS}\,1}, v_{\text{PKEBC}\,1}), \ldots, (v_{\text{MDVS}\,|\vec{v}_{\text{PKEBC}}|}, v_{\text{PKEBC}\,|\vec{v}_{\text{PKEBC}}|})\big)$
    if $\mathtt{rpk}_j \notin \vec{v}$ then
        return $\bot$
    if $\Pi_{\text{MDVS}}.\mathit{Vfy}_{\mathtt{pp}_{\text{MDVS}}}(\mathtt{spk}_i, \mathtt{vsk}_{\text{MDVS}\,j}, \text{Set}(\vec{v}_{\text{MDVS}}), (\vec{v}_{\text{PKEBC}}, m), \sigma) \neq \mathtt{valid}$ then
        return $\bot$
    return $(\mathtt{spk}_i, \vec{v}, m)$
```

mount attacks that break the Off-The-Record guarantee. For instance, consider an honest sender Alice that sends a message m to Bob. Bob, who is dishonest wants to convince a non-designated receiver, Eve, that Alice sent m. To do that, Bob could have Eve generating the keys for Bob herself, and give him only the public key (that Bob would claim as being his public key). When Alice sends m, Eve can now learn that Alice sent m as it can use Bob's secret key. Furthermore, since no one other than Eve has Bob's secret key, Eve knows that it cannot be a fake message, implying that it must be Alice's message. Current composable notions capturing the security of MDVS schemes solve this problem by assuming a trusted third party which generates all key-pairs and gives everyone access to their own key-pair [28][8]. This in particular implies that Bob would have access to its own secret key, and so even if Eve would know Bob's secret key, she would not be able to tell if Alice was the one sending messages or if Bob was faking Alice's messages.

6.1 Security Analysis of the MDRS-PKE Construction

Due to space restrictions, the full proofs of the following results are in the appendix (see Appendix H).

Theorem 6. *If Π_{PKEBC} is*

$$\begin{aligned}(&\varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{Corr}}, \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{Rob}}, \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{Cons}}, \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{IND}\text{-}\mathsf{CCA}\text{-}\mathsf{2}}, \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{IK}\text{-}\mathsf{CCA}\text{-}\mathsf{2}},\\ &t_{\mathrm{PKEBC}}, n_{\mathrm{PKEBC}}, d_{E\mathrm{PKEBC}}, q_{E\mathrm{PKEBC}}, q_{D\mathrm{PKEBC}})\text{-}\textit{secure},\end{aligned} \tag{6.1}$$

and Π_{MDVS} is

$$\begin{aligned}(&\varepsilon_{\mathrm{MDVS}\text{-}\mathsf{Corr}}, \varepsilon_{\mathrm{MDVS}\text{-}\mathsf{Cons}}, \varepsilon_{\mathrm{MDVS}\text{-}\mathsf{Unforg}}, \varepsilon_{\mathrm{MDVS}\text{-}\mathsf{OTR}}, \varepsilon_{\mathrm{MDVS}\text{-}\mathsf{PI}},\\ &t_{\mathrm{MDVS}}, n_{S\mathrm{MDVS}}, n_{V\mathrm{MDVS}}, d_{S\mathrm{MDVS}},\\ &q_{S\mathrm{MDVS}}, q_{V\mathrm{MDVS}}, \mathit{Forge}_{\mathrm{MDVS}})\text{-}\textit{secure},\end{aligned} \tag{6.2}$$

then no adversary $\mathbf{A}$ (ε, t)-breaks Π's

$$\begin{aligned}(n_S &\coloneqq n_{S\mathrm{MDVS}},\\ n_R &\coloneqq \min(n_{\mathrm{PKEBC}}, n_{V\mathrm{MDVS}}),\\ d_E &\coloneqq \min(d_{E\mathrm{PKEBC}}, d_{S\mathrm{MDVS}}),\\ q_E &\coloneqq \min(q_{E\mathrm{PKEBC}}, q_{S\mathrm{MDVS}}),\\ q_D &\coloneqq \min(q_{D\mathrm{PKEBC}}, q_{V\mathrm{MDVS}}))\text{-}\textit{Correctness},\end{aligned}$$

with $\varepsilon > \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{Corr}} + \varepsilon_{\mathrm{MDVS}\text{-}\mathsf{Corr}}$, and $t_{\mathrm{PKEBC}}, t_{\mathrm{MDVS}} \approx t + t_{\mathsf{Corr}}$, where t_{Corr} is the time to run Π's $\mathbf{G}^{\mathsf{Corr}}$ game.

[8] The composable notions capturing the security of MDVS given in [28] actually assume something even stronger: every dishonest party has access to the secret keys of every other dishonest party.

Remark 9. Similarly to Remark 3, if Π_{PKEBC}'s correctness holds even when the adversary is allowed to query for the secret key of any receiver, and Π_{MDVS}'s correctness holds even when the adversary is allowed to query for the secret keys of any signer or verifier, then Π's Correctness holds even when the adversary is allowed to query for the secret keys of any sender and receiver.

Theorem 7. *If* Π_{PKEBC} *is*

$$\begin{gathered}(\varepsilon_{\text{PKEBC-}\mathsf{Corr}}, \varepsilon_{\text{PKEBC-}\mathsf{Rob}}, \varepsilon_{\text{PKEBC-}\mathsf{Cons}}, \varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}}, \varepsilon_{\text{PKEBC-}\mathsf{IK\text{-}CCA\text{-}2}}, \\ t_{\text{PKEBC}}, n_{\text{PKEBC}}, d_{E\text{PKEBC}}, q_{E\text{PKEBC}}, q_{D\text{PKEBC}})\text{-}\textit{secure,}\end{gathered} \tag{6.3}$$

and Π_{MDVS} *is*

$$\begin{gathered}(\varepsilon_{\text{MDVS-}\mathsf{Corr}}, \varepsilon_{\text{MDVS-}\mathsf{Cons}}, \varepsilon_{\text{MDVS-}\mathsf{Unforg}}, \varepsilon_{\text{MDVS-}\mathsf{OTR}}, \varepsilon_{\text{MDVS-}\mathsf{PI}}, \\ t_{\text{MDVS}}, n_{S\text{MDVS}}, n_{V\text{MDVS}}, d_{S\text{MDVS}}, \\ q_{S\text{MDVS}}, q_{V\text{MDVS}}, \textit{Forge}_{\text{MDVS}})\text{-}\textit{secure,}\end{gathered} \tag{6.4}$$

then no adversary $\mathbf{A}$ (ε, t)*-breaks* Π*'s*

$$\begin{gathered}(n_S := n_{S\text{MDVS}}, n_R := \min(n_{\text{PKEBC}}, n_{V\text{MDVS}}), d_E := d_{S\text{MDVS}}, \\ q_E := q_{S\text{MDVS}}, q_D := \min(q_{D\text{PKEBC}}, q_{V\text{MDVS}}))\text{-}\textit{Consistency,}\end{gathered}$$

with $\varepsilon > \varepsilon_{\text{PKEBC-}\mathsf{Cons}} + \varepsilon_{\text{MDVS-}\mathsf{Cons}}$*, and* $t_{\text{PKEBC}}, t_{\text{MDVS}} \approx t + t_{\mathsf{Cons}}$*, where* t_{Cons} *is the time to run* Π*'s* $\mathbf{G}^{\mathsf{Cons}}$ *game.*

Theorem 8. *If* Π_{MDVS} *is*

$$\begin{gathered}(\varepsilon_{\text{MDVS-}\mathsf{Corr}}, \varepsilon_{\text{MDVS-}\mathsf{Cons}}, \varepsilon_{\text{MDVS-}\mathsf{Unforg}}, \varepsilon_{\text{MDVS-}\mathsf{OTR}}, \varepsilon_{\text{MDVS-}\mathsf{PI}}, \\ t_{\text{MDVS}}, n_{S\text{MDVS}}, n_{V\text{MDVS}}, d_{S\text{MDVS}}, \\ q_{S\text{MDVS}}, q_{V\text{MDVS}}, \textit{Forge}_{\text{MDVS}})\text{-}\textit{secure,}\end{gathered} \tag{6.5}$$

then no adversary $\mathbf{A}$ (ε, t)*-breaks* Π*'s*

$$\begin{gathered}(n_S := n_{S\text{MDVS}}, n_R := n_{V\text{MDVS}}, d_E := d_{S\text{MDVS}}, \\ q_E := q_{S\text{MDVS}}, q_D := q_{V\text{MDVS}})\text{-}\textit{Unforgeability,}\end{gathered}$$

with $\varepsilon > \varepsilon_{\text{MDVS-}\mathsf{Unforg}}$*, and* $t_{\text{MDVS}} \approx t + t_{\mathsf{Unforg}}$*, where* t_{Unforg} *is the time to run* Π*'s* $\mathbf{G}^{\mathsf{Unforg}}$ *game.*

Theorem 9. *If* Π_{PKEBC} *is*

$$\begin{gathered}(\varepsilon_{\text{PKEBC-}\mathsf{Corr}}, \varepsilon_{\text{PKEBC-}\mathsf{Rob}}, \varepsilon_{\text{PKEBC-}\mathsf{Cons}}, \varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}}, \varepsilon_{\text{PKEBC-}\mathsf{IK\text{-}CCA\text{-}2}}, \\ t_{\text{PKEBC}}, n_{\text{PKEBC}}, d_{E\text{PKEBC}}, q_{E\text{PKEBC}}, q_{D\text{PKEBC}})\text{-}\textit{secure,}\end{gathered} \tag{6.6}$$

then no adversary $\mathbf{A}$ (ε, t)*-breaks* Π*'s*

$$\begin{gathered}(n_R := n_{\text{PKEBC}}, d_E := d_{E\text{PKEBC}}, \\ q_E := q_{E\text{PKEBC}}, q_D := q_{D\text{PKEBC}})\text{-}\mathsf{IND\text{-}CCA\text{-}2}\ \textit{security,}\end{gathered}$$

with $\varepsilon > \varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}}$*, and* $t_{\text{PKEBC}} \approx t + t_{\mathsf{IND\text{-}CCA\text{-}2}}$*, where* $t_{\mathsf{IND\text{-}CCA\text{-}2}}$ *is the time to run* Π*'s* $\mathbf{G}^{\mathsf{IND\text{-}CCA\text{-}2}}$ *games.*

Remark 10. Note that Definitions 9 and 10 do not allow an adversary to query for the secret keys of any sender A_i that is given as input to a query to $\mathcal{O}_E$. Yet, the proofs of Theorems 9 and 10 actually show something stronger. Namely, that Π is secure according to even the stronger $\mathsf{IND\text{-}CCA\text{-}2}$ and $\mathsf{IK\text{-}CCA\text{-}2}$ security notions in which an adversary is allowed to query for the secret key of any sender.

Theorem 10. *If Π_{PKEBC} is*

$$\begin{aligned}(&\varepsilon_{\text{PKEBC-}\mathsf{Corr}}, \varepsilon_{\text{PKEBC-}\mathsf{Rob}}, \varepsilon_{\text{PKEBC-}\mathsf{Cons}}, \varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}}, \varepsilon_{\text{PKEBC-}\mathsf{IK\text{-}CCA\text{-}2}},\\ &t_{\text{PKEBC}}, n_{\text{PKEBC}}, d_{E\text{PKEBC}}, q_{E\text{PKEBC}}, q_{D\text{PKEBC}})\text{-secure,}\end{aligned} \tag{6.7}$$

then no adversary $\mathbf{A}$ (ε, t)-breaks Π's

$$\begin{aligned}(&n_R := n_{\text{PKEBC}}, d_E := d_{E\text{PKEBC}},\\ &q_E := q_{E\text{PKEBC}}, q_D := q_{D\text{PKEBC}})\text{-}\mathsf{IK\text{-}CCA\text{-}2}\ \textit{security,}\end{aligned}$$

with $\varepsilon > \varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}} + \varepsilon_{\text{PKEBC-}\mathsf{IK\text{-}CCA\text{-}2}}$, and $t_{\text{PKEBC}} \approx t + t_{\mathsf{IK\text{-}CCA\text{-}2}}$, where $t_{\mathsf{IK\text{-}CCA\text{-}2}}$ is the time to run Π's $\mathbf{G}^{\mathsf{IK\text{-}CCA\text{-}2}}$ games.

Theorem 11. *In the following let Forge denote Algorithm 4. If Π_{MDVS} is*

$$\begin{aligned}(&\varepsilon_{\text{MDVS-}\mathsf{Corr}}, \varepsilon_{\text{MDVS-}\mathsf{Cons}}, \varepsilon_{\text{MDVS-}\mathsf{Unforg}}, \varepsilon_{\text{MDVS-}\mathsf{OTR}}, \varepsilon_{\text{MDVS-}\mathsf{PI}},\\ &t_{\text{MDVS}}, n_{S\text{MDVS}}, n_{V\text{MDVS}}, d_{S\text{MDVS}},\\ &q_{S\text{MDVS}}, q_{V\text{MDVS}}, \mathit{Forge}_{\text{MDVS}})\text{-secure,}\end{aligned} \tag{6.8}$$

then no adversary $\mathbf{A}$ (ε, t)-breaks Π's

$$\begin{aligned}(&n_S := n_{S\text{MDVS}}, n_R := n_{V\text{MDVS}}, d_E := d_{S\text{MDVS}},\\ &q_E := q_{S\text{MDVS}}, q_D := q_{V\text{MDVS}}, \mathit{Forge})\text{-Off-The-Record security,}\end{aligned}$$

with $\varepsilon > \varepsilon_{\text{MDVS-}\mathsf{OTR}}$, and $t_{\text{MDVS}} \approx t + t_{\mathsf{OTR}}$, where t_{OTR} is the time to run Π's $\mathbf{G}^{\mathsf{OTR}}$ games.

Remark 11. It is easy to see from the proof of Theorem 11 that if Π_{MDVS} satisfies a stronger Off-The-Record notion in which the adversary is allowed to query for the secret key of any sender, then Π would also satisfy the analogous stronger Off-The-Record notion for MDRS-PKE schemes in which the adversary is allowed to query for the secret key of any sender.

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Appendix

A Game-Based Security Definitions for Public Key Encryption Schemes

A Public Key Encryption (PKE) scheme Π with message space $\mathcal{M}$ is a triple of PPTs $\Pi = (G, E, D)$. Below we state the multi-user multi-challenge variants of Correctness and $\mathsf{IND\text{-}CPA}$ and $\mathsf{IK\text{-}CPA}$ security for PKE schemes (first introduced in [18] and [7], respectively). Throughout the rest of this section, let $\Pi = (G, E, D)$ be a PKE scheme. As before, we assume the game systems of the following definitions have (an implicitly defined) security parameter k.

Definition 12, which captures the correctness of PKE schemes, provides adversaries with access to oracles $\mathcal{O}_{PK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$:

Public Key Generation Oracle: $\mathcal{O}_{PK}(B_j)$

1. On the first call on B_j, compute and store $(\mathtt{pk}_j, \mathtt{sk}_j) \leftarrow G(1^k)$; output $\mathtt{pk}_j$;
2. On subsequent calls, simply output $\mathtt{pk}_j$.

Encryption Oracle: $\mathcal{O}_E(B_j, m; r)$

1. If r is given as input, encrypt m under $\mathtt{pk}_j$ (B_j's public key, as generated by $\mathcal{O}_{PK}$) using r as random tape; if r is not given as input create a fresh encryption of m under $\mathtt{pk}_j$;
2. Output the resulting ciphertext back to the adversary.

Decryption Oracle: $\mathcal{O}_D(B_j, c)$

1. Decrypt c using $\mathtt{sk}_j$ (B_j's secret key, as generated by $\mathcal{O}_{PK}$);
2. Output the resulting plaintext back to the adversary (or $\perp$ if decryption failed).

Definition 12. *Consider a* PKE *scheme $\Pi = (G, E, D)$ with message space $\mathcal{M}$, and consider the following game played between between an adversary $\mathbf{A}$ and game system $\mathbf{G}^{\mathsf{Corr}}$:*

- $\mathbf{A}^{\mathcal{O}_{PK}, \mathcal{O}_E, \mathcal{O}_D}$

$\mathbf{A}$ *wins the game if there are two queries q_E and q_D to $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, where q_E has input (B_j, m) and q_D has input $({B_j}', c)$, the input c in q_D is the output of q_E, $B_j = {B_j}'$, and the output of q_D is m' with $m' \neq m$.*

The advantage of $\mathbf{A}$ in winning the Correctness game, denoted $Adv^{\mathsf{Corr}}(\mathbf{A})$, is the probability that $\mathbf{A}$ wins game $\mathbf{G}^{\mathsf{Corr}}$ as described above.

An adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Corr}}, t)$-breaks the (n, q_E, q_D)-Correctness of a PKE scheme Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{PK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n different parties, makes at most q_E queries to oracle $\mathcal{O}_E$ and at most q_D queries $\mathcal{O}_D(B_j, c)$ such that c was previously output by a query $\mathcal{O}_E(B_j, m)$ for the same party B_j, and satisfies $Adv^{\mathsf{Corr}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Corr}}$. If $\mathbf{A}$ is computationally unbounded, we write instead that $(\varepsilon_{\mathsf{Corr}})$-breaks the (n)-Correctness of Π if $\mathbf{A}$ queries $\mathcal{O}_{PK}$, $\mathcal{O}_E$ and $\mathcal{O}_D$ on at most n different parties and satisfies $Adv^{\mathsf{Corr}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Corr}}$.

The $\mathsf{IND\text{-}CPA}$ game systems provide adversaries with access to oracle $\mathcal{O}_{PK}$ described above, and to an additional oracle $\mathcal{O}_E$ which behaves as follows:

Encryption Oracle: $\mathcal{O}_E(B_j, m_0, m_1)$

1. For game system $\mathbf{G}_{\mathbf{b}}^{\textsf{IND-CPA}}$, the oracle encrypts $m_{\mathbf{b}}$ under B_j's public key, $\texttt{pk}_j$, creating a fresh ciphertext c;
2. The oracle outputs the resulting ciphertext c back to the adversary.

Definition 13. *For* $\mathbf{b} \in \{0,1\}$*, consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}_{\mathbf{b}}^{\textsf{IND-CPA}}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PK}, \mathcal{O}_E}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query* $\mathcal{O}_E(B_j, m_0, m_1)$*,* $|m_0| = |m_1|$*.*

We define the advantage of $\mathbf{A}$ *in winning the* IND-CPA *security game as*

$$Adv^{\textsf{IND-CPA}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_{\mathbf{0}}^{\textsf{IND-CPA}} = \texttt{win}] + \Pr[\mathbf{A}\mathbf{G}_{\mathbf{1}}^{\textsf{IND-CPA}} = \texttt{win}] - 1 \right|.$$

Similarly to the IND-CPA game systems, the IK-CPA game systems provide adversaries with access to oracle $\mathcal{O}_{PK}$ and to an oracle $\mathcal{O}_E$ which behaves as follows:

Encryption Oracle: $\mathcal{O}_E(B_{j,0}, B_{j,1}, m)$

1. For game system $\mathbf{G}_{\mathbf{b}}^{\textsf{IK-CPA}}$, encrypt m under $B_{j,\mathbf{b}}$'s public key, $\texttt{pk}_{j,\mathbf{b}}$, creating a fresh ciphertext c;
2. Output the resulting ciphertext c back to the adversary.

Definition 14. *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}_{\mathbf{b}}^{\textsf{IK-CPA}}$*, with* $\mathbf{b} \in \{0,1\}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PK}, \mathcal{O}_E}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$*.*

We define the advantage of $\mathbf{A}$ *in winning the* IK-CPA *security game as*

$$Adv^{\textsf{IK-CPA}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_{\mathbf{0}}^{\textsf{IK-CPA}} = \texttt{win}] + \Pr[\mathbf{A}\mathbf{G}_{\mathbf{1}}^{\textsf{IK-CPA}} = \texttt{win}] - 1 \right|.$$

We say $\mathbf{A}$ $(\varepsilon_{\textsf{IND-CPA}}, t)$-breaks (resp. $(\varepsilon_{\textsf{IK-CPA}}, t)$-breaks) the (n, q_E)-IND-CPA (resp. (n, q_E)-IK-CPA) security of a PKE scheme Π if $\mathbf{A}$ runs in time at most t, queries the oracles it has access to on at most n different parties, makes at most q_E queries to oracle $\mathcal{O}_E$, and satisfies $Adv^{\textsf{IND-CPA}}(\mathbf{A}) \geq \varepsilon_{\textsf{IND-CPA}}$ (resp. $Adv^{\textsf{IK-CPA}}(\mathbf{A}) \geq \varepsilon_{\textsf{IK-CPA}}$).

Finally, we say that Π is $(\varepsilon_{\textsf{Corr}}, \varepsilon_{\textsf{IND-CPA}}, \varepsilon_{\textsf{IK-CPA}}, t, n, q_E, q_D)$-secure if no adversary $\mathbf{A}$ $(\varepsilon_{\textsf{Corr}}, t)$-breaks the (n, q_E, q_D)-Correctness of Π, $(\varepsilon_{\textsf{IND-CPA}}, t)$-breaks the (n, q_E)-IND-CPA security of Π, or $(\varepsilon_{\textsf{IK-CPA}}, t)$-breaks the (n, q_E)-IK-CPA security of Π. Similarly, we say that Π is $(\varepsilon_{\textsf{Corr}}, \varepsilon_{\textsf{IND-CPA}}, \varepsilon_{\textsf{IK-CPA}}, t, n, q_E, q_D, \textsf{Corr})$-secure if no adversary $\mathbf{A}$ $(\varepsilon_{\textsf{IND-CPA}}, t)$-breaks the (n, q_E)-IND-CPA security of Π nor $(\varepsilon_{\textsf{IK-CPA}}, t)$-breaks the (n, q_E)-IK-CPA security of Π, and no (possibly computationally unbounded) adversary $(\varepsilon_{\textsf{Corr}})$-breaks the (n)-Correctness of Π.

B Game-Based Security Definitions for Binding Commitment Schemes

A Commitment Scheme (CS) for a message space $\mathcal{M}$ is a protocol consisting of a pair of PPT algorithms $\Pi = (G_{CRS}, \mathit{Commit})$. We now move to introduce game-based notions capturing the security of CS protocols. We assume the game systems ahead have (an implicitly defined) security parameter k.

The game systems of Definitions 15 and 16 provide adversaries with access to an oracle $\mathcal{O}_S$, defined as:

CRS Generation Oracle: $\mathcal{O}_S$

1. On the first call, compute and store $\mathtt{crs} \leftarrow G_{CRS}(1^k)$; output $\mathtt{crs}$;
2. On subsequent calls, output the previously generated $\mathtt{crs}$.

Definition 15 captures the hiding property of Commitment Schemes. We give a game-based notion capturing this property which resembles the IND-CPA notion for PKE schemes. $\mathbf{G}_\mathbf{b}^{\mathsf{Hiding}}$ provides adversaries with access to oracle $\mathcal{O}_S$ defined above, and to an oracle $\mathcal{O}_{Commit}$ whose behavior is defined below:

Encryption Oracle: $\mathcal{O}_{Commit}(m_0, m_1)$

1. Pick randomness ρ uniformly at random;
2. For game system $\mathbf{G}_\mathbf{b}^{\mathsf{Hiding}}$, compute $\mathtt{comm} \leftarrow \mathit{Commit}_{\mathtt{crs}}(m_\mathbf{b}; \rho)$; output $\mathtt{comm}$.

Definition 15. *Consider the following game played between an adversary* $\mathbf{A}$ *and a game system* $\mathbf{G}_\mathbf{b}^{\mathsf{Hiding}}$*, with* $\mathbf{b} \in \{0,1\}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_S, \mathcal{O}_{Commit}}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$.

We define the advantage of $\mathbf{A}$ *in winning the* Hiding *game as*

$$Adv^{\mathsf{Hiding}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_\mathbf{0}^{\mathsf{Hiding}} = \mathtt{win}] + \Pr[\mathbf{A}\mathbf{G}_\mathbf{1}^{\mathsf{Hiding}} = \mathtt{win}] - 1 \right|.$$

An adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Hiding}}, t)$-breaks the (q)-Hiding property of a CS Π if it runs in time t, makes at most q queries to $\mathcal{O}_{Commit}$, and satisfies $Adv^{\mathsf{Hiding}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Hiding}}$.

Definition 16, which captures the binding property of Commitment Schemes, provides adversaries with access to an oracle $\mathcal{O}_{Commit}$ defined as follows:

Commit Oracle: $\mathcal{O}_{Commit}(m, \rho)$

1. Compute $\mathtt{comm} = \mathit{Commit}_{\mathtt{crs}}(m; \rho)$;[9] output $\mathtt{comm}$.

Definition 16. *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Binding}}$*:*

- $\mathbf{A}^{\mathcal{O}_S, \mathcal{O}_{Commit}}$

[9] Here, ρ denotes the random coins used by *Commit*, meaning that $\mathit{Commit}_{(\cdot)}(\cdot; \rho)$ is a deterministic algorithm.

$\mathbf{A}$ *wins the game if there are two queries* q *and* q' *to* $\mathcal{O}_{Commit}$ *where* q *has input* (m, ρ) *and outputs* `comm` *and* q' *has input* (m', ρ') *and outputs* `comm'`*, satisfying* $m \neq m'$ *and* $\texttt{comm} = \texttt{comm}'$.

The advantage of $\mathbf{A}$ *in winning the Binding game is denoted* $Adv^{\mathsf{Binding}}(\mathbf{A})$ *and corresponds to the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Binding}}$ *as described above.*

An adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Binding}}, t)$-breaks the Binding property of a CS $\mathit{\Pi}$ if $\mathbf{A}$ runs in time at most t and satisfies $Adv^{\mathsf{Binding}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Binding}}$. If $\mathbf{A}$ is computationally unbounded, we instead write that $\mathbf{A}$ $(\varepsilon_{\mathsf{Binding}})$-breaks the Binding property of $\mathit{\Pi}$ if $Adv^{\mathsf{Binding}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Binding}}$.

We say a CS $\mathit{\Pi}$ is $(\varepsilon_{\mathsf{Hiding}}, \varepsilon_{\mathsf{Binding}}, t, q)$-secure if no adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Hiding}}, t)$-breaks the (q)-Hiding property of $\mathit{\Pi}$, or $(\varepsilon_{\mathsf{Binding}}, t)$-breaks the Binding property of $\mathit{\Pi}$. For a Statistically Binding Commitment Scheme (i.e. one for which the Binding property holds against computationally unbounded adversaries), we say instead that $\mathit{\Pi}$ is $(\varepsilon_{\mathsf{Hiding}}, \varepsilon_{\mathsf{Binding}}, t, q, \mathsf{Binding})$-secure if no adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Hiding}}, t)$-breaks the (q)-Hiding property of $\mathit{\Pi}$, and no (possibly computationally unbounded) adversary $(\varepsilon_{\mathsf{Binding}})$-breaks the Binding property of $\mathit{\Pi}$.

As is well known, any perfectly correct and IND-CPA secure PKE scheme yields a non-interactive Commitment Scheme scheme, where the `crs` is the public key of the PKE scheme (which is honestly generated by a trusted party), and commitments are encryptions of messages under the `crs` (i.e. under the public key). It is easy to see that the ElGamal PKE scheme [14], which is perfectly Correct and tightly Multi-Party Multi-Challenge IND-CPA secure under DDH (see [8]) yields a tightly Multi-Challenge hiding commitment scheme.

C Game-Based Security Definitions for Non Interactive Zero Knowledge Schemes

For a binary relation R, let L_R be the language $L_R := \{x \mid \exists w, (x, w) \in R\}$ induced by R. A *Non Interactive Proof System* (NIPS) for L_R is a triple of PPT algorithms $\mathit{\Pi} = (G_{CRS}, \mathit{Prove}, \mathit{Verify})$ where:

- $G_{CRS}(1^k)$: given security parameter 1^k, outputs a common reference string `crs`;
- $\mathit{Prove}_{\texttt{crs}}(x, w)$: given a common reference string `crs` and a statement-witness pair $(x, w) \in R$, outputs a proof p;
- $\mathit{Verify}_{\texttt{crs}}(x, p)$: given a common reference string `crs`, a statement x and a proof p, either accepts, outputting `valid` $(= 1)$ or rejects, outputting `invalid` $(= 0)$.

In the following definitions, let $\mathit{\Pi} = (G_{CRS}, \mathit{Prove}, \mathit{Verify})$ be a NIPS for a relation R, and let k be the security parameter. The security notions below (Definitions 17 and 18) provide adversaries with access to oracles $\mathcal{O}_S$ and $\mathcal{O}_V$, defined as:

CRS Generation Oracle: $\mathcal{O}_S$

1. On the first call, compute and store $\texttt{crs} \leftarrow G_{CRS}(1^k)$; output $\texttt{crs}$;
2. On subsequent calls, output the previously generated $\texttt{crs}$.

Verify Oracle: $\mathcal{O}_V(x,p)$

1. Compute $b = \mathit{Verify}_{\texttt{crs}}(x,p)$; output b.

Definition 17 additionally provides adversaries with access to an oracle $\mathcal{O}_P$:

Prove Oracle: $\mathcal{O}_P(x,w)$

1. Compute $p = \mathit{Prove}_{\texttt{crs}}(x,w)$; output p.

Definition 17. *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Complete}}$*:*

- $\mathbf{A}^{\mathcal{O}_S,\mathcal{O}_P,\mathcal{O}_V}$

$\mathbf{A}$ *wins the game if there are two queries* q_P *and* q_V *to* $\mathcal{O}_P$ *and* $\mathcal{O}_V$*, respectively, where* q_P *has input* (x,w) *and* q_V *has input* (x',p)*, satisfying* $x = x'$*, the input* p *in* q_V *is the output of* q_P*, the output of* q_V *is* $\texttt{invalid}$*, and* $(x,w) \in R$*.*

The advantage of $\mathbf{A}$ *in winning the Completeness game, denoted* $\mathit{Adv}^{\mathsf{Complete}}(\mathbf{A})$*, corresponds to the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Complete}}$ *as described above.*

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Complete}}, t)$-breaks the (q_P, q_V)-Completeness of a NIPS scheme $\mathit{\Pi}$ if $\mathbf{A}$ runs in time at most t, makes at most q_P and q_V queries to oracles $\mathcal{O}_P$ and $\mathcal{O}_V$, respectively, and satisfies $\mathit{Adv}^{\mathsf{Complete}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Complete}}$.

Definition 18. *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{Sound}}$*:*

- $\mathbf{A}^{\mathcal{O}_S,\mathcal{O}_V}$

$\mathbf{A}$ *wins the game if there is a query to* $\mathcal{O}_V$ *on input* (x,p)*, satisfying* $x \notin L_R$*, such that the oracle outputs* $\texttt{valid}$*.*

The advantage of $\mathbf{A}$ *in winning the Soundness game corresponds to the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Sound}}$ *as described above and is denoted* $\mathit{Adv}^{\mathsf{Sound}}(\mathbf{A})$*.*

An adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Sound}}, t)$-breaks the (q_V)-Soundness of a NIPS scheme $\mathit{\Pi}$ if $\mathbf{A}$ runs in time at most t, makes at most q_V queries to $\mathcal{O}_V$ and satisfies $\mathit{Adv}^{\mathsf{Sound}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Sound}}$.

A NIZK scheme $\mathit{\Pi} = (G_{CRS}, \mathit{Prove}, \mathit{Verify}, S = (S_{CRS}, S_{Sim}))$ for a relation R consists of a NIPS scheme $\mathit{\Pi}' = (G_{CRS}, \mathit{Prove}, \mathit{Verify})$ for R and a simulator $S = (S_{CRS}, S_{Sim})$, where:

- $S_{CRS}(1^k)$: given security parameter 1^k, outputs a pair $(\texttt{crs}, \tau)$;
- $S_{Sim(\texttt{crs},\tau)}(x)$: given a pair $(\texttt{crs}, \tau)$ and a statement x, outputs a proof p.

Consider a NIZK scheme $\mathit{\Pi} = (G_{CRS}, \mathit{Prove}, \mathit{Verify}, S = (S_{CRS}, S_{Sim}))$. The following security notion, which defines game systems $\mathbf{G}_{\mathbf{0}}^{\mathsf{ZK}}$ and $\mathbf{G}_{\mathbf{1}}^{\mathsf{ZK}}$, provides adversaries with access to two oracles, $\mathcal{O}_S$ and $\mathcal{O}_P$, whose behavior depends on the underlying game system. For $\mathbf{G}_{\mathbf{b}}^{\mathsf{ZK}}$ (with $\mathbf{b} \in \{0,1\}$):

CRS Generation Oracle: $\mathcal{O}_S$

1. On the first call, compute and store $\texttt{crs} \leftarrow G_{CRS}(1^k)$ if $\mathbf{b} = \mathbf{0}$, and $(\texttt{crs}, \tau) \leftarrow S_{CRS}(1^k)$ if $\mathbf{b} = \mathbf{1}$; output $\texttt{crs}$;
2. On subsequent calls, output the previously generated $\texttt{crs}$.

Prove Oracle: $\mathcal{O}_P(x, w)$

- If $\mathbf{b} = \mathbf{0}$, output $\pi = \mathit{Prove}_{\texttt{crs}}(x, w)$;
- If $\mathbf{b} = \mathbf{1}$, output $\pi \leftarrow S_{Sim(\texttt{crs},\tau)}(x)$.

Definition 19. *For* $\mathbf{b} \in \{0, 1\}$*, consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{ZK}}_{\mathbf{b}}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_S,\mathcal{O}_P}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query to* $\mathcal{O}_P$*, the input* (x, w) *given to* $\mathcal{O}_P$ *satisfies* $(x, w) \in R$*.*

The advantage of $\mathbf{A}$ *in winning the Zero-Knowledge security game for* $\mathit{\Pi}$ *is*

$$\mathit{Adv}^{\mathsf{ZK}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}^{\mathsf{ZK}}_{\mathbf{0}} = \texttt{win}] + \Pr[\mathbf{A}\mathbf{G}^{\mathsf{ZK}}_{\mathbf{1}} = \texttt{win}] - 1 \right|.$$

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{ZK}}, t)$-breaks the (q_P)-ZK security of a NIZK scheme $\mathit{\Pi}$ if it makes at most q_P queries to $\mathcal{O}_P$ and satisfies $\mathit{Adv}^{\mathsf{ZK}}(\mathbf{A}) \geq \varepsilon_{\mathsf{ZK}}$.

We now introduce Simulation Soundness for NIZK [31]. The game system defined by this notion provides adversaries with access to oracles $\mathcal{O}_S$, $\mathcal{O}_P$ and $\mathcal{O}_V$ defined as:

CRS Generation Oracle: $\mathcal{O}_S$

1. On the first call, compute and store $(\texttt{crs}, \tau) \leftarrow S_{CRS}(1^k)$; output $\texttt{crs}$;
2. On subsequent calls, output the previously generated $\texttt{crs}$.

Prove Oracle: $\mathcal{O}_P(x)$

1. Compute $p = S_{Sim(\texttt{crs},\tau)}(x)$; output p.

Verify Oracle: $\mathcal{O}_V(x, p)$

1. Compute $b = \mathit{Verify}_{\texttt{crs}}(x, p)$; output b.

Definition 20. *Consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}^{\mathsf{SS}}$*:*

- $\mathbf{A}^{\mathcal{O}_S,\mathcal{O}_P,\mathcal{O}_V}$

$\mathbf{A}$ *wins the game if it makes a query to* $\mathcal{O}_V$ *on input* (x, p) *such that* p *was not output by any query to* $\mathcal{O}_P$*,* $x \notin L_R$ *and* $\mathcal{O}_V$ *outputs* $\texttt{valid}$*.*

The advantage of $\mathbf{A}$ *in winning the Simulation Soundness game, denoted* $\mathit{Adv}^{\mathsf{SS}}(\mathbf{A})$*, is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{SS}}$ *as described above.*

An adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{SS}}, t)$-breaks the (q_P, q_V)-Simulation Soundness of a NIZK scheme $\mathit{\Pi}$ if it makes at most q_P and q_V queries to $\mathcal{O}_P$ and $\mathcal{O}_V$, respectively, and satisfies $\mathit{Adv}^{\mathsf{SS}}(\mathbf{A}) \geq \varepsilon_{\mathsf{SS}}$.

Finally, we say that a NIZK scheme $\mathit{\Pi}$ is $(\varepsilon_{\mathsf{Complete}}, \varepsilon_{\mathsf{Sound}}, \varepsilon_{\mathsf{ZK}}, \varepsilon_{\mathsf{SS}}, t, q_P, q_V)$-secure if no adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Complete}}, t)$-breaks the (q_P, q_V)-Completeness of $\mathit{\Pi}$, $(\varepsilon_{\mathsf{Sound}}, t)$-breaks the (q_V)-Soundness of $\mathit{\Pi}$, $(\varepsilon_{\mathsf{ZK}}, t)$-breaks the (q_P)-Zero-Knowledge of $\mathit{\Pi}$, or $(\varepsilon_{\mathsf{SS}}, t)$-breaks the (q_P, q_V)-Simulation Soundness of $\mathit{\Pi}$.

D Game-Based Security Definitions for Multi-Designated Verifier Signature Schemes

A Multi-Designated Verifier Signature scheme (MDVS) $\mathit{\Pi}$ is a 5-tuple $\mathit{\Pi} = (\mathit{Setup}, G_S, G_V, \mathit{Sign}, \mathit{Vfy})$, following the definition of [24]. The security games for MDVS schemes have an implicitly defined security parameter k, and provide adversaries with access to some of the following oracles:

Public Parameter Generation Oracle: $\mathcal{O}_{PP}$

1. On the first call to $\mathcal{O}_{PP}$, compute $\mathtt{pp} \leftarrow \mathit{Setup}(1^k)$; output $\mathtt{pp}$;
2. On subsequent calls, simply output $\mathtt{pp}$.

Signer Key-Pair Generation Oracle: $\mathcal{O}_{SK}(A_i)$

1. On the first call to $\mathcal{O}_{SK}$ on input A_i, compute $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow G_S(\mathtt{pp})$, and output $(\mathtt{spk}_i, \mathtt{ssk}_i)$;
2. On subsequent calls, simply output $(\mathtt{spk}_i, \mathtt{ssk}_i)$.

Verifier Key-Pair Generation Oracle: $\mathcal{O}_{VK}(B_j)$

1. Analogous to the Signer Key-Pair Generation Oracle.

Signer Public-Key Oracle: $\mathcal{O}_{SPK}(A_i)$

1. $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow \mathcal{O}_{SK}(A_i)$; output $\mathtt{spk}_i$.

Verifier Public-Key Oracle: $\mathcal{O}_{VPK}(B_j)$

1. Analogous to the Signer Public-Key Oracle.

Signing Oracle: $\mathcal{O}_S(A_i, \mathcal{V}, m)$

1. $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow \mathcal{O}_{SK}(A_i)$;
2. For all $B_j \in \mathcal{V}$: $\mathtt{vpk}_j \leftarrow \mathcal{O}_{VPK}(B_j)$;
3. Output $\sigma \leftarrow \mathit{Sign}_{\mathtt{pp}}(\mathtt{ssk}_i, \{\mathtt{vpk}_j\}_{B_j \in \mathcal{V}}, m)$.

Verification Oracle: $\mathcal{O}_V(A_i, B_j \in \mathcal{V}, \mathcal{V}, m, \sigma)$

1. $\mathtt{spk}_i \leftarrow \mathcal{O}_{SPK}(A_i)$;
2. For all $B_l \in \mathcal{V}$: $\mathtt{vpk}_l \leftarrow \mathcal{O}_{VPK}(B_l)$;
3. $(\mathtt{vpk}_j, \mathtt{vsk}_j) \leftarrow \mathcal{O}_{VK}(B_j)$;
4. Output $d \leftarrow \mathit{Vfy}_{\mathtt{pp}}(\mathtt{spk}_i, \mathtt{vsk}_j, \{\mathtt{vpk}_l\}_{B_l \in \mathcal{V}}, m, \sigma)$.

We now introduce the relevant game-based notions for MDVS schemes. Let $\mathit{\Pi} = (\mathit{Setup}, G_S, G_V, \mathit{Sign}, \mathit{Vfy})$ be an MDVS scheme.

Definition 21. *Consider the following game played between an adversary* $\mathbf{A}$ *and the game system* $\mathbf{G}^{\mathsf{Corr}}$*:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SK}, \mathcal{O}_{VK}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S, \mathcal{O}_V}$

$\mathbf{A}$ *wins the game if there are two queries* q_S *and* q_V *to* $\mathcal{O}_S$ *and* $\mathcal{O}_V$*, respectively, where* q_S *has input* $(A_i, \mathcal{V}, m)$ *and* q_V *has input* $(A_i', B_j, \mathcal{V}', m', \sigma)$*, satisfying* $A_i = A_i'$*,* $\mathcal{V} = \mathcal{V}'$*,* $B_j \in \mathcal{V}$*, the input* σ *in* q_V *is the output of* q_S*, the output of* q_V *is* 0*, and* $\mathbf{A}$ *did not query* $\mathcal{O}_{SK}$ *on (input)* A_i *nor* $\mathcal{O}_{VK}$ *on* B_j*.*

The advantage of $\mathbf{A}$ *in winning the Correctness game, denoted* $\mathit{Adv}^{\mathsf{Corr}}(\mathbf{A})$*, is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Corr}}$ *as described above.*

We say an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{Corr}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Correctness of Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{SPK}$, $\mathcal{O}_{SK}$, $\mathcal{O}_S$ and $\mathcal{O}_V$ on at most n_S different signers, queries $\mathcal{O}_{VPK}$, $\mathcal{O}_{VK}$, $\mathcal{O}_S$ and $\mathcal{O}_V$ on at most n_V different verifiers, makes at most q_S and q_V queries to $\mathcal{O}_S$ and $\mathcal{O}_V$, respectively, with sum of the sizes of the sets of verifiers input to $\mathcal{O}_S$ being at most d_S and satisfies $Adv^{\mathsf{Corr}}(\mathbf{A}) \geq \varepsilon_{\mathsf{Corr}}$.

The following security notions, Definitions 22 and 23, correspond to multi-challenge variants of existing security notions from the literature [13, 35]. The multi-challenge versions of both of these security notions are asymptotically equivalent to the single challenge counterparts (meaning that if a scheme asymptotically satisfies the single-challenge version of either of these notions then it also asymptotically satisfies the corresponding multi-challenge version). To allow for the multiple challenges, we will introduce an additional oracle $\mathcal{O}_{Challenge}$ that adversaries use to submit the (possibly multiple) challenges to the games. Inputs to oracle $\mathcal{O}_{Challenge}$ are quadruples of the form $(m^*, {A_i}^*, \mathcal{V}^*, \sigma^*)$; the oracle does not output any value. The exact behavior of the oracle, and in particular the definition of when the adversary wins the underlying game, depends on the security notion. On any query to this oracle, and regardless of whether it is a game-winning one, the oracle does not give any output.

The following security notion is the multi-challenge version of the Consistency security notion for MDVS, introduced by Damgård et al. in [13].

Definition 22. *Consider the following game played between an adversary $\mathbf{A}$ and the game system $\mathbf{G}^{\mathsf{Cons}}$:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SK}, \mathcal{O}_{VK}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S, \mathcal{O}_V, \mathcal{O}_{Challenge}}$

where oracles $\mathcal{O}_{PP}$, $\mathcal{O}_{SK}$, $\mathcal{O}_{VK}$, $\mathcal{O}_{SPK}$, $\mathcal{O}_{VPK}$, $\mathcal{O}_S$, and $\mathcal{O}_V$ are as defined above, and oracle $\mathcal{O}_{Challenge}$ receives as input a pair (m^*, σ^*) and does not give any output. We say that $\mathbf{A}$ wins the game if it queries $\mathcal{O}_{Challenge}$ on a pair (m^*, σ^*) such that there are two later queries to $\mathcal{O}_V$ with inputs $(A_i, B_j, \mathcal{V}, m^*, \sigma^*)$ and $(A_i, {B_j}', \mathcal{V}, m^*, \sigma^*)$, and outputs b, b' satisfying $B_j, {B_j}' \in \mathcal{V}$ and $b \neq b'$, and $\mathcal{O}_{VK}$ was not queried on B_j nor on ${B_j}'$.

The advantage of $\mathbf{A}$ in winning the Consistency game, denoted $Adv^{\mathsf{Cons}}(\mathbf{A})$, is the probability that $\mathbf{A}$ wins game $\mathbf{G}^{\mathsf{Cons}}$ as described above.

Definition 23. *Consider the following game played between adversary $\mathbf{A}$ and game system $\mathbf{G}^{\mathsf{Unforg}}$:*

- $\mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SK}, \mathcal{O}_{VK}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S, \mathcal{O}_V, \mathcal{O}_{Challenge}}$

where oracles $\mathcal{O}_{PP}$, $\mathcal{O}_{SK}$, $\mathcal{O}_{VK}$, $\mathcal{O}_{SPK}$, $\mathcal{O}_{VPK}$, $\mathcal{O}_S$, and $\mathcal{O}_V$ are as defined above, and oracle $\mathcal{O}_{Challenge}$ receives as input a pair (m^*, σ^*) and does not give any output. We say that $\mathbf{A}$ wins the game if there is a query q to $\mathcal{O}_{Challenge}$ on an input (m^*, σ^*) and there is a later query to $\mathcal{O}_V$ with input $(A_i, B_j, \mathcal{V}, m^*, \sigma^*)$ that outputs 1 with $B_j \in \mathcal{V}$ such that all of the following conditions hold:

1. *$\mathcal{O}_{SK}$ was not queried on A_i;*

2. $\mathcal{O}_{VK}$ was not queried on B_j;
3. for every query $({A_i}', \mathcal{V}', m')$ to $\mathcal{O}_S$ before q, $(A_i, \mathcal{V}, m^*) \neq ({A_i}', \mathcal{V}', m')$;

The advantage of $\mathbf{A}$ *in winning the Unforgeability game, denoted* $Adv^{\mathsf{Unforg}}(\mathbf{A})$, *is the probability that* $\mathbf{A}$ *wins game* $\mathbf{G}^{\mathsf{Unforg}}$ *as described above.*

The following security notion was first introduced in [28], and is the multi-challenge variant of the *Off-The-Record* (game-based) security notion introduced in [13]. This notion defines two game systems, $\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}Forge}$ and $\mathbf{G}_{\mathbf{1}}^{\mathsf{OTR}\text{-}Forge}$, which are parameterized by an algorithm *Forge*. The game system also defines an oracle $\mathcal{O}_{ChallengeSign}$ whose behavior varies depending on the underlying game system, i.e. depending on $\mathbf{b} \in \{0, 1\}$ the oracle $\mathcal{O}_{ChallengeSign}$ provided by $\mathbf{G}_{\mathbf{b}}^{\mathsf{OTR}\text{-}Forge}$ behaves differently, as described below.

ChallengeSign Oracle: $\mathcal{O}_{ChallengeSign}(\mathtt{type} \in \{\mathtt{sign}, \mathtt{forge}\}, A_i, \mathcal{V}, m, \mathcal{D})$
For game system $\mathbf{G}_{\mathbf{b}}^{\mathsf{OTR}\text{-}Forge}$, the oracle behaves as follows:

1. $\mathtt{pp} \leftarrow \mathcal{O}_{PP}$;
2. $(\mathtt{spk}_i, \mathtt{ssk}_i) \leftarrow \mathcal{O}_{SK}(A_i)$;
3. For each $B_j \in \mathcal{V}$: $\mathtt{vpk}_j \leftarrow \mathcal{O}_{VPK}(B_j)$;
4. For each $B_j \in \mathcal{D}$: $(\mathtt{vpk}_j, \mathtt{vsk}_j) \leftarrow \mathcal{O}_{VK}(B_j)$;
5. $\sigma_{\mathbf{0}} \leftarrow Sign_{\mathtt{pp}}(\mathtt{ssk}_i, \{\mathtt{vpk}_j\}_{B_j \in \mathcal{V}}, m)$;
6. $\sigma_{\mathbf{1}} \leftarrow Forge_{\mathtt{pp}}(\mathtt{spk}_i, \{\mathtt{vpk}_j\}_{B_j \in \mathcal{V}}, m, \{\mathtt{vsk}_j\}_{B_j \in \mathcal{D}})$;
7. If $\mathbf{b} = 0$, output $\sigma_{\mathbf{0}}$ if $\mathtt{type} = \mathtt{sign}$ and $\sigma_{\mathbf{1}}$ if $\mathtt{type} = \mathtt{forge}$;
8. Otherwise, if $\mathbf{b} = 1$, output $\sigma_{\mathbf{1}}$.

Definition 24. *Consider an* MDVS *scheme* $\Pi = (Setup, G_S, G_V, Sign, Vfy)$. *In the following, let Forge be a PPT algorithm that on input* $\mathtt{pp}$, $\mathtt{spk}_{i^*}$, $\{\mathtt{vpk}_l\}_{B_l \in \mathcal{V}^*}$, m^* *and* $\{\mathtt{vsk}_j\}_{B_j \in \mathcal{D}^*}$, *outputs a forged signature* σ'. *For* $\mathbf{b} \in \{0, 1\}$, *consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}_{\mathbf{b}}^{\mathsf{OTR}\text{-}Forge}$*:*

- $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SK}, \mathcal{O}_{VK}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_V, \mathcal{O}_{ChallengeSign}}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query* $(\mathtt{type}, A_i, \mathcal{V}, m, \mathcal{D})$ *to* $\mathcal{O}_{ChallengeSign}$ *all of the following hold:*

1. $\mathcal{D} \subseteq \mathcal{V}$;
2. for every query B_j to $\mathcal{O}_{VK}$, $B_j \notin \mathcal{V} \setminus \mathcal{D}$;
3. for every query A_l to $\mathcal{O}_{SK}$, $A_l \neq A_i$; and
4. letting σ denote the output of $\mathcal{O}_{ChallengeSign}$ to the query above, for all queries $(A_l, B_j, \mathcal{V}', m', \sigma')$ to oracle $\mathcal{O}_V$, $\sigma' \neq \sigma$.

The advantage of $\mathbf{A}$ *in winning the Off-The-Record security game with respect to Forge is*

$$Adv^{\mathsf{OTR}\text{-}Forge}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}Forge} = \mathtt{win}] + \Pr[\mathbf{A}\mathbf{G}_{\mathbf{1}}^{\mathsf{OTR}\text{-}Forge} = \mathtt{win}] - 1 \right|.$$

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{OTR}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Off-The-Record security of Π with respect to algorithm *Forge* if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{SPK}$, $\mathcal{O}_{SK}$, $\mathcal{O}_S$ and $\mathcal{O}_V$ on at most n_S different signers, queries $\mathcal{O}_{VPK}$, $\mathcal{O}_{VK}$, $\mathcal{O}_S$ and $\mathcal{O}_V$ on at most n_V different verifiers, makes at most q_S and q_V queries to $\mathcal{O}_S$ and $\mathcal{O}_V$, respectively, with sum of the sizes of the sets of verifiers input to $\mathcal{O}_S$ being at most d_S and satisfies $Adv^{\mathsf{OTR}}(\mathbf{A}) \geq \varepsilon_{\mathsf{OTR}}$.

The following security notion is the multi-challenge variant of the *Privacy of Identities* (game-based) security notion introduced in [13]. Similarly to the Off-The-Record security notion, this new notion defines two game systems, $\mathbf{G}_{\mathbf{0}}^{\mathrm{PI}}$ and $\mathbf{G}_{\mathbf{1}}^{\mathrm{PI}}$. The game system also defines an oracle $\mathcal{O}_{ChallengeSign}$ whose behavior varies depending on the underlying game system, i.e. depending on $\mathbf{b} \in \{0,1\}$ the oracle $\mathcal{O}_{ChallengeSign}$ provided by $\mathbf{G}_{\mathbf{b}}^{\mathrm{PI}}$ behaves differently, as described below.

ChallengeSign Oracle: $\mathcal{O}_{ChallengeSign}((A_0, \mathcal{V}_0), (A_1, \mathcal{V}_1), m)$

For game system $\mathbf{G}_{\mathbf{b}}^{\mathrm{PI}}$, the oracle behaves as follows:

1. $\mathtt{pp} \leftarrow \mathcal{O}_{PP}$;
2. $(\mathtt{spk}_{\mathbf{b}}, \mathtt{ssk}_{\mathbf{b}}) \leftarrow \mathcal{O}_{SK}(A_{\mathbf{b}})$;
3. For each $B_j \in \mathcal{V}_{\mathbf{b}}$: $\mathtt{vpk}_j \leftarrow \mathcal{O}_{VPK}(B_j)$;
4. $\sigma \leftarrow Sign_{\mathsf{pp}}(\mathtt{ssk}_{\mathbf{b}}, \{\mathtt{vpk}_j\}_{B_j \in \mathcal{V}_{\mathbf{b}}}, m)$; output σ.

Definition 25 (Privacy of Identities). *Consider an* MDVS *scheme* $\Pi = (Setup, G_S, G_V, Sign, Vfy)$. *For* $\mathbf{b} \in \{0,1\}$*, consider the following game played between an adversary* $\mathbf{A}$ *and game system* $\mathbf{G}_{\mathbf{b}}^{\mathsf{PI}}$*:*

– $b' \leftarrow \mathbf{A}^{\mathcal{O}_{PP}, \mathcal{O}_{SK}, \mathcal{O}_{VK}, \mathcal{O}_{SPK}, \mathcal{O}_{VPK}, \mathcal{O}_S, \mathcal{O}_V, \mathcal{O}_{ChallengeSign}}$

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$ *and for every query* $\mathcal{O}_{ChallengeSign}((A_0, \mathcal{V}_0), (A_1, \mathcal{V}_1), m)$ *all of the following hold:*

1. $|\mathcal{V}_{\mathbf{0}}| = |\mathcal{V}_{\mathbf{1}}|$;
2. *for all queries* A_i *to* $\mathcal{O}_{SK}$, $A_i \notin \{A_{\mathbf{0}}, A_{\mathbf{1}}\}$;
3. *for all queries* B_j *to* $\mathcal{O}_{VK}$, $B_j \notin \mathcal{V}_{\mathbf{0}} \cup \mathcal{V}_{\mathbf{1}}$;
4. *for all queries* $(A_i, B_j, \mathcal{V}, m, \sigma)$ *to oracle* $\mathcal{O}_V$, $A_i \notin \{A_{\mathbf{0}}, A_{\mathbf{1}}\}$ *or* $\mathcal{V} \nsubseteq \mathcal{V}_{\mathbf{0}} \cup \mathcal{V}_{\mathbf{1}}$.

We define the advantage of $\mathbf{A}$ *as*

$$Adv^{\mathsf{PI}}(\mathbf{A}) := \left| \Pr[\mathbf{A}\mathbf{G}_{\mathbf{0}}^{\mathsf{PI}} = \mathtt{win}] + \Pr[\mathbf{A}\mathbf{G}_{\mathbf{1}}^{\mathsf{PI}} = \mathtt{win}] - 1 \right|.$$

An adversary $\mathbf{A}$ (ε, t)-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Consistency, Unforgeability, or Identity-Privacy of Π if $\mathbf{A}$ runs in time at most t, queries $\mathcal{O}_{SPK}$, $\mathcal{O}_{SK}$, $\mathcal{O}_S$ and $\mathcal{O}_V$ on at most n_S different signers, queries $\mathcal{O}_{VPK}$, $\mathcal{O}_{VK}$, $\mathcal{O}_S$ and $\mathcal{O}_V$ on at most n_V different verifiers, makes at most q_S and q_V queries to $\mathcal{O}_S$ and $\mathcal{O}_V$, respectively, with sum of the sizes of the sets of verifiers input to $\mathcal{O}_S$ being at most d_S and the advantage of $\mathbf{A}$ in winning the (corresponding) security game being at least ε.

Finally, we say that Π is

$$(\varepsilon_{\mathsf{Corr}}, \varepsilon_{\mathsf{Cons}}, \varepsilon_{\mathsf{Unforg}}, \varepsilon_{\mathsf{OTR}}, \varepsilon_{\mathsf{PI}}$$
$$t, n_S, n_V, d_S, q_S, q_V, Forge)\text{-secure,}$$

if no adversary $\mathbf{A}$:

- $(\varepsilon_{\mathsf{Corr}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Correctness of $\mathit{\Pi}$;
- $(\varepsilon_{\mathsf{Cons}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Consistency of $\mathit{\Pi}$;
- $(\varepsilon_{\mathsf{Unforg}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Unforgeability of $\mathit{\Pi}$;
- $(\varepsilon_{\mathsf{OTR}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Off-The-Record security of $\mathit{\Pi}$ with respect to *Forge*; or
- $(\varepsilon_{\mathsf{IND\text{-}CCA\text{-}2}}, t)$-breaks the $(n_S, n_V, d_S, q_S, q_V)$-Privacy of $\mathit{\Pi}$.

E Multi-Designated Verifier Signature Scheme with Privacy from Standard Assumptions

The construction of an MDVS scheme achieving Privacy of Identities (see Definition 25) from standard assumptions is straightforward from the MDRS-PKE scheme construction given in Algorithm 2. For completeness, we give an explicit construction in Algorithm 3. Note that the signature forgery algorithm that is required to exist so that the MDVS scheme is Off-The-Record is the same as for the MDRS-PKE scheme (see Algorithm 4).

Algorithm 3 Construction of an MDVS scheme $\mathit{\Pi} = (\mathit{Setup}, G_S, G_V, \mathit{Sign}, \mathit{Vfy})$ with privacy from an MDRS-PKE scheme $\mathit{\Pi}_{\text{MDRS-PKE}} = (S, G_S, G_R, E, D)$.

```
$\mathit{Setup}(1^k)$
    return $\mathit{\Pi}_{\text{MDRS-PKE}}.S(1^k)$

$G_S(\mathtt{pp})$
    return $\mathit{\Pi}_{\text{MDRS-PKE}}.G_S(\mathtt{pp})$

$G_V(\mathtt{pp})$
    return $\mathit{\Pi}_{\text{MDRS-PKE}}.G_V(\mathtt{pp})$

$\mathit{Sign}(\mathtt{pp}, \mathtt{ssk}_i, \{\mathtt{vpk}_j\}_{j \in \mathcal{V}}, m)$
    Let $\vec{v}$ be an arbitrary (but fixed) vector satisfying $|\vec{v}| = |\mathcal{V}|$ and $\mathcal{V} = \{v_i\}_{i \in \{1,\ldots,|\vec{v}|\}}$
    return $\mathit{\Pi}_{\text{MDRS-PKE}}.E_{\mathsf{pp}}(\vec{v}, m)$

$\mathit{Vfy}(\mathtt{pp}, \mathtt{spk}_i, \mathtt{vsk}_j, \{\mathtt{vpk}_l\}_{l \in \mathcal{V}}, m, c)$
    $(\vec{v}, m') \leftarrow \mathit{\Pi}_{\text{MDRS-PKE}}.D_{\mathsf{pp}}(\mathtt{vsk}_j, c)$
    if $(\vec{v}, m') = \perp \ \vee \ m \neq m' \ \vee \ \mathrm{Set}(\vec{v}) \neq \{\mathtt{vpk}_l\}_{l \in \mathcal{V}}$ then
        return invalid
    else
        return valid
```

F Tight Multi-User Multi-Challenge $\mathsf{IK\text{-}CPA}$ Security of ElGamal

Throughout this section, let $G = \langle g \rangle$ be a group with $|G| = q$ prime.

Definition 26. *For* $\mathbf{b} \in \{0, 1\}$*, consider the following game between an adversary* $\mathbf{A}$ *and* $\mathbf{G}^{\mathsf{DDH}}_{\mathbf{b}}$*:*

1. $(x, y, z) \xleftarrow{\$} \mathbb{Z}_q^* \times \mathbb{Z}_q^* \times \mathbb{Z}_q^*$;
2. $(X, Y, Z_0, Z_1) = (g^x, g^y, g^{xy}, g^z)$;
3. $b' \leftarrow \mathbf{A}(g, q, X, Y, Z_{\mathbf{b}})$.

$\mathbf{A}$ *wins the game if* $b' = \mathbf{b}$.

$\mathbf{A}$*'s advantage in winning* DDH *for* G *is:*

$$Adv^{\mathsf{DDH}}(\mathbf{A}) := \left|\Pr[\mathbf{AG}_{\mathbf{0}}^{\mathsf{DDH}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{DDH}} = \mathtt{win}] - 1\right|.$$

We say that an adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{DDH}}, t)$-breaks the DDH assumption for a group G if $\mathbf{A}$ runs in time at most t and satisfies $Adv^{\mathsf{DDH}}(\mathbf{A}) \geq \varepsilon_{\mathsf{DDH}}$; conversely we say that the DDH assumption $(\varepsilon_{\mathsf{DDH}}, t)$-holds for G if no adversary $\mathbf{A}$ $(\varepsilon_{\mathsf{DDH}}, t)$-breaks the DDH assumption for G.

For a cyclic group $G = \langle g \rangle$ with $|G| = q$ prime, the ElGamal [14] PKE scheme works as follows:

Key-Pair Generation
1. Pick $b \leftarrow \mathbb{Z}_q^*$ uniformly at random, and compute $B = g^b$;
2. $\mathtt{pk} := (g, q, B)$, and $\mathtt{sk} := (g, q, b)$.

Encryption
1. On input $(\mathtt{pk} := (g, q, B), m)$, pick $r \leftarrow \mathbb{Z}_q^*$ uniformly at random;
2. Output as ciphertext $c := (R := g^r, C := (g^b)^r \cdot m)$.

Decryption
1. On input $(\mathtt{sk} := (g, q, b), c := (R, C))$, compute R^{-b};
2. Output $R^{-b} \cdot C$.

It is well known that ElGamal is tightly Multi-User Multi-Challenge IND-CPA secure under DDH [8]: if an adversary $\mathbf{A}$ (ε, t)-breaks the (n, q_E)-IND-CPA security of ElGamal for a (cyclic) group G (with $|G| = q$ prime), then there is an adversary $\mathbf{A}'$ that (ε', t')-breaks the DDH assumption for the same group G with $\varepsilon' \geq \varepsilon/2 - 1/q$ and $t' = t + O(T^{\exp} \cdot (n + q_E))$, where $T^{\exp}$ is an upper bound on the time to compute an exponentiation. However, to the best of our knowledge only the Single-User Single-Challenge IK-CPA security of ElGamal has been proven [7]. For completeness, we now show that ElGamal is also tightly Multi-User Multi-Challenge IK-CPA secure under DDH.

Theorem 12. *If there is an adversary* $\mathbf{A}$ *that* (ε, t)*-breaks the* (n, q_E)-IK-CPA *security of the ElGamal* PKE *scheme for a cyclic group* $G = \langle g \rangle$ *with* $|G| = q$ *prime, then there is an adversary* $\mathbf{A}'$ *that* (ε', t')*-breaks the* DDH *assumption for the same group* G*, with*

- $\varepsilon' \geq \frac{\varepsilon - 1/q}{2}$*; and*
- $t' = t + O(T^{exp} \cdot (n + q_E))$*, where* T^{exp} *is an upper bound on the time to compute an exponentiation.*

Proof. Consider the IK-CPA game systems $\mathbf{G}_{\mathbf{0}}^{\mathsf{IK-CPA}}$ and $\mathbf{G}_{\mathbf{1}}^{\mathsf{IK-CPA}}$ for the ElGamal PKE scheme using G as the underlying group. We give reductions $\mathbf{C}_0$ and $\mathbf{C}_1$ satisfying:

(1) $\mathbf{C}_0\mathbf{G}_\mathbf{1}^{\mathsf{DDH}} \equiv \mathbf{C}_1\mathbf{G}_\mathbf{1}^{\mathsf{DDH}}$ with probability at least ${}^{q-1}/_q$;
(2) $\mathbf{C}_0\mathbf{G}_\mathbf{0}^{\mathsf{DDH}} \equiv \mathbf{G}_\mathbf{0}^{\mathsf{IK-CPA}}$;
(3) $\mathbf{C}_1\mathbf{G}_\mathbf{0}^{\mathsf{DDH}} \equiv \mathbf{G}_\mathbf{1}^{\mathsf{IK-CPA}}$.

Reductions $\mathbf{C}_0$ and $\mathbf{C}_1$ work as follows:

– Initialization:
  1. $(g, q, X, Y, Z) \leftarrow \mathbf{G}^{\mathsf{DDH}}$.
– $\mathbf{A}$ queries $\mathcal{O}_{PK}$ on input B_j:
  1. On the first call on B_j, pick r_j uniformly at random from $\mathbb{Z}_q^*$, and store $(B_j, r_j, (g, q, X^{r_j}))$; output $\mathtt{pk}_j := (g, q, X^{r_j})$;
  2. On subsequent calls for B_j, simply output $\mathtt{pk}_j := (g, q, X^{r_j})$.
– $\mathbf{A}$ queries $\mathcal{O}_E$ of $\mathbf{C}_b$ on input $(B_{j,0}, B_{j,1}, m)$:
  1. Pick r uniformly at random from $\mathbb{Z}_q^*$, and compute $R := Y^r$ and $C := Z^{r_{j,b}\cdot r}.m$, where $r_{j,b}$ is the value created by $\mathcal{O}_{PK}$ for $B_{j,b}$; output (R, C).
– When $\mathbf{A}$ outputs a guess b':
  1. $\mathbf{C}_b$ outputs $b' \oplus b$ as the guess.

It is easy to see that $\mathbf{C}_0$ and $\mathbf{C}_1$ satisfy Conditions (1), (2) and (3) specified above. Finally, by considering a reduction system $\mathbf{C}$ that initially picks $b \leftarrow \{0,1\}$ uniformly at random and then behaves as $\mathbf{C}_b$, it follows that adversary $\mathbf{AC}$ $(\frac{\varepsilon - 1/q}{2}, t')$-breaks the DDH assumption, where $t' \approx t + O(T^{\exp} \cdot (n + 2 \cdot q_E))$. □

G PKEBC Construction Security Proofs

In this section we give the (missing) full security proofs for the PKEBC construction given in Sect. 4.

G.1 Proof of Theorem 1

Proof. We prove a stronger result. Namely, we consider an alternative Correctness security notion for PKEBC schemes that only differs from Definition 1 in that it allows the adversary to query for the secret key of any receiver and still win the game.

This proof proceeds in a sequence of games [9,32].

Game 1. This is the original $\mathbf{G}^{\mathsf{Corr}}$ Correctness game as described above.

Game 2. This game is just like the original Game 1, except that now the $\mathtt{crs}_{\mathrm{CS}}$ output by $\mathcal{O}_S$ is perfectly binding (see Appendix B).

Let

$$Adv^{(\text{Game 2})\text{-}\mathsf{Corr}}(\mathbf{A}) := \Pr[\mathbf{A}\mathbf{G}^{(\text{Game 2})\text{-}\mathsf{Corr}} = \mathtt{win}],$$

where the conditions for $\mathbf{A}$ winning $\mathbf{G}^{(\text{Game 2})\text{-}\mathsf{Cons}}$ are the same as for the original game. It follows from Eq. (4.5), that no adversary $(\varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}})$-breaks the Binding property of Π_{CS}, implying

$$Adv^{\mathsf{Corr}}(\mathbf{A}) \leq \varepsilon_{\mathrm{CS}\text{-}\mathsf{Binding}} + Adv^{(\text{Game 2})\text{-}\mathsf{Corr}}(\mathbf{A}).$$

Game 3. This game is just like the original Game 2, except that now for each ciphertext $c \coloneqq (p, \texttt{comm}, \vec{c})$ output by $\mathcal{O}_E$ for a query with input $(\vec{V}, m)$, if $\mathcal{O}_D$ is queried on input (B_j, c) with $B_j \in \vec{V}$, $\mathcal{O}_D$ no longer verifies p's validity using $\Pi_{\mathrm{NIZK}}.V$, and instead simply proceeds as if p would verify as being valid.

Let

$$Adv^{(\text{Game 3})\text{-}\mathsf{Corr}}(\mathbf{A}) \coloneqq \Pr[\mathbf{AG}^{(\text{Game 3})\text{-}\mathsf{Corr}} = \texttt{win}],$$

where the conditions for $\mathbf{A}$ winning $\mathbf{G}^{(\text{Game 3})\text{-}\mathsf{Corr}}$ are the same as for the original game. Since $\mathbf{A}$ can only make up to $q_E \le q_{P\mathrm{NIZK}}$ queries to $\mathcal{O}_E$ and $q_D \le q_{V\mathrm{NIZK}}$ queries to $\mathcal{O}_D$, it follows from Eq. (4.4), that no adversary $(\varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Complete}}, t_{\mathrm{NIZK}})$-breaks the $(q_{P\mathrm{NIZK}}, q_{V\mathrm{NIZK}})$-Completeness of Π_{NIZK}, implying

$$Adv^{(\text{Game 2})\text{-}\mathsf{Corr}}(\mathbf{A}) \le \varepsilon_{\mathrm{NIZK}\text{-}\mathsf{Complete}} + Adv^{(\text{Game 3})\text{-}\mathsf{Corr}}(\mathbf{A}).$$

Game 4. This game is just like Game 3, except that when $\mathcal{O}_D$ is queried on an input (B_j, c), where $c \coloneqq (p, \texttt{comm}, \vec{c})$ was output by a query $\mathcal{O}_E(\vec{V}, m)$ such that $B_j \in \vec{V}$, $\mathcal{O}_D$ no longer tries decrypting each $c_{l,0}$ of $\vec{c}$ satisfying $v_l = \texttt{pk}_j$ using $\Pi_{\mathrm{PKE}}.D$, and instead simply assumes the output is $(\rho, \vec{v}, m)$, the tuple encrypted by $\Pi_{\mathrm{PKE}}.E$—with ρ being the random coins used by Π_{CS} to compute the commitment $\texttt{comm}$.

The probability of winning Game 4 is 0: consider any query $\mathcal{O}_E(\vec{V}, m)$ and any later query $\mathcal{O}_D(B_j, c)$ where $c = (p, \texttt{comm}, \vec{c})$ is the output of the first query and where $B_j \in \vec{V}$:

- Since Π_{PKE} is now assumed to be a correct PKE scheme, then for the least $l \in \{1, \ldots, |\vec{c}|\}$ satisfying $V_l = B_j$, B_j's decryption of $c_{l,0}$ of $\vec{c}$ is going to be $(\vec{v}, m)$, where $\vec{v}$ is the vector of public keys corresponding to $\vec{V}$. By the definition of $\Pi.D$ this then implies that if no $(\vec{v}'', m'') \neq (\vec{v}, m)$ is output—corresponding to the decryption of some $c_{l',0}$ where $l' < l$—then $\Pi.D$ outputs $(\vec{v}, m)$;
- Since $\texttt{crs}_{\mathrm{CS}}$ is binding, for any $(\vec{v}', m') \neq (\vec{v}, m)$ (with $(\vec{v}', m') \neq \bot$) and any ρ':
$$\texttt{comm} \neq \Pi_{\mathrm{CS}}.Commit_{\texttt{crs}_{\mathrm{CS}}}(\vec{v}', m'; \rho'),$$
implying $\mathcal{O}_D$ does not output $(\vec{v}', m') \neq (\vec{v}, m)$.

To conclude, since $n \le n_{\mathrm{PKE}}$, $d_E \le q_{E\mathrm{PKE}}$ and $q_D \le q_{D\mathrm{PKE}}$[10], it follows from Eq. (4.3), that no adversary $(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}, t_{\mathrm{PKE}})$-breaks the $(n, q_{E\mathrm{PKE}}, q_{D\mathrm{PKE}})$-Correctness of Π_{PKE}, implying

$$Adv^{(\text{Game 3})\text{-}\mathsf{Corr}}(\mathbf{A}) \le \varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}.$$

□

[10] Note that, as since Game 3 Π_{NIZK} is assumed to be complete, each query $\mathcal{O}_D(B_j, c)$, where $c = (p, \texttt{comm}, \vec{c})$ is the output of some prior query $\mathcal{O}_E(\vec{V}, m)$ satisfying $B_j \in \vec{V}$, will entail a query to $\Pi_{\mathrm{PKE}}.D$ for B_j's decryption of a ciphertext $c_{l,0}$ of $\vec{c}$ that was encrypted in the $\mathcal{O}_E$ query using $\texttt{pk}_j$ (B_j's public key).

G.2 Proof of Theorem 2

Proof. First, note that an adversary $\mathbf{A}$ wins this stronger Robustness game if there are two queries q_{E} and q_{D} to $\mathcal{O}_E$ and $\mathcal{O}_D$, respectively, where q_{E} has input $(\vec{V}, m)$ and q_{D} has input (B_j, c), satisfying $B_j \notin \vec{V}$, the input c in q_{D} is the output of q_{E}, and the output of q_{D} is $(\vec{v}', m')$ with $(\vec{v}', m') \neq \bot$. By the definition of $\Pi.D$, the output of q_{D} being some pair $(\vec{v}', m') \neq \bot$ implies $\mathtt{pk}_j \in \vec{v}'$; since $B_j \notin \vec{V}$, $\mathtt{pk}_j \notin \vec{v}$ (where $\mathtt{pk}_j$ is B_j's public key and $\vec{v}$ is the vector of public keys corresponding to the vector of parties $\vec{V}$), implying $(\vec{v}', m') \neq (\vec{v}, m)$. Also by the definition of $\Pi.D$, the output of q_{D} being some pair $(\vec{v}', m') \neq \bot$ implies there is a sequence of random coins ρ' such that

$$\mathtt{comm} = \Pi_{\mathrm{CS}}.Commit_{\mathtt{crs}_{\mathrm{CS}}}(\vec{v}', m'; \rho').$$

Noting that this is only possible if $\mathtt{crs}_{\mathrm{CS}}$ is non-binding, it follows from Eq. (4.6), that no adversary ($\varepsilon_{\mathsf{CS\text{-}Binding}}$)-breaks the Binding property of Π_{CS}, implying

$$Adv^{\mathsf{Rob}}(\mathbf{A}) \leq \varepsilon_{\mathsf{CS\text{-}Binding}}.$$

□

G.3 Proof of Theorem 3

Proof. We prove a stronger result. Namely, we consider an alternative Consistency security notion for PKEBC schemes that only differs from Definition 3 in that it allows the adversary to query for the secret key of any receiver (and still win the game).

This proof proceeds in a sequence of games [9, 32].

Game 1. This is the original $\mathbf{G}^{\mathsf{Cons}}$ Consistency game as described above.

Game 2. This game is just like the original Game 1, except that now the $\mathtt{crs}_{\mathrm{CS}}$ output by $\mathcal{O}_S$ is perfectly binding (see Appendix B).

Let

$$Adv^{(\text{Game 2})\text{-}\mathsf{Cons}}(\mathbf{A}) := \Pr[\mathbf{A}\mathbf{G}^{(\text{Game 2})\mathsf{Cons}} = \mathtt{win}],$$

where the conditions for $\mathbf{A}$ winning $\mathbf{G}^{(\text{Game 2})\text{-}\mathsf{Cons}}$ are the same as for the original game. It follows from Eq. (4.9), that no adversary ($\varepsilon_{\mathsf{CS\text{-}Binding}}$)-breaks the Binding property of Π_{CS}, implying

$$Adv^{\mathsf{Cons}}(\mathbf{A}) \leq \varepsilon_{\mathsf{CS\text{-}Binding}} + Adv^{(\text{Game 2})\text{-}\mathsf{Cons}}(\mathbf{A}).$$

Game 3. This game is just like Game 2, except that whenever $\mathcal{O}_D$ is queried on an input (B_j, c), with $c := (p, \texttt{comm}, \vec{c})$, such that $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c}) \notin L_{\text{Cons}}$ ($\texttt{crs}_{\text{CS}}$ being the one generated by $\mathcal{O}_{PP}$), $\mathcal{O}_D$ outputs $\perp$.

Game 3 is perfectly indistinguishable from Game 2 unless $\mathbf{A}$ makes a decryption query on a ciphertext $c := (p, \texttt{comm}, \vec{c})$ such that the NIZK proof p verifies as being a valid one for statement $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c}) \in L_{\text{Cons}}$, and with respect to the $\texttt{crs}_{\text{CS}}$ output by $\mathcal{O}_S$, but $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c}) \notin L_{\text{Cons}}$. Let

$$Adv^{(\text{Game 3})\textsf{-Cons}}(\mathbf{A}) := \Pr[\mathbf{A}\mathbf{G}^{(\text{Game 3})\textsf{-Cons}} = \texttt{win}],$$

where the conditions for $\mathbf{A}$ winning $\mathbf{G}^{(\text{Game 3})\textsf{-Cons}}$ are the same as for winning Game 2. Noting that $\mathbf{A}$ makes at most $q_D = q_{V\text{NIZK}}$ decryption queries, it follows from Eq. (4.8) that no adversary $(\varepsilon_{\text{NIZK-}\textsf{Sound}}, t_{\text{NIZK}})$-breaks the $(q_{V\text{NIZK}})$-Soundness of $\mathit{\Pi}_{\text{NIZK}}$, implying

$$Adv^{(\text{Game 2})\textsf{-Cons}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\textsf{Sound}} + Adv^{(\text{Game 3})\textsf{-Cons}}(\mathbf{A}).$$

To conclude this proof, we will prove the following claim:

Claim. For any adversary $\mathbf{A}$ such that $n \leq n_{\text{PKE}}$,

$$Adv^{(\text{Game 3})\textsf{-Cons}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\textsf{Corr}}.$$

Proof. Recall that an adversary $\mathbf{A}$ wins Game 3, if it queries $\mathcal{O}_D$ on inputs (B_i, c) and (B_j, c) for some B_i and B_j (possibly with $B_i = B_j$) and some ciphertext c, and the first query outputs $(\vec{v}, m) \neq \perp$ with $\texttt{pk}_j \in \vec{v}$ (where $\texttt{pk}_j$ is B_j's public key), whereas the second outputs either $\perp$ or some $(\vec{v}', m')$ with $(\vec{v}', m') \neq (\vec{v}, m)$.

Consider any two queries $q_{D,i}$ and $q_{D,j}$ that $\mathbf{A}$ makes to $\mathcal{O}_D$ on inputs (B_i, c) and (B_j, c'), respectively, satisfying $c = c'$, and such that $q_{D,i}$ outputs $(\vec{v}_i, m_i)$ with $(\vec{v}_i, m_i) \neq \perp$ and $\texttt{pk}_j \in \vec{v}_i$. First, note that if $\mathbf{A}$ does not make any two queries satisfying these conditions, then it does not win Game 3. In the following, let $c := (p, \texttt{comm}, \vec{c})$ be the ciphertext input to $q_{D,i}$ and $q_{D,j}$.

By the soundness of $\mathit{\Pi}_{\text{NIZK}}$, there is a vector of public keys $\vec{v}$ and a message m such that $\texttt{comm}$ is a commitment to $(\vec{v}, m)$, and for every ciphertext $c_{x,b}$ of $\vec{c}$ there is a sequence of random coins $r_{x,b}$ such that $c_{x,b}$ is the $\mathit{\Pi}_{\text{PKE}}$ encryption of $(\rho, \vec{v}, m)$ under key $v_{x,b}$ using $r_{x,b}$ as the encryption's (sequence of) random coins; by the binding of $\texttt{crs}_{\text{CS}}$, both $\vec{v}$ and m are unique; the definition of $\mathit{\Pi}.D$ implies $(\vec{v}_i, m_i) = (\vec{v}, m)$ and implies the existence of $l, l' \in \{1, \ldots, |\vec{v}|\}$ satisfying, respectively, $\texttt{pk}_i = v_l$ and $\texttt{pk}_j = v_{l'}$. Furthermore, by the definition of $\mathit{\Pi}.D$, $q_{D,i}$ (resp. $q_{D,j}$) will not output $(\vec{v}_{i,\alpha}, m_{i,\alpha})$ from $(\rho_{i,\alpha}, \vec{v}_{i,\alpha}, m_{i,\alpha}) \leftarrow \mathit{\Pi}_{\text{PKE}}.D_{\texttt{sk}_i}(c_{\alpha,0})$ (resp. $(\vec{v}_{j,\beta}, m_{j,\beta})$ from $(\rho_{j,\beta}, \vec{v}_{j,\beta}, m_{j,\beta}) \leftarrow \mathit{\Pi}_{\text{PKE}}.D_{\texttt{sk}_j}(c_{\beta,0})$) for any α with $v_\alpha \neq \texttt{pk}_i$ (resp. any β with $v_\beta \neq \texttt{pk}_j$), because either $\vec{v}_{j,\alpha} \neq \vec{v}$ (resp. $\vec{v}_{j,\beta} \neq \vec{v}$), or $\texttt{pk}_i \neq v_\alpha$ (resp. $\texttt{pk}_j \neq v_\beta$). Again from the definition of $\mathit{\Pi}.D$, $q_{D,i}$ outputs, for some $l \in \{1, \ldots, |\vec{v}|\}$, $(\vec{v}_{i,l}, m_{i,l})$ from $(\rho_{i,l}, \vec{v}_{i,l}, m_{i,l}) \leftarrow \mathit{\Pi}_{\text{PKE}}.D_{\texttt{sk}_i}(c_{l,0})$, with $v_l = \texttt{pk}_i$ and where $c_{l,0} \in \vec{c}$. Similarly, $q_{D,j}$ either outputs $\perp$, or outputs, for some $l' \in \{1, \ldots, |\vec{v}|\}$, $(\vec{v}_{j,l'}, m_{j,l'})$ from $(\rho_{j,l'}, \vec{v}_{j,l'}, m_{j,l'}) \leftarrow \mathit{\Pi}_{\text{PKE}}.D_{\texttt{sk}_j}(c_{l',0})$, with $v_{l'} = \texttt{pk}_j$ and where $c_{l',0} \in \vec{c}$. Note that, since given a fixed sequence of random coins ρ, $\mathit{\Pi}_{\mathrm{CS}}.\mathit{Commit}$ is a deterministic algorithm, if $(\rho_{i,l}, \vec{v}_{i,l}, m_{i,l}) = (\rho_{j,l'}, \vec{v}_{j,l'}, m_{j,l'})$ then the outputs of $q_{D,i}$ and $q_{D,j}$ are the same. Recall from before that the soundness of $\mathit{\Pi}_{\mathrm{NIZK}}$ implies all ciphertexts $c_{x,b}$ of $\vec{c}$ are encryptions of the same triple $(\rho, \vec{v}, m)$ under some sequence of random coins $r_{x,b}$. Thus, the only way for $\mathbf{A}$ to win Game 3 is by breaking the Correctness of the underlying $\mathit{\Pi}_{\mathrm{PKE}}$ scheme. However, since $\mathbf{A}$ queries for at most $n \coloneqq n_{\mathrm{PKE}}$ different parties, it follows from Eq. (4.7) that $\mathbf{A}$ does not (n_{PKE})-break the $(\varepsilon_{\mathsf{Corr}})$-correctness of $\mathit{\Pi}_{\mathrm{PKE}}$[11]. This implies the advantage of $\mathbf{A}$ in winning Game 3—or in other words, the advantage of $\mathbf{A}$ in making any two queries $q_{D,i}$ and $q_{D,j}$ satisfying the conditions described above so that the outputs of $q_{D,i}$ and $q_{D,j}$ differ—is bounded by $\varepsilon_{\mathsf{Corr}}$.

This concludes the proof of the claim, and thus also of Theorem 3. □

G.4 Proof of Theorem 4

Proof. We also proceed in a sequence of games [9, 32].

Game 1. This is the original $\mathbf{G_0^{\textsf{IK-CCA-2}}}$ security game from Definition 5, meaning that each challenge ciphertext is an encryption of the corresponding challenge plaintext m^* under $\vec{V}_0^*$.

Game 2. This game is just like the original Game 1, except that the $\mathit{\Pi}_{\mathrm{PKE}}$ instance that generates every ciphertext $c_{l,0}$, with $l \in \{1, \ldots, |\vec{c}|\}$, of the challenge ciphertext's vector $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is now assumed to be correct, meaning that if one would decrypt any such ciphertext one would obtain the plaintext that was initially encrypted.

Let

$$\mathit{Adv}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}}(\mathbf{A}) \coloneqq \\ \left|\Pr[\mathbf{AG}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}} = \mathtt{win}] + \Pr[\mathbf{AG_1^{\textsf{IK-CCA-2}}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}}$ if it outputs $b' = 0$. Since $n \leq n_{\mathrm{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}, t_{\mathrm{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\mathrm{PKE}}$, implying

$$\mathit{Adv}^{\textsf{IK-CCA-2}}(\mathbf{A}) \leq \varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}} + \mathit{Adv}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}}(\mathbf{A}).$$

Game 3. This game is just like Game 2, except that the $\mathit{\Pi}_{\mathrm{PKE}}$ instance that generates every ciphertext $c_{l,1}$, with $l \in \{1, \ldots, |\vec{c}|\}$, of the challenge ciphertext's vector of ciphertexts $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is now assumed to be correct, meaning that if one would decrypt any such ciphertext one would obtain whatever plaintext had been initially encrypted.

[11] Note that since there is a sequence of random coins $r_{x,b}$ such that $c_{x,b}$ of $\vec{c}$ is an encryption of $(\rho, \vec{v}, m)$ under $\mathtt{pk}_j$, then a query $\mathcal{O}_E(B_j, (\rho, \vec{v}, m); r_{x,b})$ at $\mathit{\Pi}_{\mathrm{PKE}}$'s correctness game will output $c_{x,b}$.

Let

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 3})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 3})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 3})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{Corr}}, t_{\text{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\text{PKE}}$, implying

$$\mathit{Adv}^{(\text{Game 2})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{Corr}} + \mathit{Adv}^{(\text{Game 3})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 4. This game is just like Game 3, except that now $\mathcal{O}_S$ generates $\mathtt{crs}_{\mathtt{NIZK}}$ using S_{CRS}, and for each challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$, the NIZK proof p is now simulated, meaning it is generated by S_{Sim}.

Let

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 4})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 4})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 4})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $\mathbf{A}$ can only make up to $q_E \leq q_{P\text{NIZK}}$ queries to $\mathcal{O}_E$, it follows from Eq. (4.11) that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{ZK}}, t_{\text{NIZK}})$-breaks the $(q_{P\text{NIZK}})$-Zero-Knowledge of $\mathit{\Pi}_{\text{NIZK}}$, implying

$$\mathit{Adv}^{(\text{Game 3})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{ZK}} + \mathit{Adv}^{(\text{Game 4})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 5. This game is just like Game 4, except that for each $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,1}$ of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ is now an encryption of $(\tilde{\rho}, \vec{v}_1^*, m^*)$—where $\tilde{\rho}$ is some sequence of random coins, *independent of the one* used by $\mathit{\Pi}_{\text{CS}}.\mathit{Commit}$—instead of $(\rho, \vec{v}_0^*, m^*)$—where ρ is *the sequence of random coins used by* $\mathit{\Pi}_{\text{CS}}.\mathit{Commit}$.

Let

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 5})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 5})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins Game 5 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IND\text{-}CPA}$ security of $\mathit{\Pi}_{\text{PKE}}$, implying

$$\mathit{Adv}^{(\text{Game 4})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + \mathit{Adv}^{(\text{Game 5})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 6. This game is just like Game 5, except that for each $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,1}$ (of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$) is now encrypted under public key $(v_1^*)_{l,1}$, instead of being encrypted under public key $(v_0^*)_{l,1}$.

Let

$$\begin{aligned} &\mathit{Adv}^{(\text{Game 6})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ &\qquad \left|\Pr[\mathbf{AG}^{(\text{Game 6})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|, \end{aligned}$$

where $\mathbf{A}$ wins Game 6 if it outputs $b' = 0$. Again, since $d_E \le q_{E\text{PKE}}$ and $n \le n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IK\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IK\text{-}CPA}$ security of $\mathit{\Pi}_{\text{PKE}}$, implying

$$\mathit{Adv}^{(\text{Game 5})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \le \varepsilon_{\text{PKE-}\mathsf{IK\text{-}CPA}} + \mathit{Adv}^{(\text{Game 6})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 7. This game is just like Game 6, except now a decryption query for a party B_j—with secret key $((\mathtt{pk}_{j,0}, \mathtt{sk}_{j,0}), (\mathtt{pk}_{j,1}, \mathtt{sk}_{j,1}))$—on a ciphertext $c := (p, \mathtt{comm}, \vec{c})$—with $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$—behaves slightly differently: rather than decrypting, for $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ using $\mathtt{sk}_{j,0}$, it now decrypts $c_{l,1}$ using $\mathtt{sk}_{j,1}$ instead.

It is easy to see that Game 6 is perfectly indistinguishable from Game 7 unless $\mathbf{A}$ makes a decryption query for a receiver B_j on ciphertext $c := (p, \mathtt{comm}, \vec{c})$ such that the NIZK proof p verifies with respect to $(\mathtt{crs}_{\text{CS}}, \mathtt{comm}, \vec{c})$ ($\mathtt{crs}_{\text{CS}}$ being the one generated by $\mathcal{O}_{PP}$), but $(\mathtt{crs}_{\text{CS}}, \mathtt{comm}, \vec{c}) \notin L_{\text{Cons}}$. Let

$$\begin{aligned} &\mathit{Adv}^{(\text{Game 7})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ &\qquad \left|\Pr[\mathbf{AG}^{(\text{Game 7})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|, \end{aligned}$$

where $\mathbf{A}$ wins Game 7 if it outputs $b' = 0$. Since $\mathbf{A}$ sees at most $q_E \le q_{P\text{NIZK}}$ simulated proofs (namely the ones in the challenge ciphertext) and makes at most $q_D \le q_{V\text{NIZK}}$ decryption queries, it follows from Eq. (4.11) that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{SS}}, t_{\text{NIZK}})$-breaks the $(q_{P\text{NIZK}}, q_{V\text{NIZK}})$-Simulation Soundness of $\mathit{\Pi}_{\text{NIZK}}$, implying

$$\mathit{Adv}^{(\text{Game 6})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \le \varepsilon_{\text{NIZK-}\mathsf{SS}} + \mathit{Adv}^{(\text{Game 7})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 8. This game is just like Game 7, except that for every $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ is now an encryption of $(\tilde{\rho}', \vec{v}_0^*, m^*)$—where $\tilde{\rho}'$ is some sequence of random coins *independent of the one* used by $\mathit{\Pi}_{\text{CS}}.\mathit{Commit}$—instead of $(\rho, \vec{v}_0^*, m^*)$—where ρ is *the sequence of random coins used by* $\mathit{\Pi}_{\text{CS}}.\mathit{Commit}$—similarly to Game 5.

Let

$$Adv^{(\text{Game 8})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 8})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 8 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IND\text{-}CPA}$ security of Π_{PKE}, implying

$$Adv^{(\text{Game 7})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + Adv^{(\text{Game 8})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 9. This game is just like Game 8, except that for each challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ $\mathtt{comm}$ is now a commitment to $(\vec{v}_1^*, m)$ instead of being a commitment to $(\vec{v}_0^*, m)$. Note that the sequence of random coins encrypted in each ciphertext in $\vec{c}$ is now independent from the sequence used by $\Pi_{\text{CS}}.\mathit{Commit}$.

Let

$$Adv^{(\text{Game 9})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 9})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 9 if it outputs $b' = 0$. Since $q_E \leq q_{\text{CS}}$, it follows from Eq. (4.12), that no adversary $(\varepsilon_{\text{CS-}\mathsf{Hiding}}, t_{\text{CS}})$-breaks the (q_{CS})-Hiding security property of Π_{CS}, implying

$$Adv^{(\text{Game 8})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{CS-}\mathsf{Hiding}} + Adv^{(\text{Game 9})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 10. This game is just like Game 9, except that for every $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ (of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$) is now an encryption of $(\rho, \vec{v}_1^*, m^*)$—where ρ is *the sequence of random coins used by* $\Pi_{\text{CS}}.\mathit{Commit}$—instead of $(\tilde{\rho}', \vec{v}_0^*, m^*)$—where $\tilde{\rho}'$ is some sequence of random coins, *independent of the one* used by $\Pi_{\text{CS}}.\mathit{Commit}$.

Let

$$Adv^{(\text{Game 10})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 10})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 10 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IND\text{-}CPA}$ security of Π_{PKE}, implying

$$Adv^{(\text{Game 9})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + Adv^{(\text{Game 10})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 11. This game is just like Game 10, except that, similarly to Game 6, for every $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ (of vector $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* \coloneqq (p, \texttt{comm}, \vec{c})$) is now encrypted under public key $(v_1^*)_{l,1}$, instead of being encrypted under public key $(v_0^*)_{l,1}$.

Let

$$\begin{aligned} &\mathit{Adv}^{(\text{Game 11})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \coloneqq \\ &\qquad \Big|\Pr[\mathbf{AG}^{(\text{Game 11})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \texttt{win}] - 1\Big|, \end{aligned}$$

where $\mathbf{A}$ wins Game 11 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IK\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-IK-CPA security of $\mathit{\Pi}_{\text{PKE}}$, implying

$$\mathit{Adv}^{(\text{Game 10})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IK\text{-}CPA}} + \mathit{Adv}^{(\text{Game 11})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 12. This game is just like Game 11, except now a decryption query for a party B_j—with secret key $((\texttt{pk}_{j,0}, \texttt{sk}_{j,0}), (\texttt{pk}_{j,1}, \texttt{sk}_{j,1}))$—on a ciphertext $c \coloneqq (p, \texttt{comm}, \vec{c})$—with $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$—behaves slightly differently: rather than decrypting, for $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,1}$ using $\texttt{sk}_{j,1}$, it returns to decrypting $c_{l,0}$ using $\texttt{sk}_{j,0}$.

It is easy to see that Game 11 is perfectly indistinguishable from Game 12 unless $\mathbf{A}$ makes a decryption query for a receiver B_j on ciphertext $c \coloneqq (p, \texttt{comm}, \vec{c})$ such that the NIZK proof p verifies with respect to $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c})$ ($\texttt{crs}_{\text{CS}}$ being the one generated by $\mathcal{O}_{PP}$), but $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c}) \notin L_{\text{Cons}}$. Let

$$\begin{aligned} &\mathit{Adv}^{(\text{Game 12})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \coloneqq \\ &\qquad \Big|\Pr[\mathbf{AG}^{(\text{Game 12})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \texttt{win}] - 1\Big|, \end{aligned}$$

where $\mathbf{A}$ wins Game 12 if it outputs $b' = 0$. Since $\mathbf{A}$ sees at most $q_E \leq q_{P\text{NIZK}}$ simulated proofs (namely the ones in the challenge ciphertext) and makes at most $q_D \leq q_{V\text{NIZK}}$ decryption queries, it follows from Eq. (4.11) that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{SS}}, t_{\text{NIZK}})$-breaks the $(q_{P\text{NIZK}}, q_{V\text{NIZK}})$-Simulation Soundness of $\mathit{\Pi}_{\text{NIZK}}$, implying

$$\mathit{Adv}^{(\text{Game 11})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{SS}} + \mathit{Adv}^{(\text{Game 12})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 13. This game is just like Game 12, except that for every $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,1}$ (of vector $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* \coloneqq (p, \texttt{comm}, \vec{c})$) is now an encryption of $(\rho, \vec{v}_1^*, m^*)$—where ρ is *the sequence of random coins used by* $\mathit{\Pi}_{\text{CS}}.\mathit{Commit}$—instead of $(\tilde{\rho}, \vec{v}_0^*, m^*)$—where $\tilde{\rho}$ is some sequence of random coins, *independent of the one* used by $\mathit{\Pi}_{\text{CS}}.\mathit{Commit}$.

Let

$$\begin{aligned} &\mathit{Adv}^{(\text{Game 13})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \coloneqq \\ &\qquad \Big|\Pr[\mathbf{AG}^{(\text{Game 13})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \texttt{win}] - 1\Big|, \end{aligned}$$

where $\mathbf{A}$ wins Game 13 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IND\text{-}CPA}$ security of $\mathit{\Pi}_{\text{PKE}}$, implying

$$Adv^{(\text{Game 12})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + Adv^{(\text{Game 13})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 14. This game is just like Game 13, except that the $\mathtt{crs}_{\mathtt{NIZK}}$ output by oracle $\mathcal{O}_S$ returns to the one generated by $\mathit{\Pi}_{\text{NIZK}}.G_{CRS}$, and the NIZK proof p in each challenge ciphertext returns to a real one generated by $\mathit{\Pi}_{\text{NIZK}}.P$.

Let

$$Adv^{(\text{Game 14})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 14})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 14 if it outputs $b' = 0$. Since $\mathbf{A}$ can only make up to $q_E \leq q_{P\text{NIZK}}$ queries to $\mathcal{O}_E$, it follows from Eq. (4.11), that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{ZK}}, t_{\text{NIZK}})$-breaks the $(q_{P\text{NIZK}})$-Zero-Knowledge of $\mathit{\Pi}_{\text{NIZK}}$, implying

$$Adv^{(\text{Game 13})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{ZK}} + Adv^{(\text{Game 14})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 15. This game is just like Game 14, except that the $\mathit{\Pi}_{\text{PKE}}$ instance that generates every ciphertext $c_{l,0}$, with $l \in \{1, \ldots, |\vec{c}|\}$, of the challenge ciphertext's vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is no longer assumed to be perfectly correct.

Let

$$Adv^{(\text{Game 15})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 15})\text{-}\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 15})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{Corr}}, t_{\text{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\text{PKE}}$, implying

$$Adv^{(\text{Game 14})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{Corr}} + Adv^{(\text{Game 15})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 16. This game is now $\mathbf{G}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}}$: the only difference from Game 15 is that the $\mathit{\Pi}_{\text{PKE}}$ instance that generates every ciphertext $c_{l,1}$, with $l \in \{1, \ldots, |\vec{c}|\}$, of the challenge ciphertext's vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is no longer assumed to be perfectly correct.

Since $n \leq n_{\text{PKE}}$, it follows from Eq. (4.10), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{Corr}}, t_{\text{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\text{PKE}}$, implying

$$Adv^{(\text{Game 15})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{Corr}}.$$

□

G.5 Proof of Theorem 5

Proof. This proof follows the same overall structure as the one given in [19] (in particular, we also proceed in a sequence of games [9,32]).

Game 1. This is the original $\mathbf{G}_{\mathbf{0}}^{\mathsf{IND\text{-}CCA\text{-}2}}$ security game from Definition 4, meaning that each challenge ciphertext is an encryption of the corresponding challenge plaintext m_0^*.

Game 2. This game is just like the original Game 1, except that the $\mathit{\Pi}_{\mathrm{PKE}}$ instance that generates every ciphertext $c_{l,0}$, with $l \in \{1, \dots, |\vec{c}|\}$, of the challenge ciphertext's vector $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \dots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is now assumed to be correct, meaning that if one would decrypt any such ciphertext one would obtain the plaintext that was initially encrypted.

Let

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 2})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \coloneqq \\ &\qquad \Big|\Pr[\mathbf{AG}^{(\text{Game 2})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\Big|,\end{aligned}$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 2})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $n \leq n_{\mathrm{PKE}}$, it follows from Eq. (4.13), that no adversary $(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}, t_{\mathrm{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\mathrm{PKE}}$, implying

$$\mathit{Adv}^{\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}} + \mathit{Adv}^{(\text{Game 2})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 3. This game is just like Game 2, except that the $\mathit{\Pi}_{\mathrm{PKE}}$ instance that generates every ciphertext $c_{l,1}$, with $l \in \{1, \dots, |\vec{c}|\}$, of the challenge ciphertext's vector of ciphertexts $\vec{c} \coloneqq \big((c_{1,0}, c_{1,1}), \dots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is now assumed to be correct, meaning that if one would decrypt any such ciphertext one would obtain whatever plaintext had been initially encrypted.

Let

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 3})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \coloneqq \\ &\qquad \Big|\Pr[\mathbf{AG}^{(\text{Game 3})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\Big|,\end{aligned}$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 3})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $n \leq n_{\mathrm{PKE}}$, it follows from Eq. (4.13), that no adversary $(\varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}}, t_{\mathrm{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\mathrm{PKE}}$, implying

$$\mathit{Adv}^{(\text{Game 2})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\mathrm{PKE}\text{-}\mathsf{Corr}} + \mathit{Adv}^{(\text{Game 3})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 4. This game is just like Game 3, except that now $\mathcal{O}_S$ generates $\texttt{crs}_{\texttt{NIZK}}$ using S_{CRS}, and for each challenge ciphertext $c^* := (p, \texttt{comm}, \vec{c})$, the NIZK proof p is now simulated by S_{Sim}.

Let

$$Adv^{(\text{Game 4})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 4})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] - 1\right|,$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 4})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $\mathbf{A}$ can only make up to $q_E \leq q_{P\text{NIZK}}$ queries to $\mathcal{O}_E$, it follows from Eq. (4.14), that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{ZK}}, t_{\text{NIZK}})$-breaks the $(q_{P\text{NIZK}})$-Zero-Knowledge of Π_{NIZK}, implying

$$Adv^{(\text{Game 3})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{ZK}} + Adv^{(\text{Game 4})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 5. This game is just like Game 4, except that now, for each $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,1}$ of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \texttt{comm}, \vec{c})$ is an encryption of $(\tilde{\rho}, \vec{v}^*, m_1^*)$—where $\tilde{\rho}$ is some sequence of random coins, *independent of the one* used by $\Pi_{\text{CS}}.\mathit{Commit}$—instead of $(\rho, \vec{v}^*, m_0^*)$—where ρ is the sequence of random keys used by Π_{CS}.

Let

$$Adv^{(\text{Game 5})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 5})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 5 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.13), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IND\text{-}CPA}$ security of Π_{PKE}, implying

$$Adv^{(\text{Game 4})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + Adv^{(\text{Game 5})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 6. This game is just like Game 5, except now a decryption query for a party B_j—with secret key $((\texttt{pk}_{j,0}, \texttt{sk}_{j,0}), (\texttt{pk}_{j,1}, \texttt{sk}_{j,1}))$—on a ciphertext $c := (p, \texttt{comm}, \vec{c})$—with $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$—behaves slightly differently: rather than decrypting, for $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ using $\texttt{sk}_{j,0}$, it now decrypts $c_{l,1}$ using $\texttt{sk}_{j,1}$ instead.

It is easy to see that Game 5 is perfectly indistinguishable from Game 6 unless $\mathbf{A}$ makes a decryption query for a receiver B_j on ciphertext $c := (p, \texttt{comm}, \vec{c})$ such that the NIZK proof p is valid with respect to statement $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c}) \in L_{\text{Cons}}$, but $(\texttt{crs}_{\text{CS}}, \texttt{comm}, \vec{c}) \notin L_{\text{Cons}}$ ($\texttt{crs}_{\text{CS}}$ being the one generated by $\mathcal{O}_{PP}$). Let

$$Adv^{(\text{Game 6})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 6})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \texttt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 6 if it outputs $b' = 0$. Since $\mathbf{A}$ sees at most $q_E \leq q_{P\text{NIZK}}$ simulated proofs and makes at most $q_D \leq q_{V\text{NIZK}}$ decryption queries, it follows from Eq. (4.14) that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{SS}}, t_{\text{NIZK}})$-breaks the $(q_{P\text{NIZK}}, q_{V\text{NIZK}})$-Simulation Soundness of Π_{NIZK}, implying

$$Adv^{(\text{Game 5})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{SS}} + Adv^{(\text{Game 6})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 7. This game is just like Game 6, except that, similarly to Game 5 for every $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ is now also an encryption of $(\tilde{\rho}', \vec{v}^*, m_1^*)$—where $\tilde{\rho}'$ is some sequence of random coins, *independent of the one* used by $\Pi_{\text{CS}}.\mathit{Commit}$—instead of $(\rho, \vec{v}^*, m_0^*)$—where ρ is the sequence of random keys used by Π_{CS}.

Let

$$Adv^{(\text{Game 7})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 7})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 7 if it outputs $b' = 0$. Since $d_E \leq q_{E\text{PKE}}$ and $n \leq n_{\text{PKE}}$, it follows from Eq. (4.13), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}}, t_{\text{PKE}})$-breaks the $(n_{\text{PKE}}, q_{E\text{PKE}})$-$\mathsf{IND\text{-}CPA}$ security of Π_{PKE}, implying

$$Adv^{(\text{Game 6})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + Adv^{(\text{Game 7})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 8. This game is just like Game 8, except that for each challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ $\mathtt{comm}$ is now a commitment to $(\vec{v}^*, m_1)$ instead of being a commitment to $(\vec{v}^*, m_0)$. Note that the sequence of random coins encrypted in each ciphertext in $\vec{c}$ is now independent from the sequence used by $\Pi_{\text{CS}}.\mathit{Commit}$.

Let

$$Adv^{(\text{Game 8})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) := \\ \left|\Pr[\mathbf{AG}^{(\text{Game 8})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,$$

where $\mathbf{A}$ wins Game 8 if it outputs $b' = 0$. Since $q_E \leq q_{\text{CS}}$, it follows from Eq. (4.15), that no adversary $(\varepsilon_{\text{CS-}\mathsf{Hiding}}, t_{\text{CS}})$-breaks the (q_{CS})-Hiding security property of Π_{CS}, implying

$$Adv^{(\text{Game 7})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{CS-}\mathsf{Hiding}} + Adv^{(\text{Game 8})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 9. This game is just like Game 8, except that for every $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ is now an encryption of $(\rho, \vec{v}^*, m_1^*)$—where ρ is the sequence of random keys used by Π_{CS}—instead of $(\tilde{\rho}', \vec{v}^*, m_0^*)$—where $\tilde{\rho}'$ is some sequence of random coins, *independent of the one* used by $\Pi_{\text{CS}}.\mathit{Commit}$.

Let

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 9})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 9})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins Game 9 if it outputs $b' = 0$. By an argument equivalent to the one used for Game 7, one can conclude:

$$\mathit{Adv}^{(\text{Game 8})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + \mathit{Adv}^{(\text{Game 9})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 10. This game is just like Game 9, except now a decryption query for a party B_j—with secret key $((\mathtt{pk}_{j,0}, \mathtt{sk}_{j,0}), (\mathtt{pk}_{j,1}, \mathtt{sk}_{j,1}))$—on a ciphertext $c := (p, \mathtt{comm}, \vec{c})$—with $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$—returns to decrypting, for $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,0}$ using $\mathtt{sk}_{j,0}$, rather than decrypting $c_{l,1}$ using $\mathtt{sk}_{j,1}$.

It is easy to see that, by an argument similar to the one used for Game 6, and letting

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 10})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 10})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins Game 10 if it outputs $b' = 0$, we have

$$\mathit{Adv}^{(\text{Game 9})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{SS}} + \mathit{Adv}^{(\text{Game 10})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 11. This game is just like Game 10, except that now, for each $l \in \{1, \ldots, |\vec{c}|\}$, ciphertext $c_{l,1}$ of vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ of the challenge ciphertext $c^* := (p, \mathtt{comm}, \vec{c})$ is an encryption of $(\rho, \vec{v}^*, m_1^*)$—where ρ is the sequence of random keys used by Π_{CS}—instead of $(\tilde{\rho}, \vec{v}^*, m_1^*)$—where $\tilde{\rho}$ is some sequence of random coins, *independent of the one* used by $\Pi_{\text{CS}}.\mathit{Commit}$.

By following an argument similar to the one used for Game 5, it is easy to see that, letting

$$\begin{aligned}&\mathit{Adv}^{(\text{Game 11})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 11})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins Game 11 if it outputs $b' = 0$, we have

$$\mathit{Adv}^{(\text{Game 10})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{IND\text{-}CPA}} + \mathit{Adv}^{(\text{Game 11})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 12. This game is just like Game 11, except that $\mathcal{O}_S$ returns to generating $\mathtt{crs}_{\mathtt{NIZK}}$ honestly using $\mathit{\Pi}_{\text{NIZK}}.G_{CRS}$, and the NIZK proof p of each challenge ciphertext returns to a real one generated by $\mathit{\Pi}_{\text{NIZK}}.P$.

Let

$$\begin{aligned}&Adv^{(\text{Game 12})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 12})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins Game 12 if it outputs $b' = 0$. Since $\mathbf{A}$ can only make up to $q_E \leq q_{P_{\text{NIZK}}}$ queries to $\mathcal{O}_E$, it follows from Eq. (4.14), that no adversary $(\varepsilon_{\text{NIZK-}\mathsf{ZK}}, t_{\text{NIZK}})$-breaks the $(q_{P_{\text{NIZK}}})$-Zero-Knowledge of $\mathit{\Pi}_{\text{NIZK}}$, implying

$$Adv^{(\text{Game 11})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{NIZK-}\mathsf{ZK}} + Adv^{(\text{Game 12})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 13. This game is just like Game 12, except that the $\mathit{\Pi}_{\text{PKE}}$ instance that generates every ciphertext $c_{l,0}$, with $l \in \{1, \ldots, |\vec{c}|\}$, of the challenge ciphertext's vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is no longer assumed to be perfectly correct.

Let

$$\begin{aligned}&Adv^{(\text{Game 13})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) :=\\ &\qquad\left|\Pr[\mathbf{AG}^{(\text{Game 13})\text{-}\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] + \Pr[\mathbf{AG}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}} = \mathtt{win}] - 1\right|,\end{aligned}$$

where $\mathbf{A}$ wins $\mathbf{G}^{(\text{Game 13})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}$ if it outputs $b' = 0$. Since $n \leq n_{\text{PKE}}$, it follows from Eq. (4.13), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{Corr}}, t_{\text{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\text{PKE}}$, implying

$$Adv^{(\text{Game 12})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{Corr}} + Adv^{(\text{Game 13})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}).$$

Game 14. This game is now $\mathbf{G}_{\mathbf{1}}^{\mathsf{IND\text{-}CCA\text{-}2}}$: the only difference from Game 13 is that the $\mathit{\Pi}_{\text{PKE}}$ instance that generates every ciphertext $c_{l,1}$, with $l \in \{1, \ldots, |\vec{c}|\}$, of the challenge ciphertext's vector $\vec{c} := \big((c_{1,0}, c_{1,1}), \ldots, (c_{|\vec{c}|,0}, c_{|\vec{c}|,1})\big)$ is no longer assumed to be perfectly correct.

Since $n \leq n_{\text{PKE}}$, it follows from Eq. (4.13), that no adversary $(\varepsilon_{\text{PKE-}\mathsf{Corr}}, t_{\text{PKE}})$-breaks the (n_{PKE})-Correctness of $\mathit{\Pi}_{\text{PKE}}$, implying

$$Adv^{(\text{Game 13})\text{-}\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKE-}\mathsf{Corr}}.$$

□

H MDRS-PKE Construction Security Proofs

In this section we give the full security proofs for the MDRS-PKE construction given in Sect. 6.

H.1 Proof of Theorem 6

Proof. This proof proceeds in a sequence of games [9, 32].

Game 1. This is the original $\mathbf{G}^{\mathsf{Corr}}$ Correctness game from Definition 6.

Game 2. This game is just like Game 1, except that now decryption queries work differently. More concretely, if the ciphertext c input to $\mathcal{O}_D$ was the output of a query to $\mathcal{O}_E$, say on input $(A_i, \vec{V}, m)$, $\mathcal{O}_D$ works as follows: Let $(\mathtt{spk}_i, \vec{v}_{\mathrm{MDVS}}, m, \sigma)$ be the plaintext that was encrypted by $\Pi_{\mathrm{PKEBC}}.E$ under $\vec{v}_{\mathrm{PKEBC}}$, where $\mathtt{spk}_i$ is A_i's public key,

$$\begin{aligned}\vec{v}_{\mathrm{MDVS}} &\coloneqq (\mathtt{vpk}_{\mathrm{MDVS}1}, \dots, \mathtt{vpk}_{\mathrm{MDVS}|\vec{v}|}),\\ \vec{v}_{\mathrm{PKEBC}} &\coloneqq (\mathtt{pk}_{\mathrm{PKEBC}1}, \dots, \mathtt{pk}_{\mathrm{PKEBC}|\vec{v}|})\end{aligned}$$

are, respectively, the vectors of public MDVS verifier keys and public PKEBC receiver keys corresponding to $\vec{V}$, and where

$$\sigma \leftarrow \Pi_{\mathrm{MDVS}}.Sign_{\mathtt{pp}_{\mathrm{MDVS}}}(\mathtt{ssk}_{\mathrm{MDVS}i}, \{\mathtt{vpk}_{\mathrm{MDVS}l}\}_{l \in \{1,\dots,|\vec{v}|\}}, (\vec{v}_{\mathrm{PKEBC}}, m)),$$

is an MDVS signature on $(\vec{v}_{\mathrm{PKEBC}}, m)$, with $\mathtt{ssk}_{\mathrm{MDVS}i}$ being A_i's secret MDVS sender key and $\{\mathtt{vpk}_{\mathrm{MDVS}l}\}_{l \in \{1,\dots,|\vec{v}|\}}$ being the set of public MDVS verifier keys of the parties in $\vec{V}$. Then, oracle $\mathcal{O}_D$ no longer decrypts c using $\Pi_{\mathrm{PKEBC}}.D$, and instead simply assumes the decryption is the pair $(\vec{v}_{\mathrm{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\mathrm{MDVS}}, m, \sigma))$, where $(\mathtt{spk}_i, \vec{v}_{\mathrm{MDVS}}, m, \sigma)$ is the plaintext that was encrypted by $\mathcal{O}_E$ using $\Pi_{\mathrm{PKEBC}}.E$.

Let

$$Adv^{(\text{Game 2})\text{-}\mathsf{Corr}}(\mathbf{A}) \coloneqq \Pr[\mathbf{A}\mathbf{G}^{(\text{Game 2})\text{-}\mathsf{Corr}} = \mathtt{win}],$$

where the conditions for $\mathbf{A}$ winning $\mathbf{G}^{(\text{Game 2})\text{-}\mathsf{Corr}}$ are the same as for the original game.

Since $\mathbf{A}$ only queries for at most $n_R \leq n_{\mathrm{PKEBC}}$ different receiver keys, makes up to $q_E \leq q_{E\mathrm{PKEBC}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{D\mathrm{PKEBC}}$ queries to $\mathcal{O}_D$, and the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{E\mathrm{PKEBC}}$, it follows from Eq. (6.1), that $\mathbf{A}$ does not $(\varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{Corr}}, t_{\mathrm{PKEBC}})$-break the

$$(n_{\mathrm{PKEBC}}, d_{E\mathrm{PKEBC}}, q_{E\mathrm{PKEBC}}, q_{D\mathrm{PKEBC}})\text{-Correctness}$$

of Π_{PKEBC}, implying

$$Adv^{\mathsf{Corr}}(\mathbf{A}) \leq \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{Corr}} + Adv^{(\text{Game 2})\text{-}\mathsf{Corr}}(\mathbf{A}).$$

Game 3. This game is just like Game 2, except that decryption queries once again work differently. Essentially, in the same way that Game 2 differed from Game 1—assuming that each ciphertext c output by a query to $\mathcal{O}_E$ decrypts correctly when $\mathcal{O}_D$ is queried on (B_j, c), B_j being one of the parties in the vector input to $\mathcal{O}_E$—now Game 3 differs from Game 2 in that it assumes that also each MDVS signature σ generated by $\mathcal{O}_E$ using $\mathit{\Pi}_{\text{MDVS}}.\mathit{Sign}$ also verifies as being valid when $\mathcal{O}_D$ is queried on (B_j, c), B_j being one of the parties in the vector input to $\mathcal{O}_E$, and σ being (part of) the plaintext corresponding to c.

Note that the winning probability of any adversary in winning Game 3 is 0. So, since $\mathbf{A}$ only queries for at most $n_S \leq n_{S\text{MDVS}}$ (resp. $n_R \leq n_{V\text{MDVS}}$) different sender keys (resp. different receiver keys), makes up to $q_E \leq q_{S\text{MDVS}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{V\text{MDVS}}$ queries to $\mathcal{O}_D$, and the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{S\text{MDVS}}$, it follows from Eq. (6.2), that $\mathbf{A}$ does not $(\varepsilon_{\text{MDVS-}\mathsf{Corr}}, t_{\text{MDVS}})$-break the

$$(n_{S\text{MDVS}}, n_{V\text{MDVS}}, d_{S\text{MDVS}}, q_{S\text{MDVS}}, q_{V\text{MDVS}})\text{-Correctness}$$

of $\mathit{\Pi}_{\text{MDVS}}$, implying

$$\mathit{Adv}^{(\text{Game 2})\text{-}\mathsf{Corr}}(\mathbf{A}) \leq \varepsilon_{\text{MDVS-}\mathsf{Corr}}.$$

□

H.2 Proof of Theorem 7

Proof. Assume there is a ciphertext c and two queries $\mathcal{O}_D(B_i, c)$ and $\mathcal{O}_D(B_j, c)$ (possibly with $B_i = B_j$) such that $\mathcal{O}_{RK}$ was not queried on either B_i or B_j, and $\mathcal{O}_D(B_i, c)$ outputs some $(\mathtt{spk}_l, \vec{v}, m)$ satisfying $(\mathtt{spk}_l, \vec{v}, m) \neq \bot$, $\mathtt{spk}_l$ is some party A_l's public sender key (i.e. $\mathcal{O}_{SPK}(A_l) = \mathtt{spk}_l$) and $\mathtt{rpk}_j \in \vec{v}$ (where $\mathtt{rpk}_j$ is B_j's public key), and query $\mathcal{O}_D(B_j, c)$ does not output the same triple $(\mathtt{spk}_l, \vec{v}, m)$. Note that any adversary $\mathbf{A}$ which does not make any two queries satisfying these conditions has advantage 0 in winning the Consistency game.

Consider the query $\mathcal{O}_D(B_i, c)$ mentioned above. On input (B_i, c), $\mathcal{O}_D$ first fetched B_i's secret key $\mathtt{rsk}_i$, and then used this key to decrypt c using $\mathit{\Pi}$'s PKEBC scheme $\mathit{\Pi}_{\text{PKEBC}}$. Since $\mathcal{O}_D(B_i, c)$ output some $(\mathtt{spk}_l, \vec{v}, m) \neq \bot$, then $\mathit{\Pi}_{\text{PKEBC}}$'s decryption of c using B_j's secret PKEBC key must have output some pair

$$\big(\vec{v}_{\text{PKEBC}}, (\mathtt{spk}_{l'}, \vec{v}_{\text{MDVS}}, m, \sigma)\big) \neq \bot,$$

with $\mathtt{spk}_{l'} = \mathtt{spk}_l$ and $|\vec{v}_{\text{PKEBC}}| = |\vec{v}_{\text{MDVS}}|$. Furthermore, we have that

$$\vec{v} := \big((v_{\text{MDVS}1}, v_{\text{PKEBC}1}), \ldots, (v_{\text{MDVS}|\vec{v}_{\text{PKEBC}}|}, v_{\text{PKEBC}|\vec{v}_{\text{PKEBC}}|})\big).$$

Since $\mathtt{rpk}_j \in \vec{v}$ (where $\mathtt{rpk}_j$ is B_j's public key), it follows $\mathtt{rpk}_j = v_k$ for some $k \in \{1, \ldots, |\vec{v}|\}$. Letting $\mathtt{rpk}_j = (\mathtt{vpk}_{\text{MDVS}j}, \mathtt{pk}_{\text{PKEBC}j})$, we then have $v_{\text{MDVS}k} = \mathtt{vpk}_{\text{MDVS}j}$ and $v_{\text{PKEBC}k} = \mathtt{pk}_{\text{PKEBC}j}$, implying $\mathtt{vpk}_{\text{MDVS}j} \in \vec{v}_{\text{MDVS}}$ and $\mathtt{pk}_{\text{PKEBC}j} \in \vec{v}_{\text{PKEBC}}$.

By Eq. (6.3), since $\mathbf{A}$ sees at most $n_R \leq n_{\text{PKEBC}}$ different public keys and makes at most $q_D \leq q_{D\text{PKEBC}}$ decryption queries to oracle $\mathcal{O}_D$, $\mathbf{A}$ does not $(\varepsilon_{\text{PKEBC-}\mathsf{Cons}}, t_{\text{PKEBC}})$-break the $(n_{\text{PKEBC}}, q_{D\text{PKEBC}})$-Consistency of the underlying Π_{PKEBC} scheme; since $\mathtt{pk}_{\text{PKEBC}j} \in \vec{v}_{\text{PKEBC}}$ this implies the probability that query $\mathcal{O}_D(B_j, c)$'s Π_{PKEBC} decryption of c does not match

$$\big(\vec{v}_{\text{PKEBC}}, (\mathtt{spk}_{l'}, \vec{v}_{\text{MDVS}}, m, \sigma)\big)$$

is bounded by $\varepsilon_{\text{PKEBC-}\mathsf{Cons}}$. Let us now assume that $\mathcal{O}_D(B_j, c)$'s Π_{PKEBC} decryption of c is the same as $\mathcal{O}_D(B_i, c)$'s decryption of c. This in particular implies that if $\mathbf{A}$ wins the Consistency game then the output of query $\mathcal{O}_D(B_j, c)$ is $\perp$, as the only way for $\mathbf{A}$ to win the game is Π_{MDVS}'s signature verification of σ for message $(\vec{v}_{\text{PKEBC}}, m)$ with respect to A_l's sender public key $\mathtt{spk}_l$, set of MDVS verifier public keys $\text{Set}(\vec{v}_{\text{MDVS}})$, and B_j's secret MDVS verifier key $\mathtt{vsk}_{\text{MDVS}j}$ outputting $\mathtt{invalid}$.

Since $\mathcal{O}_D(B_i, c)$ did output some $(\mathtt{spk}_l, \vec{v}, m) \neq \perp$, it follows that Π_{MDVS}'s signature verification of σ for message $(\vec{v}_{\text{PKEBC}}, m)$ with respect to A_l's sender public key $\mathtt{spk}_l$, set of MDVS verifier public keys $\text{Set}(\vec{v}_{\text{MDVS}})$, and B_i's secret MDVS verifier key $\mathtt{vsk}_{\text{MDVS}i}$ output $\mathtt{valid}$. But then this means that if Π_{MDVS}'s signature verification of σ outputs $\mathtt{invalid}$, the Π_{MDVS} scheme underlying Π's construction is not consistent. However, since $\mathbf{A}$ only queries for at most $n_S \leq n_{S\text{MDVS}}$ (resp. $n_R \leq n_{V\text{MDVS}}$) different sender keys (resp. different receiver keys), the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{S\text{MDVS}}$, $\mathbf{A}$ makes up to $q_E \leq q_{S\text{MDVS}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{V\text{MDVS}}$ queries to $\mathcal{O}_D$, it follows from Eq. (6.4), that $\mathbf{A}$ does not $(\varepsilon_{\text{MDVS-}\mathsf{Cons}}, t_{\text{MDVS}})$-break the

$$(n_{S\text{MDVS}}, n_{V\text{MDVS}}, d_{S\text{MDVS}}, q_{S\text{MDVS}}, q_{V\text{MDVS}})\text{-Consistency}$$

of Π's underlying MDVS scheme Π_{MDVS}, implying

$$Adv^{\mathsf{Cons}}(\mathbf{A}) \leq \varepsilon_{\text{PKEBC-}\mathsf{Cons}} + \varepsilon_{\text{MDVS-}\mathsf{Cons}}.$$

□

H.3 Proof of Theorem 8

Proof. To prove this, we give a (trivial) reduction from winning the Unforgeability game of the MDRS-PKE scheme Π to winning the Unforgeability game for underlying the MDVS scheme Π_{MDVS}.

```
$\mathcal{O}_{PP}()$ :
    1. $\mathtt{pp}_{\text{PKEBC}} \leftarrow \Pi_{pkebc}.S(1^k)$;
    2. Output $(\Pi_{\text{MDVS}}.\mathcal{O}_{PP}(), \mathtt{pp}_{\text{PKEBC}})$.
$\mathcal{O}_{SPK}(A_i)$ :
    1. Output $\Pi_{\text{MDVS}}.\mathcal{O}_{SPK}(A_i)$.
$\mathcal{O}_{SK}(A_i)$ :
    1. Output $\Pi_{\text{MDVS}}.\mathcal{O}_{SK}(A_i)$.
```

$\mathcal{O}_{RPK}(B_j)$:

1. On the first query on input B_j (to either $\mathcal{O}_{RPK}$ or $\mathcal{O}_{RK}$), compute and store $(\mathtt{pk}_{\mathrm{PKEBC}j}, \mathtt{sk}_{\mathrm{PKEBC}j}) \leftarrow \mathit{\Pi}_{\mathrm{PKEBC}}.G(\mathtt{pp}_{\mathrm{PKEBC}})$;
2. Output $(\mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_{VPK}(B_j), \mathtt{pk}_{\mathrm{PKEBC}j})$.

$\mathcal{O}_{RK}(B_j)$:

1. On the first query (to either $\mathcal{O}_{RPK}$ or $\mathcal{O}_{RK}$) on input B_j, compute and store $(\mathtt{pk}_{\mathrm{PKEBC}j}, \mathtt{sk}_{\mathrm{PKEBC}j}) \leftarrow \mathit{\Pi}_{\mathrm{PKEBC}}.G(\mathtt{pp}_{\mathrm{PKEBC}})$;
2. Let $(\mathtt{vpk}_{\mathrm{MDVS}j}, \mathtt{vsk}_{\mathrm{MDVS}j}) \leftarrow \mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_{VK}(B_j)$;
3. Let $\mathtt{rpk}_j \leftarrow (\mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_{VPK}(B_j), \mathtt{pk}_{\mathrm{PKEBC}j})$;
4. Let $\mathtt{rsk}_j \leftarrow (\mathtt{rpk}_j, (\mathtt{vsk}_{\mathrm{MDVS}j}, \mathtt{sk}_{\mathrm{PKEBC}j}))$;
5. Output $(\mathtt{rpk}_j, \mathtt{rsk}_j)$.

$\mathcal{O}_E(A_i, \vec{V}, m)$:

1. Let $\vec{v}_{\mathrm{PKEBC}}$ and $\vec{v}_{\mathrm{MDVS}}$ be, respectively, the vectors of public PKEBC keys and public MDVS verifier keys corresponding to party vector $\vec{V}$[12];
2. Let $\mathtt{spk}_i \leftarrow \mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_{SPK}(A_i)$;
3. $\sigma \leftarrow \mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_S(A_i, \mathrm{Set}(\vec{V}), (\vec{v}_{\mathrm{PKEBC}}, m))$;
4. Output $\mathit{\Pi}_{\mathrm{PKEBC}}.E(\mathtt{pp}_{\mathrm{PKEBC}}, \vec{v}_{\mathrm{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\mathrm{MDVS}}, m, \sigma))$.

$\mathcal{O}_D(B_j, c)$:

1. Let $\mathtt{sk}_{\mathrm{PKEBC}j}$ be as generated before for B_j;
2. Let $(\vec{v}_{\mathrm{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\mathrm{MDVS}}, m, \sigma)) \leftarrow \mathit{\Pi}_{\mathrm{PKEBC}}.D(\mathtt{sk}_{\mathrm{PKEBC}j}, c)$;
3. If $(\vec{v}_{\mathrm{PKEBC}}, (\mathtt{spk}_i, \vec{v}_{\mathrm{MDVS}}, m, \sigma)) = \bot$, output $\bot$;
4. $\mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_{Challenge}((\vec{v}_{\mathrm{PKEBC}}, m), \sigma)$;
5. Let $\mathcal{V}$ be the set of parties whose set of public PKEBC keys is the same as the set induced by $\vec{v}_{\mathrm{PKEBC}}$;
6. If $\mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_V(A_i, B_j, \mathcal{V}, (\vec{v}_{\mathrm{PKEBC}}, m), \sigma)$ outputs $\mathtt{invalid}$, output $\bot$;
7. Otherwise, output $(\mathtt{spk}_i, \vec{v}, m)$, where $\mathtt{spk}_i$ is the public key of A_i and $\vec{v}$ is the vector of public keys induced by $\vec{v}_{\mathrm{MDVS}}$ and $\vec{v}_{\mathrm{PKEBC}}$.

Noting that $\mathbf{A}$ only queries for at most $n_S \le n_{S\mathrm{MDVS}}$ (resp. $n_R \le n_{V\mathrm{MDVS}}$) different sender keys (resp. different receiver keys), makes up to $q_E \le q_{S\mathrm{MDVS}}$ queries to $\mathcal{O}_E$ and up to $q_D \le q_{V\mathrm{MDVS}}$ queries to $\mathcal{O}_D$, and the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \le d_{S\mathrm{MDVS}}$, it follows from Eq. (6.5), that $\mathbf{A}$ does not $(\varepsilon_{\mathrm{MDVS}\text{-}\mathsf{Unforg}}, t_{\mathrm{MDVS}})$-break the

$$(n_{S\mathrm{MDVS}}, n_{V\mathrm{MDVS}}, d_{S\mathrm{MDVS}}, q_{S\mathrm{MDVS}}, q_{V\mathrm{MDVS}})\text{-Unforgeability}$$

of $\mathit{\Pi}_{\mathrm{MDVS}}$, implying

$$Adv^{\mathsf{Unforg}}(\mathbf{A}) \le \varepsilon_{\mathrm{MDVS}\text{-}\mathsf{Unforg}}.$$

□

[12] Note that these vectors can be obtained by querying $\mathit{\Pi}_{\mathrm{MDVS}}.\mathcal{O}_{VPK}$ on each $V_i \in \vec{V}$

H.4 Proof of Theorem 9

Proof. We prove a stronger result. In the following, we consider an alternative $\mathsf{IND\text{-}CCA\text{-}2}$ security notion for MDRS-PKE schemes, whose only difference from Definition 9 is that now the adversary is allowed to query for the secret keys of the senders (i.e. it can query $\mathcal{O}_{SK}(A_i)$ and still win the game even if it makes a query $\mathcal{O}_E(A_i, \vec{V}, m_0, m_1)$).

The only difference between $\mathbf{G}_\mathbf{0}^{\mathsf{IND\text{-}CCA\text{-}2}}$ and $\mathbf{G}_\mathbf{1}^{\mathsf{IND\text{-}CCA\text{-}2}}$ is that $\mathbf{G}_\mathbf{0}^{\mathsf{IND\text{-}CCA\text{-}2}}$'s $\mathcal{O}_E$ oracle, on an input $((A_i, \vec{V}), m_0, m_1)$, outputs a ciphertext c that is an encryption of $(\mathtt{spk}_{\mathrm{MDVS}i}, \vec{v}_{\mathrm{MDVS}}, m_0, \sigma_0)$—where $\mathtt{spk}_{\mathrm{MDVS}i}$ is A_i's public signing MDVS key, $\vec{v}_{\mathrm{MDVS}}$ is the vector of public verifier MDVS keys corresponding to $\vec{V}$, and σ_0 is an MDVS signature of $(\vec{v}_{\mathrm{PKEBC}}, m_0)$ from A_i to $\vec{V}$, with $\vec{v}_{\mathrm{PKEBC}}$ being the vector of public PKEBC keys corresponding to the parties in $\vec{V}$—while $\mathbf{G}_\mathbf{1}^{\mathsf{IND\text{-}CCA\text{-}2}}$'s $\mathcal{O}_E$ oracle outputs an encryption of $(\mathtt{spk}_{\mathrm{MDVS}i}, \vec{v}_{\mathrm{MDVS}}, m_1, \sigma_1)$—where σ_1 is an MDVS signature of $(\vec{v}_{\mathrm{PKEBC}}, m_1)$ from A_i to $\vec{V}$.

Note that, since the MDRS-PKE's $\mathsf{IND\text{-}CCA\text{-}2}$ games do not provide the adversary with access to the $\mathcal{O}_{RK}$ oracle, the adversary cannot query for the secret keys of receivers. Thus, one can trivially reduce breaking the $\mathsf{IND\text{-}CCA\text{-}2}$ security of the MDRS-PKE scheme to breaking the $\mathsf{IND\text{-}CCA\text{-}2}$ security of the underlying PKEBC scheme. For instance, just consider a reduction that generates an MDVS signer key-pair for each sender, an MDVS verifier key-pair for each receiver, and then uses these key-pairs to answer $\mathcal{O}_{SK}$, $\mathcal{O}_{SPK}$ and $\mathcal{O}_{RPK}$ queries (for the case of $\mathcal{O}_{RPK}$ queries, the reduction also relies on the $\mathcal{O}_{PK}$ oracle of the underlying $\mathsf{IND\text{-}CCA\text{-}2}$ game for the PKEBC scheme). In the case of an $\mathcal{O}_E$ query, the reduction would simply generate an MDVS signature with the MDVS keys it generated on each of the input messages (σ_0 and σ_1 above), and then use the underlying $\mathcal{O}_E$ provided by the game system to generate the final ciphertext c as the encryption of one of $(\mathtt{spk}_{\mathrm{MDVS}i}, \vec{v}_{\mathrm{MDVS}}, m_0, \sigma_0)$ or $(\mathtt{spk}_{\mathrm{MDVS}i}, \vec{v}_{\mathrm{MDVS}}, m_1, \sigma_1)$. For $\mathcal{O}_D$ queries, the reduction would rely on the underlying $\mathcal{O}_D$ oracle provided by the $\mathsf{IND\text{-}CCA\text{-}2}$ game of the PKEBC scheme to obtain the pair $(\vec{v}_{\mathrm{PKEBC}}, (\mathtt{spk}_{\mathrm{MDVS}i}, \vec{v}_{\mathrm{MDVS}}, m_0, \sigma_0))$, which the reduction then would use to mimic the MDRS-PKE construction's D algorithm.

Finally, since $\mathbf{A}$ only queries for at most $n_R \leq n_{\mathrm{PKEBC}}$ different receiver public keys, the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{E\mathrm{PKEBC}}$, $\mathbf{A}$ makes up to $q_E \leq q_{E\mathrm{PKEBC}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{D\mathrm{PKEBC}}$ queries to $\mathcal{O}_D$, it follows from Eq. (6.6), that $\mathbf{A}$ does not $(\varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{IND\text{-}CCA\text{-}2}}, t_{\mathrm{PKEBC}})$-break the $(n_{\mathrm{PKEBC}}, d_{E\mathrm{PKEBC}}, q_{E\mathrm{PKEBC}}, q_{D\mathrm{PKEBC}})$-$\mathsf{IND\text{-}CCA\text{-}2}$ security of Π_{PKEBC}, implying

$$Adv^{\mathsf{IND\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\mathrm{PKEBC}\text{-}\mathsf{IND\text{-}CCA\text{-}2}}.$$

□

H.5 Proof of Theorem 10

Proof. Similarly to the proof of Theorem 9, we prove a stronger result. In the following, we consider an alternative $\mathsf{IK\text{-}CCA\text{-}2}$ security notion for MDRS-PKE schemes that only differs from Definition 10 in that the adversary is now allowed to query for the secret keys of the senders (i.e. it can query $\mathcal{O}_{SK}(A_i)$ and still win the game even if it makes a query $\mathcal{O}_E((A_{i,0}, \vec{V}_0), (A_{i,1}, \vec{V}_1), m)$ with $A_i \in \{A_{i,0}, A_{i,1}\}$).

This proof proceeds in a sequence of games [9, 32].

Game 1. This is the original $\mathbf{G}_{\mathbf{0}}^{\textsf{IK-CCA-2}}$ game from Definition 10.

Game 2. This game is just like Game 1, except that now, on an input $((A_{i,0}, \vec{V}_0), (A_{i,1}, \vec{V}_1), m)$, letting

$$\vec{v}_{\mathrm{PKEBC},0} \coloneqq (\mathtt{pk}_{\mathrm{PKEBC},0_1}, \ldots, \mathtt{pk}_{\mathrm{PKEBC},0_{|\vec{v}_0|}})$$

and

$$\vec{v}_{\mathrm{PKEBC},1} \coloneqq (\mathtt{pk}_{\mathrm{PKEBC},1_1}, \ldots, \mathtt{pk}_{\mathrm{PKEBC},1_{|\vec{v}_1|}})$$

be the vectors of PKEBC public keys corresponding to the vectors of parties $\vec{V}_0$ and $\vec{V}_1$, respectively, letting

$$\sigma_0 \leftarrow \mathit{\Pi}_{\mathrm{MDVS}}.\mathit{Sign}_{\mathtt{pp}_{\mathrm{MDVS}}}(\mathtt{ssk}_{\mathrm{MDVS},0_i}, \{\mathtt{vpk}_{\mathrm{MDVS},0_l}\}_{l\in\{1,\ldots,|\vec{v}_0|\}}, (\vec{v}_{\mathrm{PKEBC},0}, m)),$$

where $\mathtt{ssk}_{\mathrm{MDVS},0_i}$ is $A_{i,0}$'s secret MDVS signing key, and $\{\mathtt{vpk}_{\mathrm{MDVS},0_l}\}_{l\in\{1,\ldots,|\vec{v}_0|\}}$ is the set of public MDVS verifier keys of the parties in vector $\vec{V}_0$, $\mathcal{O}_E$ computes

$$c \leftarrow \mathit{\Pi}_{\mathrm{PKEBC}}.E_{\mathtt{pp}_{\mathrm{PKEBC}}}\big(\vec{v}_{\mathrm{PKEBC},1}, (\mathtt{spk}_{\mathrm{MDVS},0_i}, \vec{v}_{\mathrm{MDVS},0}, m, \sigma_0)\big),$$

using the vector $\vec{v}_{\mathrm{PKEBC},1}$ of PKEBC public keys corresponding to the vector of parties $\vec{V}_1$, instead of using the vector $\vec{v}_{\mathrm{PKEBC},0}$ of PKEBC public keys corresponding to the vector of parties $\vec{V}_0$, where

$$\vec{v}_{\mathrm{MDVS},0} \coloneqq (\mathtt{vpk}_{\mathrm{MDVS},0_1}, \ldots, \mathtt{vpk}_{\mathrm{MDVS},0_{|\vec{v}_0|}}).$$

Let

$$\begin{aligned} \mathit{Adv}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}}(\mathbf{A}) \coloneqq & \Big| \Pr[\mathbf{A}\mathbf{G}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}} = \mathtt{win}] \\ & + \Pr[\mathbf{A}\mathbf{G}_{\mathbf{1}}^{\textsf{IK-CCA-2}} = \mathtt{win}] - 1 \Big|, \end{aligned}$$

where the conditions for $\mathbf{A}$ to win Game 2 are the same as for winning the original $\mathbf{G}_{\mathbf{0}}^{\textsf{IK-CCA-2}}$ game from Definition 10. Since $\mathbf{A}$ only queries for at most $n_R \leq n_{\mathrm{PKEBC}}$ different receiver public keys, the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{E\mathrm{PKEBC}}$, $\mathbf{A}$ makes up to $q_E \leq q_{E\mathrm{PKEBC}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{D\mathrm{PKEBC}}$ queries to $\mathcal{O}_D$, it follows from Eq. (6.7), that $\mathbf{A}$ does not $(\varepsilon_{\mathrm{PKEBC}\text{-}\textsf{IK-CCA-2}}, t_{\mathrm{PKEBC}})$-break the $(n_{\mathrm{PKEBC}}, d_{E\mathrm{PKEBC}}, q_{E\mathrm{PKEBC}}, q_{D\mathrm{PKEBC}})$-IK-CCA-2 security of $\mathit{\Pi}_{\mathrm{PKEBC}}$, implying

$$\mathit{Adv}^{\textsf{IK-CCA-2}}(\mathbf{A}) \leq \varepsilon_{\mathrm{PKEBC}\text{-}\textsf{IK-CCA-2}} + \mathit{Adv}^{(\text{Game 2})\text{-}\textsf{IK-CCA-2}}(\mathbf{A}).$$

Game 3. This game is now the $\mathbf{G}_{\mathbf{1}}^{\mathsf{IK\text{-}CCA\text{-}2}}$ game from Definition 10. The only difference from Game 2 is that now, on an input $((A_{i,0}, \vec{V}_0), (A_{i,1}, \vec{V}_1), m)$, and letting

$$\vec{v}_{\text{PKEBC},1} \coloneqq (\mathtt{pk}_{\text{PKEBC},1_1}, \ldots, \mathtt{pk}_{\text{PKEBC},1_{|\vec{v}_1|}})$$

be the vector of PKEBC public keys corresponding to $\vec{V}_1$, letting

$$\sigma_1 \leftarrow \mathit{\Pi}_{\text{MDVS}}.\mathit{Sign}_{\mathtt{pp}_{\text{MDVS}}}(\mathtt{ssk}_{\text{MDVS},1_i}, \{\mathtt{vpk}_{\text{MDVS},1_l}\}_{l\in\{1,\ldots,|\vec{v}_1|\}}, (\vec{v}_{\text{PKEBC},1}, m)),$$

where $\mathtt{ssk}_{\text{MDVS},1_i}$ is $A_{i,1}$'s secret MDVS signing key, and $\{\mathtt{vpk}_{\text{MDVS},1_l}\}_{l\in\{1,\ldots,|\vec{v}_1|\}}$ is the set of MDVS verifier public keys of the parties in vector $\vec{V}_1$, $\mathcal{O}_E$ outputs

$$c \leftarrow \mathit{\Pi}_{\text{PKEBC}}.E_{\mathtt{pp}_{\text{PKEBC}}}\big(\vec{v}_{\text{PKEBC},1}, (\mathtt{spk}_{\text{MDVS},1_i}, \vec{v}_{\text{MDVS},1}, m, \sigma_1)\big),$$

where

$$\vec{v}_{\text{MDVS},1} \coloneqq (\mathtt{vpk}_{\text{MDVS},1_1}, \ldots, \mathtt{vpk}_{\text{MDVS},1_{|\vec{v}_1|}})$$

is the vector of public MDVS verifier keys corresponding to vector of parties $\vec{V}_1$.

Since $\mathbf{A}$ only queries for at most $n_R \leq n_{\text{PKEBC}}$ different receiver public keys, the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{E\text{PKEBC}}$, $\mathbf{A}$ makes up to $q_E \leq q_{E\text{PKEBC}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{D\text{PKEBC}}$ queries to $\mathcal{O}_D$, it follows from Eq. (6.7), that $\mathbf{A}$ does not $(\varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}}, t_{\text{PKEBC}})$-break the $(n_{\text{PKEBC}}, d_{E\text{PKEBC}}, q_{E\text{PKEBC}}, q_{D\text{PKEBC}})$-$\mathsf{IND\text{-}CCA\text{-}2}$ security of $\mathit{\Pi}_{\text{PKEBC}}$, implying

$$\mathit{Adv}^{(\text{Game 2})\text{-}\mathsf{IK\text{-}CCA\text{-}2}}(\mathbf{A}) \leq \varepsilon_{\text{PKEBC-}\mathsf{IND\text{-}CCA\text{-}2}}.$$

□

H.6 Proof of Theorem 11

Proof. The only difference between $\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$ and $\mathbf{G}_{\mathbf{1}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$ is that, on an input $(\mathtt{type}, A_i, \vec{V}, m, \mathcal{D})$, $\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$'s $\mathcal{O}_E$ oracle creates fresh ciphertexts if $\mathtt{type} = \mathtt{sign}$ and creates forged ciphertexts if $\mathtt{type} = \mathtt{forge}$, whereas $\mathbf{G}_{\mathbf{1}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$'s $\mathcal{O}_E$ oracle always creates forged ciphertexts using *Forge*. Note that, by the definition of $\mathit{\Pi}$'s E algorithm, and by the definition of *Forge* (see Algorithm 4), the only difference between a fresh ciphertext created by $\mathit{\Pi}$'s E algorithm and a forged one created by *Forge* is that the signature σ of the quadruple $(\mathtt{spk}_i, \vec{v}_{\text{MDVS}}, m, \sigma)$ that is encrypted by the PKEBC scheme is a real signature in the first case, and a forged signature for the latter. This means that being able to distinguish a real ciphertext as output by $\mathbf{G}_{\mathbf{0}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$'s $\mathcal{O}_E(\mathtt{sign}, \cdot, \cdot, \cdot, \cdot)$ from a forged one as output by $\mathbf{G}_{\mathbf{a}}^{\mathsf{OTR}\text{-}\mathit{Forge}}$'s $\mathcal{O}_E(\mathtt{sign}, \cdot, \cdot, \cdot, \cdot)$ implies being able to distinguish if the (encrypted) MDVS signature σ is a forged one or a real one.

To conclude the proof, since $\mathbf{A}$ only queries for at most $n_S \leq n_{S\text{MDVS}}$ (resp. $n_R \leq n_{V\text{MDVS}}$) different sender public keys (resp. different receiver keys), the sum of lengths of the party vectors input to $\mathcal{O}_E$ is at most $d_E \leq d_{S\text{MDVS}}$, $\mathbf{A}$ makes up to $q_E \leq q_{S\text{MDVS}}$ queries to $\mathcal{O}_E$ and up to $q_D \leq q_{V\text{MDVS}}$ queries

Algorithm 4 *Forge* algorithm for the construction given in Algorithm 2. In the following, let $\mathit{\Pi}_{\text{MDVS}}$ and $\mathit{\Pi}_{\text{PKEBC}}$ respectively be the MDVS and PKEBC schemes underlying the construction given in Algorithm 2, $\mathit{Forge}_{\text{MDVS}}$ be a signature forging algorithm for $\mathit{\Pi}_{\text{MDVS}}$, and $\{\mathtt{rsk}_{j'}\}_{B_{j'} \in \mathcal{D}}$ be the set of secret receiver keys of $\mathcal{D}$, the set of dishonest parties.

```
$\mathit{Forge}_{\mathsf{pp}}(\mathtt{spk}_i, \vec{v}, m, \{\mathtt{rsk}_{j'}\}_{B_{j'} \in \mathcal{D}})$
  **With**
    $\mathtt{pp} := (\mathtt{pp}_{\text{MDVS}}, \mathtt{pp}_{\text{PKEBC}})$
    $\mathtt{spk}_i := \mathtt{spk}_{\text{MDVS}\,i}$
    **for each** $\mathtt{rsk}_j \in \{\mathtt{rsk}_{j'}\}_{B_{j'} \in \mathcal{D}}$
      $\mathtt{rsk}_j := \big((\mathtt{vpk}_{\text{MDVS}\,j}, \mathtt{pk}_{\text{PKEBC}\,j}), (\mathtt{vsk}_{\text{MDVS}\,j}, \mathtt{sk}_{\text{PKEBC}\,j})\big)$
    $\vec{v} := \big(\mathtt{rpk}_1, \ldots, \mathtt{rpk}_{|\vec{v}|}\big)$
    **for each** $i \in \{1, \ldots, |\vec{v}|\}$
      $\mathtt{rpk}_i = (\mathtt{vpk}_{\text{MDVS}\,i}, \mathtt{pk}_{\text{PKEBC}\,i})$

  $\vec{v}_{\text{PKEBC}} \leftarrow (\mathtt{pk}_{\text{PKEBC}\,1}, \ldots, \mathtt{pk}_{\text{PKEBC}\,|\vec{v}|})$
  $\vec{v}_{\text{MDVS}} \leftarrow (\mathtt{vpk}_{\text{MDVS}\,1}, \ldots, \mathtt{vpk}_{\text{MDVS}\,|\vec{v}|})$
  $\sigma_{\text{MDVS}} \leftarrow \mathit{Forge}_{\text{MDVS}_{\mathsf{pp}_{\text{MDVS}}}}(\mathtt{spk}_{\text{MDVS}\,i}, \text{Set}(\vec{v}_{\text{MDVS}}), (\vec{v}_{\text{PKEBC}}, m), \{\mathtt{vsk}_{\text{MDVS}\,j'}\}_{B_{j'} \in \mathcal{D}})$
  **return** $\mathit{\Pi}_{\text{PKEBC}}.E_{\mathsf{pp}_{\text{PKEBC}}}\big(\vec{v}_{\text{PKEBC}}, (\mathtt{spk}_{\text{MDVS}\,i}, \vec{v}_{\text{MDVS}}, m, \sigma_{\text{MDVS}})\big)$
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to $\mathcal{O}_D$, it follows from Eq. (6.8), that $\mathbf{A}$ does not $(\varepsilon_{\text{MDVS-}\mathsf{OTR}}, t_{\text{MDVS}})$-break the $(n_{S\text{MDVS}}, n_{V\text{MDVS}}, d_{S\text{MDVS}}, q_{S\text{MDVS}}, q_{V\text{MDVS}})$-Off-The-Record security of $\mathit{\Pi}_{\text{MDVS}}$ with respect to $\mathit{Forge}_{\text{MDVS}}$, implying

$$\mathit{Adv}^{\mathsf{OTR}\text{-}\mathit{Forge}}(\mathbf{A}) \leq \varepsilon_{\text{MDVS-}\mathsf{OTR}}.$$

□