

Algebraic isomorphic spaces of ideal lattices, reduction of Ring-SIS problem, and new reduction of Ring-LWE problem

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Abstract

The main focus of this article is on an open problem, namely the Ring-SIS reduction problem. We first utilize a spatial isomorphism approach to reduce the Ring-SIS problem to the classic SIS problem in lattices, indirectly reducing it to the classic SIVP in lattices. This provides theoretical assurance to some extent for the difficulty and resistance against quantum attacks of the Ring-SIS problem.

Additionally, we reduce the Ring-LWE problem to the Ring-SIS problem, which guarantees the security of encryption schemes based on Ring-LWE to a certain degree. Finally, this article proves that the difficulty of the Ring-SIS problem and the Ring-LWE problem is relatively average with respect to the spatial dimension or polynomial degree.

Keywords: reduction of the ring-SIS problem; shortest trapdoor in ideal lattices; ring-LWE problem; univariate polynomial ring; SIVP.

1 Introduction

The main research problem of this article is to reduce the Ring-SIS problem [1] to the SIS problem, and the Ring-SIS problem refers to $f_1, \dots, f_m \in \mathcal{R}_q$, where \mathcal{R}_q is a polynomial ring with modulus q , find m polynomials g_1, \dots , whose coefficients are not all 0, $g_m \in \mathcal{R}_{\{0,1\}}$, such that

$$f_1g_1 + \dots + f_mg_m = 0 \text{ mod } q\mathcal{R}.$$

Currently, lattice cryptography is an important research field in post-quantum cryptography. In 2005, Regev completed the reduction work of the learning with error problem (LWE), reducing it to a difficult problem in the classic lattice, that is, the closet vector problem and the shortest vector problem [2]. Regev's work ensured the theoretical foundation of lattice cryptography. In 1996, Ajtai gave a new lattice difficulty problem, namely the shortest integer problem (SIS, [3]). Subsequently, Micciancio and Peikert gave a more concise conclusion and combined it with the LWE problem to serve as a one-way trapdoor function for the encryption scheme based on the LWE problem [4]. Moreover, the SIS problem itself provides the security guarantee of the trapdoor function. This achievement makes the world of lattice cryptography more dynamic.

With in-depth research on encryption schemes based on the LWE problem, everyone found that the computational overhead of this type of scheme is not very ideal. Therefore, it is hoped that there will be

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a new difficult problem that can inherit the difficulty of LWE and at the same time ensure the operating efficiency of the solution. In 2010, Lyubashevsky, Peikert, and Regev proposed a learning with error problem on polynomial rings, namely the Ring-LWE problem [1], establishing an isomorphic relationship with the ideal lattice. And reduce it to the bounded coding problem on the ideal lattice (Ideal-BDD). The Ring-LWE problem is similar in form to the LWE problem, and its calculation time is much less than that of the LWE problem.

However, the Ideal-BDD problem itself is also a variant of the BDD problem, and the reduction work has not yet been completed. Therefore, simply reducing the Ring-LWE problem to the Ideal-BDD problem may not necessarily explain the difficulty. In addition, similar to the encryption scheme based on LWE, the encryption scheme based on Ring-LWE also requires the Ring-SIS problem as a trapdoor function, and Lyubashevsky, Peikert, and Regev mentioned “Indeed, the perspectives and techniques that have so far been employed for the Ring-SIS problem appear insufficient for adapting the more involved hardness proofs for LWE to the ring setting” in the article [1]. Steven Yue mentioned on the Zhihu website that the Ring-SIS problem is still an open problem [4]. Therefore, the reduction of the Ring-SIS problem is an urgent problem that needs to be solved in current lattice cryptography.

1.1 Our work

The main work of this paper is to reduce the Ring-SIS problem to the SIS problem. Similar to the article by Micciancio and Peikert [4], we also divide the article into three parts, namely Ring-SIS to Ring-SIS Reduction, Direct Reduction and Ring-SIS to Ring-LWE Reduction.

1.1.1 Ring-SIS to Ring-SIS Reduction

Let Ring-SIS(m, n, q, β) be a problem, where m, n are positive integers, q is a prime number, and β is a positive number. This part is divided into two steps, namely

1. When there is an oracle that can solve Ring-SIS(m, n, q, β) in polynomial time, there is also an efficient algorithm that can solve Ring text-SIS($m' = tm, n, q, \beta$), and the number of times the algorithm queries the oracle is t times;
2. When there is an oracle that can solve Ring-SIS(m, n, q, β) in polynomial time, there is also an efficient algorithm that can solve Ring text-SIS($m, n' = tn, q, \beta$), and the number of times this algorithm queries the oracle is $m(t - 1) + 1$ times;

For **first step**, we can divide it according to the number of polynomials in Ring-SIS($m' = tm, n, q, \beta$). More specifically, $f_1, \dots, f_{m'} \in \mathcal{R}_q$ is divided into t parts, each part is exactly Ring-SIS(m, n, q, β), at this time, use the Ring-SIS(m, n, q, β) oracle to solve its sutras. The solution for each part together, it is the solution of Ring-SIS($m' = tm, n, q, \beta$). At this time, we need to ask t times Ring-SIS(m, n, q, β) oracle.

For **second step**, according to the properties of polynomials, that is, when $f(x)g(x) = (a_0 + a_1x + \dots + a_{n-1}x^{n-1})(b_0 + b_1x + \dots + b_{n-1}x^{n-1}) \bmod (x^n + 1) = 0$, then there is $f_{t,k}(x)g_{t,k}(x) = (a_0x^k + a_1x^{k+t} + \dots + a_{n-1}x^{k+(n-1)t})(b_0x^k + b_1x^{k+t} + \dots + b_{n-1}x^{k+(n-1)t}) \bmod (x^{nt} + 1) = 0$, where $k \in [0, t)$.

According to this property, we have obtained a very important theorem, that is, when $F^T(x) := (f^{(1)}(x), f^{(2)}(x), \dots, f^{(m)}(x))$, $G(x) := (g^{(1)}(x), g^{(2)}(x), \dots, g^{(m)}(x))$, if there is

$$F^T(x)G(x) = \sum_i^m f^{(i)}(x)g^{(i)}(x) \bmod (x^n + 1) = 0,$$

then there is

$$F_{t,k}^T(x)G_{t,k}(x) = \sum_l^m f_{t,k}^{(l)}(x)g_{t,k}^{(l)}(x) \bmod (x^{nt} + 1) = 0$$

for $F_{t,k}^T(x) := (f_{t,k}^{(1)}(x), f_{t,k}^{(2)}(x), \dots, f_{t,k}^{(m)}(x))$, $G(x) := (g_{t,k}^{(1)}(x), g_{t,k}^{(2)}(x), \dots, g_{t,k}^{(m)}(x))$. Among them, $k \in [0, t)$. This conclusion is very important. According to this conclusion, we can extract an n order polynomial in Ring-SIS($m, n' = tn, q, \beta$), thus forming a Ring-SIS(m, n, q, β) problem. We can get the solution to a part of the Ring-SIS($m, n' = tn, q, \beta$) problem that is divided into parts of the Ring-SIS(m, n, q, β) problem by asking the Ring-SIS(m, n, q, β) oracle m times.

According to the above properties, it can be seen that after the first query m times Ring-SIS(m, n, q, β) oracle gets the solution, it will be compared with Ring-SIS($m, n' = tn, q, \beta$) problem, the n' terms of the $n' - 1$ degree polynomial of the original problem are reduced by n terms, because we asked m times Ring-SIS(m, n, q, β) oracle, so after action, it is still Ring-SIS($m, n' = tn, q, \beta$) problem, except that each polynomial is missing n terms.

Repeat this method there are only n terms left in each polynomial of the Ring-SIS($m, n' = tn, q, \beta$) problem, and we can directly use Ring-SIS(m, n, q, β) is solved by the oracle. Combining these solutions is the final solution to the Ring-SIS($m, n' = tn, q, \beta$) problem. Then a total of $m(tn - n)/n + 1 = m(t - 1) + 1$ times are asked Ring-SIS(m, n, q, β) oracle.

1.1.2 Direct Reduction

We reduce the Ring-SIS problem to the SIS problem by establishing an isomorphic relationship between the two, thus proving the difficulty of the Ring-SIS problem. However, we cannot directly reduce the Ring-SIS problem to the SIS problem, so we thought of a way. We consider the variants of the SIS problem and the Ring-SIS problem, that is, the one-dimensional SIS problem and the Ring-SIS $_{|x=0}$ problem. The so-called one-dimensional SIS problem refers to $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{M}_{mn \times 1}(\mathbb{Z}_q)$, $\alpha_i^T \in \mathcal{M}_{n \times 1}(\mathbb{Z}_q)$, find $z = (z_1, z_2, \dots, z_n) \in \mathcal{M}_{1 \times mn}(\mathbb{Z}_q)$, $z_i \in \mathcal{M}_{1 \times n}(\mathbb{Z}_q)$, $\|z\| \leq \beta$, such that

$$z\alpha = \sum_i^m z_i \alpha_i = (z_1, z_2, \dots, z_m)(\alpha_1, \alpha_2, \dots, \alpha_m)^T = 0.$$

Where $\mathcal{M}_{m \times n}(\mathbb{Z}_q)$ represents a matrix of order $m \times n$ and the coefficients are elements in $\mathbb{Z}_q[5]$. The corresponding Ring-SIS $_{|x=0}$ problem refers to $f_1, \dots, f_m \in \mathcal{R}_q$, where \mathcal{R}_q is a polynomial ring with modulus q , find a set of polynomials $g_1, \dots, g_m \in \mathcal{R}_{\{0,1\}}$ whose coefficients are not all 0, such that

$$f_1 g_1 + \dots + f_m g_m|_{x=0} = 0 \text{ mod } q\mathcal{R}.$$

The purpose of this is to establish an isomorphic relationship between the Ring-SIS $_{|x=0}$ problem and the one-dimensional SIS problem.

When the isomorphism relationship is established, we follow the conclusion of Lemma 16, that is, if space A and space B are isomorphic, then if for \mathcal{A} in space A is an abstract hard problem if and only if \mathcal{B} in the corresponding space B is also a hard problem. When the isomorphic relationship between the Ring-SIS $_{|x=0}$ problem and the one-dimensional SIS problem is established, we assume that there are collision-resolving oracles for the two, and obtain the collision-resolving oracle of the Ring-SIS $_{|x=0}$ problem that needs to be asked to solve the Ring-SIS problem. The number of times the oracle needs to be asked and the number of times the one-dimensional SIS question that needs to be asked to crack the SIS problem collides with the oracle. In the end, the number of times the two are obtained is ‘‘roughly the same’’ in terms of probability, so this illustrates the connection between the Ring-SIS problem and the SIS problem.

Although the demonstration of this proof in this article may not be sufficient, one thing we are sure of is that the one-dimensional SIS problem is difficult, and the difficulty of the Ring-SIS problem is not lower than that of the one-dimensional SIS problem, but not higher than that of the SIS problem.

1.1.3 Ring-SIS to Ring-LWE Reduction

Let $n \geq 1$, and $p = p(n) \leq \text{poly}(n)$ be prime numbers, now consider a set of ‘equations of with error’

$$\begin{aligned} \langle s, a_1 \rangle &\approx_{\chi} b_1 \pmod{q}, \\ \langle s, a_2 \rangle &\approx_{\chi} b_2 \pmod{q}, \\ &\vdots \\ \langle s, a_n \rangle &\approx_{\chi} b_n \pmod{q}. \end{aligned}$$

Among them, $s \in \mathcal{M}_{m \times 1}(\mathbb{Z}_q)$, a_i is in $\mathcal{M}_{m \times 1}(\mathbb{Z}_q)$ independently selected uniformly, $b_i \in \mathbb{Z}_q$. There is a perturbation $e_i \in_R \chi \subset \mathbb{Z}_q$ in the above equation such that for each i , there are $b_i = \langle s, a_i \rangle + e_i$. We put these n ‘equations of with error’ together and get

$$b = As + e, \text{ here } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, A = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix}, e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}. \quad (1)$$

The learning with error problem $\text{LWE}_{q,\chi}$ refers to finding s from (1). This problem occupies a very important position in the hardness assumptions of lattice cryptography, so that various variants based on the LWE problem have been proposed accordingly. For example, the learning with rounding problem [6], the Evasive learning with error problem [7], and the learning parity with noise problem [4] etc.

In 2005, Regev gave a proof of the difficulty of the LWE problem. So for the connection between the LWE problem and the SIS problem [8, 9], we have that when there is an oracle that can solve the LWE problem within a time polynomial, there is also an effective algorithm that can solve the SIS problem. This is because for the LWE problem $b = As + e$, where $A \in \mathcal{M}_{m \times n}(\mathbb{Z}_q)$, $s \in \mathcal{M}_{m \times 1}(\mathbb{Z}_q)$, and $e_i \in \chi$. So we can use the collision-resolving oracle of the LWE problem to query $b - e = As$ twice, and get $As_1 = b - e = As_2$, so we have $Au = A(s_1 - s_2) = 0$, at this time u is the solution to the SIS problem.

Then the same relationship exists between Ring-SIS and Ring-LWE. The difference is that the Ring-LWE problem is only ‘one-dimensional’, that is, finding the polynomial s from $b = as + e$, where $a, s \in \mathcal{R}_q$, and $x \leftarrow \chi$. The Ring-SIS problem is to solve m polynomials. In fact, the encryption scheme based on the Ring-LWE problem also requires multiple $b = as + e$ to set the public key and private key, so the relationship between m Ring-LWE problems and Ring-SIS is the same as the LWE problem and SIS. The relationship between the problems is consistent.

2 Preliminaries

Lattice. Each element of a lattice in \mathbb{R}^n can be expressed linearly by n linearly independent vector integer coefficients. This set of linearly independent vectors is called a lattice basis, and we know that the lattice basis is not unique. Given a set of lattice bases (v_1, \dots, v_n) in the lattice \mathcal{L} , then the fundamental parallelepiped is

$$\mathcal{P}(v_1, \dots, v_n) = \left\{ \sum_{i=1}^n k_i v_i \mid k_i \in [0, 1) \right\}.$$

If the lattice base (v_1, \dots, v_n) is determined, we can use the symbol $\mathcal{P}(\mathcal{L})$ to replace $\mathcal{P}(v_1, \dots, v_n)$. $\forall x \in \mathbb{R}^n$, we can project it onto $\mathcal{P}(\mathcal{L})$. According to the properties of projection [10], there is a unique $y \in \mathcal{P}(\mathcal{L})$ makes $y - x \in \mathcal{L}$. We use the symbol $\det(\mathcal{L})$ to represent the volume of the fundamental parallelepiped of the lattice \mathcal{L} . In other words, the symbol $\det(\mathcal{L})$ represents the determinant of a matrix composed of a set of lattice bases (v_1, \dots, v_n) . For a given n dimensional lattice, the $\det(\mathcal{L})$ size of any set of lattice bases of the lattice is constant. We simply prove this theorem.

Given n lattice \mathcal{L} , (v_1, \dots, v_n) and (u_1, \dots, u_n) are two arbitrary groups of lattice \mathcal{L} respectively lattice bases. Therefore we have $v_i = \sum_{j=1}^n m_{ij}u_j$ and $u_i = \sum_{j=1}^n m'_{ij}v_j, i \in \{1, \dots, n\}$, therefore there are two integer matrices M and M' such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = M \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = M' \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

It is easy to prove that M and M' are inverse to each other, and M and M' are both integer matrices, so there are $\det(M) \det(M') = 1$ and $\det(M) = \det(M') = \pm 1$, so $\det(v_1, \dots, v_n) = \pm \det(u_1, \dots, u_n)$.

Isomorphic mapping of polynomial $\mathbb{Z}[x]/\langle x^n + 1 \rangle$ to ideal lattice \mathcal{I} .

Definition 1. An ideal lattice is a subset of rings or domains that satisfies the following two properties:

1. *Additive closure: If any two elements in the ideal are added, the result is still in the ideal. In other words, for any elements a and b in the ideal, $a + b$ also belongs to that ideal.*
2. *Multiplicative absorptivity: If an element in the ideal is multiplied by any element in the ring (or field), the result is still in the ideal. In other words, for any element a in the ideal and any element r in the ring (or field), ar and ra belong to that ideal.*

For a commutative ring, we can further require that the ideal be closed for both addition and multiplication. Such an ideal is called a true ideal.

Definition 2. Referring to the definition of ideal, the ideal lattice \mathcal{I} is a subset of the lattice \mathcal{L} that satisfies the following two properties:

1. *Additive closure: If any two elements in an ideal lattice are added, the result is still in the ideal lattice. In other words, for any elements a and b in an ideal lattice, $a + b$ also belongs to that ideal lattice.*
2. *Multiplicative absorptivity: If an element in an ideal lattice is multiplied by an element in any other ideal lattice, the result remains in the ideal lattice. In other words, for any element a in the ideal and any element r in another ideal lattice, both ar and ra belong to that ideal lattice.*

Corollary 3. The ideal lattice \mathcal{I} is a true idea of the lattice \mathcal{L} .

For $f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ is mapped to

$$\text{Rot}(f) = a_0I + a_1X + \dots + a_{n-1}X^{n-1} \in \tilde{\mathcal{R}}.$$

Among them, $\tilde{\mathcal{R}}$ is the mapping of all $\mathbb{Z}[x]/\langle x^n + 1 \rangle$ to the elements in the ideal lattice \mathcal{I} collection, and

$$X = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

So there is

$$\text{Rot}(f) = \begin{pmatrix} a_0 & -a_{n-1} & \cdots & -a_1 \\ a_1 & a_0 & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix},$$

it is easy to prove that this mapping relationship is isomorphic.

SIS problem [3, 11]. Given the integers n, m, q and the positive number β . The shortest integer solution problem is to randomly select vector $\alpha_i \in \mathcal{M}_{n \times 1}(\mathbb{Z}_q)$, $m \in \{1, \dots, m\}$. The matrix $A \in \mathcal{M}_{n \times m}(\mathbb{Z}_q)$, find the non-zero integer coefficient vector $z \in \mathcal{M}_{m \times 1}(\mathbb{Z}_q)$, $\|z\| \leq \beta$, such that

$$f_A(z) := Az = \sum_i^m \alpha_i z_i = 0 \in \mathbb{Z}_q^n.$$

Given the lattice \mathcal{L} , the representation of the SIS problem on the lattice is

$$\mathcal{L}^\perp(A) = \{z \in \mathbb{Z}^m : Az = 0 \in \mathbb{Z}_q^n\}.$$

A variant of the SIS problem

$$\mathcal{L}_u^\perp(A) = \{z \in \mathbb{Z}^m : Az = u \in \mathbb{Z}_q^n\} = c + \mathcal{L}^\perp(A).$$

Among them, c is the solution of any non-homogeneous SIS, that is, $Ac = u$. The variant of the SIS problem are usually used to construct the one-way trapdoor function of encryption schemes.

Ring-SIS problem [12, 1]. In $\mathbb{Z}[x]/\langle f(x) \rangle$, $f(x) = x^n - 1$, the Ring-SIS problem is not difficult. The reason is that $f(x) = x^n - 1$ is reducible, that is

$$x^n - 1 = (1 - x)(1 + x + x^2 + \dots + x^{n-1}).$$

We let $\tilde{g}(x) := 1 + x + x^2 + \dots + x^{n-1} \in \mathbb{Z}[x]/\langle x^n - 1 \rangle$, so there is

$$(1 - x)\tilde{g}(x) = x^n - 1 = 0. \quad (2)$$

On the other hand, for the Ring-SIS problem $\mathcal{F}(x) = (f_1(x), f_2(x), \dots, f_m(x))$. That is, find $\mathcal{G}(x) = (g_1(x), g_2(x), \dots, g_m(x))$, such that

$$\mathcal{F}(x)\mathcal{G}(x) = \sum_{i=1}^m f_i g_i = 0.$$

We let $\mathcal{G}(x) = (\tilde{g}(x), 0, \dots, 0)$, if for the solution of $\mathcal{F}(x)$ is $\mathcal{G}(x)$, only $f_1(x)\tilde{g}(x) = 0 \pmod{q\mathcal{R}}$. So what kind of $\tilde{g}(x)$ can satisfy this condition? In fact, we assume that $f_1(x)$ is a multiple of the polynomial $x - 1$, that is, $f_1(x) = f'(x)(x - 1)$ then there is

$$f_1(x)\tilde{g}(x) = f'(x)(x - 1)\tilde{g}(x) = 0 \pmod{q\mathcal{R}}.$$

In other words, as long as $f_1(x) = f'(x)(x - 1)$ is satisfied, it is the solution of $\mathcal{F}(x)$. So what is the probability of this happening? We have the following lemma.

Lemma 4. *If $f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ is a multiple of $(x - 1)$, then $\sum_{i=0}^{n-1} a_i = 0$.*

Proof. We use mathematical induction. When $n = 1$, if $f(x) = f'(x)(x - 1)$, then $f'(x) = \lambda \in \mathbb{Z}_q$. At this time, there is

$$f(x) = \lambda - \lambda x = a_0 + a_1x,$$

so there is $a_0 + a_1 = 0$. Assume that it is true when $n = k$. When $n = k + 1$, we assume that $f'(x) = b_0 + b_1x + \dots + b_kx^k$ and

$$\begin{aligned} f(x) &= f'(x)(x - 1) = (b_0 + b_1x + \dots + b_kx^k)(x - 1) \\ &= (b_0 + b_1x + \dots + b_{k-1}x^{k-1})(x - 1) + b_kx^k(x - 1) \\ &= \underbrace{a_0 + a_1x + \dots + a_kx^k}_{(a)} + \underbrace{b_kx^k(x - 1)}_{(b)}. \end{aligned}$$

Because it is true when $k = n$, then the sum of the coefficients of the (a) equation is 0, and it is easy to prove that the sum of the coefficients of the (b) equation is also 0. Therefore, the proposition is true. \square

Lemma 5 ([12]). *If $f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in \mathbb{Z}_q[x]$, then the probability of $\sum_{i=0}^{n-1} a_i = 0$ occurring is $1/q$.*

Since the first $n - 1$ coefficients are all random numbers in the integer ring \mathbb{Z}_q , so $\sum_{i=0}^{n-2} a_i$ is also in the integer ring \mathbb{Z}_q random number. Randomly select a_{n-1} , then the probability of satisfying $\sum_{i=0}^{n-1} a_i = 0$ is $1/q$. The cracking probability of $1/q$ is very large for password security, so the Ring-SIS problem of polynomial $\mathbb{Z}[x]/\langle f(x) \rangle, f(x) = x^n - 1$ is not difficult for the security of the password scheme.

Lemma 6. *If $f(x)g(x) = (a_0 + a_1x + \dots + a_{n-1}x^{n-1})(b_0 + b_1x + \dots + b_{n-1}x^{n-1}) \bmod (x^n + 1) = 0$, then there is $f_{t,k}(x)g_{t,k}(x) = (a_0x^k + a_1x^{k+t} + \dots + a_{n-1}x^{k+(n-1)t})(b_0x^k + b_1x^{k+t} + \dots + b_{n-1}x^{k+(n-1)t}) \bmod (x^{nt} + 1) = 0$, where $k \in [0, t)$.*

Proof. According to the conditions, because $f(x)g(x) = (a_0 + a_1x + \dots + a_{n-1}x^{n-1})(b_0 + b_1x + \dots + b_{n-1}x^{n-1}) \bmod (x^n + 1) = 0$, so there is

$$\begin{aligned} & \left(a_0b_0 + (-1) \sum_{j=1}^{n-1} a_j b_{n-j} \right) \\ + & \left(\sum_{i=1}^2 a_{i-1} b_{2-i} + (-1) \sum_{j=2}^{n-1} a_j b_{n+1-j} \right) x \\ + & \left(\sum_{i=1}^3 a_{i-1} b_{3-i} + (-1) \sum_{j=3}^{n-1} a_j b_{n+2-j} \right) x^2 \quad \bmod (x^n + 1) = 0. \\ & \vdots \\ + & \left(\sum_{i=1}^n a_{i-1} b_{n-i} \right) x^{n-1} \end{aligned}$$

So we get

$$\begin{aligned} & \left(a_0b_0 + (-1) \sum_{j=1}^{n-1} a_j b_{n-j} \right) x^{2k} \bmod (x^{nt} + 1) \\ + & \left(\sum_{i=1}^2 a_{i-1} b_{2-i} + (-1) \sum_{j=2}^{n-1} a_j b_{n+1-j} \right) x^{2k+t} \bmod (x^{nt} + 1) \\ + & \left(\sum_{i=1}^3 a_{i-1} b_{3-i} + (-1) \sum_{j=3}^{n-1} a_j b_{n+2-j} \right) x^{2k+2t} \bmod (x^{nt} + 1) = 0. \\ & \vdots \\ + & \left(\sum_{i=1}^n a_{i-1} b_{n-i} \right) x^{2k+(n-1)t} \bmod (x^{nt} + 1) \end{aligned}$$

□

Corollary 7. *Let $F^T(x) := (f^{(1)}(x), f^{(2)}(x), \dots, f^{(m)}(x))$, $G(x) := (g^{(1)}(x), g^{(2)}(x), \dots, g^{(m)}(x))$, if there is*

$$F^T(x)G(x) = \sum_i^m f^{(i)}(x)g^{(i)}(x) \bmod (x^n + 1) = 0.$$

Then for $F_{t,k}^T(x) := (f_{t,k}^{(1)}(x), f_{t,k}^{(2)}(x), \dots, f_{t,k}^{(m)}(x))$, $G(x) := (g_{t,k}^{(1)}(x), g_{t,k}^{(2)}(x), \dots, g_{t,k}^{(m)}(x))$, there are

$$F_{t,k}^T(x)G_{t,k}(x) = \sum_l^m f_{t,k}^{(l)}(x)g_{t,k}^{(l)}(x) \bmod (x^{nt} + 1) = 0.$$

Among them, $k \in [0, t)$.

Proof. Let the l th component (polynomial) of $F^T(x)$ be

$$f^{(l)}(x) := a_0^{(l)} + a_1^{(l)}x + \dots + a_{n-1}^{(l)}x^{n-1}.$$

So we get

$$\begin{aligned} F^T(x)G(x) &= \sum_l^n f^{(l)}(x)g^{(l)}(x) \\ &= \sum_l^m \left(a_0^{(l)}b_0^{(l)} + (-1) \sum_{j=1}^{n-1} a_j^{(l)}b_{n-j}^{(l)} \right) \\ + & \sum_l^m \left(\sum_{i=1}^2 a_{i-1}^{(l)}b_{2-i}^{(l)} + (-1) \sum_{j=2}^{n-1} a_j^{(l)}b_{n+1-j}^{(l)} \right) x \\ + & \sum_l^m \left(\sum_{i=1}^3 a_{i-1}^{(l)}b_{3-i}^{(l)} + (-1) \sum_{j=3}^{n-1} a_j^{(l)}b_{n+2-j}^{(l)} \right) x^2 \quad \bmod (x^n + 1) = 0. \\ & \vdots \\ + & \sum_l^m \left(\sum_{i=1}^n a_{i-1}^{(l)}b_{n-i}^{(l)} \right) x^{n-1} \end{aligned}$$

And we have

$$\begin{aligned}
F_{t,k}^T(x)G_{t,k}(x) &= \sum_l^m f_{t,k}^{(l)}(x)g_{t,k}^{(l)}(x) \bmod (x^{nt} + 1) \\
&= \sum_l^m \left(a_0^{(l)}b_0^{(l)} + (-1) \sum_{j=1}^{n-1} a_j^{(l)}b_{n-j}^{(l)} \right) x^{2k} \bmod (x^{nt} + 1) \\
&+ \sum_l^m \left(\sum_{i=1}^2 a_{i-1}^{(l)}b_{2-i}^{(l)} + (-1) \sum_{j=2}^{n-1} a_j^{(l)}b_{n+1-j}^{(l)} \right) x^{2k+t} \bmod (x^{nt} + 1) \\
&+ \sum_l^m \left(\sum_{i=1}^3 a_{i-1}^{(l)}b_{3-i}^{(l)} + (-1) \sum_{j=3}^{n-1} a_j^{(l)}b_{n+2-j}^{(l)} \right) x^{2k+2t} \bmod (x^{nt} + 1) = 0. \\
&\vdots \\
&+ \sum_l^m \left(\sum_{i=1}^n a_{i-1}^{(l)}b_{n-i}^{(l)} \right) x^{2k+(n-1)t} \bmod (x^{nt} + 1)
\end{aligned}$$

□

Lemma 8. When $f(x)g(x) \bmod (x^n + 1) = 0$, $f(x)g(x)x \bmod (x^n + 1) = 0$. More generally, if $\sum_{i=1}^m f(x)g(x) \bmod (x^n + 1) = 0$, $\sum_{i=1}^m f(x)g(x)x \bmod (x^n + 1) = 0$.

Proof. When $f(x)g(x) \bmod (x^n + 1) = 0$, we have

$$\begin{aligned}
&\left(a_0b_0 + (-1) \sum_{j=1}^{n-1} a_jb_{n-j} \right) \\
&+ \left(\sum_{i=1}^2 a_{i-1}b_{2-i} + (-1) \sum_{j=2}^{n-1} a_jb_{n+1-j} \right) x \\
&+ \left(\sum_{i=1}^3 a_{i-1}b_{3-i} + (-1) \sum_{j=3}^{n-1} a_jb_{n+2-j} \right) x^2 \bmod (x^n + 1) = 0. \\
&\vdots \\
&+ \left(\sum_{i=1}^n a_{i-1}b_{n-i} \right) x^{n-1}
\end{aligned}$$

Therefore, there is

$$\begin{aligned}
0 &= \left(a_0b_0 + (-1) \sum_{j=1}^{n-1} a_jb_{n-j} \right) \\
&= \left(\sum_{i=1}^2 a_{i-1}b_{2-i} + (-1) \sum_{j=2}^{n-1} a_jb_{n+1-j} \right) \\
&\vdots \\
&= \left(\sum_{i=1}^n a_{i-1}b_{n-i} \right).
\end{aligned}$$

So, we know that

$$\left(\begin{array}{l} \left(a_0b_0 + (-1) \sum_{j=1}^{n-1} a_jb_{n-j} \right) \\ + \left(\sum_{i=1}^2 a_{i-1}b_{2-i} + (-1) \sum_{j=2}^{n-1} a_jb_{n+1-j} \right) x \\ + \left(\sum_{i=1}^3 a_{i-1}b_{3-i} + (-1) \sum_{j=3}^{n-1} a_jb_{n+2-j} \right) x^2 \\ \vdots \\ + \left(\sum_{i=1}^n a_{i-1}b_{n-i} \right) x^{n-1} \end{array} \right) x \bmod (x^n + 1) = 0.$$

According to the Corollary 7, if $\sum_{i=1}^m f(x)g(x) \bmod (x^n + 1) = 0$, then $\sum_{i=1}^m f(x)g(x)x \bmod (x^n + 1) = 0$. □

Corollary 9. When $f(x)g(x) \bmod (x^n + 1) = 0$, $f(x)g(x)x^k \bmod (x^n + 1) = 0$. More generally, if $\sum_{i=1}^m f(x)g(x) \bmod (x^n + 1) = 0$, $\sum_{i=1}^m f(x)g(x)x^k \bmod (x^n + 1) = 0$, $k \in \mathbb{Z}$.

3 Hardness of Ring-SIS with Small Modulus

Lemma 10. For any integer m , q , even number n and $\mathcal{X} \subset \mathcal{M}_{m \times 1}(\tilde{\mathcal{R}}_q)$, such that $\forall x, x' \in \mathcal{X}$, there are $\gcd(x - x', q) = 1$ and $\|x - x'\| \leq \beta$. Then if there is a collision-resolving query oracle \mathcal{W} for Ring-SIS(m, n^k, q, \mathcal{X}), then there is also a solution Ring-SIS($m, n^{k+1}, q^{k+1}, \beta^{k+1}$) algorithm, and ask the number of oracle \mathcal{W} is $m(n-1) + 1$.

Proof. According to the definition of the Ring-SIS($m, n^{k+1}, q^{k+1}, \beta^{k+1}$) problem, we can think that we find solution

$$G(x) := (g^1(x), g^2(x), \dots, g^m(x)), g^i(x) \in \mathbb{Z}_q[x]/\langle x^{n^{k+1}} + 1 \rangle, i \in \{1, \dots, m\}.$$

Make $F^T(x)G(x) = 0$, where

$$F(x) := (f^1(x), f^2(x), \dots, f^m(x)), f^i(x) \in \mathbb{Z}_q[x]/\langle x^{n^{k+1}} + 1 \rangle, i \in \{1, \dots, m\}.$$

We use the method of Definition 34 in the Appendix to Ring-SIS($m, n^{k+1}, q^{k+1}, \beta^{k+1}$) extracts the first Ring-SIS(m, n^k, q, \mathcal{X}), that is

$$(\underline{F}^{(0)})^T(x) := (\underline{f}^{(1,0)}(x), \underline{f}^{(2,0)}(x), \dots, \underline{f}^{(m,0)}(x)), \underline{f}^{(i,0)}(x) \in \mathbb{Z}_q[x]/\langle x^{n^k} + 1 \rangle, i \in \{1, \dots, m\}.$$

Here,

$$\underline{f}^{(i,0)}(x) = \underline{a}_0^{(i,0)} + \underline{a}_1^{(i,0)}x + \dots + \underline{a}_{n^k-1}^{(i,0)}x^{n^k-1}.$$

And use the collision-resolving query oracle \mathcal{W} of Ring-SIS(m, n^k, q, \mathcal{X}) to $\underline{G}^{(0)}(x)$ is used to find m solutions, and

$$\begin{aligned} \underline{G}_j^{(0)}(x) &:= (\underline{g}^{(1,j,0)}(x), \underline{g}^{(2,j,0)}(x), \dots, \underline{g}^{(m,j,0)}(x)), \\ \underline{g}^{(i,j,0)}(x) &\in \mathbb{Z}_q[x]/\langle x^{n^k} + 1 \rangle, i \in \{1, \dots, n\}, j \in \{1, \dots, m\} \end{aligned}$$

is obtained. Where,

$$\underline{g}^{(i,j,0)}(x) = b_0^{(i,j,0)} + b_1^{(i,j,0)}x + \dots + b_{n^k-1}^{(i,j,0)}x^{n^k-1}.$$

We will turn it into

$$g^{(i,j,0)}(x) = b_0^{(i,j,0)} + b_1^{(i,j,0)}x^n + \dots + b_{n^k-1}^{(i,j,0)}x^{(n^k-1)n}.$$

Construct

$$G_j^{(0)}(x) := (g^{(1,j,0)}(x), g^{(2,j,0)}(x), \dots, g^{(m,j,0)}(x)), g^{(i,j,0)}(x) \in \mathbb{Z}_q[x]/\langle x^{n^k} + 1 \rangle, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.$$

According to the Corollary 7, we have $(\underline{F}^{(0)}(x))^T \underline{G}_j^{(0)}(x) = 0$. We let

$$F(x) := (f^{(1)}(x), f^{(2)}(x), \dots, f^{(m)}(x)), f^{(i)}(x) \in \mathbb{Z}_q[x]/\langle x^{n^{k+1}} + 1 \rangle, i \in \{1, \dots, m\}.$$

Among them,

$$f^{(i)}(x) = \mathcal{F}_0^{(i)} + \dots + \mathcal{F}_{n-1}^{(i)}, \mathcal{F}_\lambda = a_0x^\lambda + a_1x^{\lambda+n} + \dots + a_{n^k-1}x^{\lambda+(n^k-1)n}, \lambda \in \{1, \dots, n\}.$$

So we set $F(x) = F^{(0)}(x)$, and calculate

$$\begin{aligned} F^{(1)}(x) &= ((F^{(0)}(x))^T G_1^{(0)}(x), \dots, (F^{(0)}(x))^T G_m^{(0)}(x)), \\ (F^{(0)}(x))^T G_i^{(0)}(x) &\in \mathbb{Z}_q[x]/\langle x^{n^{k+1}} + 1 \rangle, i \in \{1, \dots, m\}. \end{aligned}$$

This is still a Ring-SIS($m, n^{k+1}, q^{k+1}, \beta^{k+1}$) problem. But what is different from $F(x)$ is that since $(\underline{F}^{(0)}(x))^T \underline{G}_j^{(0)}(x) = 0$, so the polynomials in $F^{(1)}(x)$ all have less n^k terms.

Continue to follow the above method until finally each polynomial of $F^{(n-1)}(x)$ has only n^k terms left, so it can be solved by using the oracle again. According to this method to solve the Ring-SIS($m, n^{k+1}, q^{k+1}, \beta^{k+1}$) problem, then we ask the oracle

$$m \left(\frac{n^{k+1} - n^k}{n^k} \right) + 1 = m(n-1) + 1(\text{times}).$$

□

Remark 11. According to the above process, it can be summarized as

$$G(x) = (\tilde{g}_1, \dots, \tilde{g}_m).$$

Here,

$$\tilde{g}_i = \left(\sum_{j_1=1}^m g_i^{j_1} \sum_{j_2=1}^m g_{j_1}^{j_2} \cdots \sum_{j_{n-1}=1}^m g_{j_{n-1}} \right) g_{i,j_n}.$$

3.1 Ring-SIS to Ring-SIS Reduction

Lemma 12. For any integer n, m, q and $\mathcal{X} \subset \mathcal{M}_{m \times 1}(\tilde{\mathcal{R}}_q)$, such that $\forall x, x' \in \mathcal{X}$, there are $\gcd(x - x', q) = 1$ and $\|x - x'\| \leq \beta$. Then for the integer c , we have a direct reduction from the collision-resolving query algorithm of Ring-SIS(m, n^c, q^c, β^c) to the collision-resolving query algorithm of Ring-SIS(m, n, q, \mathcal{X}). This algorithm reduces in polynomial time of its input size and makes $m(n^{c-1} - 1) + 1$ calls to its oracle.

Proof. See the appendix for detailed proof. □

Lemma 13. For any integer m, q , even number n and $\mathcal{X} \subset \mathcal{M}_{m \times 1}(\tilde{\mathcal{R}}_q)$, such that $\forall x, x' \in \mathcal{X}$, there are $\gcd(x - x', q) = 1$ and $\|x - x'\| \leq \beta$. Then for the integer c , we have a direct reduction from the collision-resolving query algorithm of Ring-SIS(m^c, n, q, β) to the collision-resolving query algorithm of Ring-SIS(m, n, q, \mathcal{X}). This algorithm reduces in polynomial time of its input size and makes m^{c-1} calls to its oracle.

Proof. Suppose $F(x) = (f_1(x), f_2(x), \dots, f_{m^c}(x))$, which is divided into m^{c-1} parts, that is

$$\begin{aligned} F_1(x) &= (f_1(x), f_2(x), \dots, f_m(x)), \\ F_2(x) &= (f_{m+1}(x), f_{m+2}(x), \dots, f_{2m}(x)), \\ &\vdots \\ F_{m^{c-1}}(x) &= (f_{m^{c-m+1}}(x), f_{m^{c-m+2}}(x), \dots, f_{m^c}(x)). \end{aligned}$$

So $F_i(x)$ is Ring-SIS(m, n, q, \mathcal{X}) question, $i \in \{1, 2, \dots, m^{c-1}\}$. So we only need to ask m^{c-1} times Ring-SIS(m, n, q, \mathcal{X}) collision-resolving oracle to solve Ring-SIS(m^c, n, q, β) question. □

Remark 14. When n and m are close, the conclusions of Lemma 12 and Lemma 13 are “roughly the same” from the perspective of the collision-resolving oracle querying Ring-SIS(m, n, q, \mathcal{X}). But in fact, the computational complexity of Lemma 12 is much greater than that of Lemma 13. Therefore, assuming that Ring-SIS(m, n, q, \mathcal{X}) is attacked one day, it is far safer to increase the order of the polynomial than to increase the number of polynomials (or the dimension of Ring-SIS(m, n, q, \mathcal{X})).

3.2 Direct Reduction

Lemma 15. $As = 0$, is equivalent to $xB = 0$. Where $A \in \mathcal{M}_{n \times m}(\mathbb{Z})$, $s \in \mathcal{M}_{m \times 1}(\mathbb{Z})$; $x \in \mathcal{M}_{1 \times m}(\mathbb{Z})$, $B \in \mathcal{M}_{m \times n}(\mathbb{Z})$.

Lemma 16. *If space A and space B are isomorphic, then if \mathcal{A} in space A is an abstract difficult problem, if and only if \mathcal{B} in the corresponding space B is also a difficult question.*

Proof. Assuming that spaces A and B are isomorphic, we denote an abstract difficult problem in space A as \mathcal{A} and the corresponding problem in space B as \mathcal{B} . By the definition of isomorphism, there exists a bijective function $f : A \rightarrow B$ that preserves the structure and properties in A . Therefore, we can map elements in A to elements in B through f . Now let us prove that if \mathcal{A} is a hard problem, then \mathcal{B} is also a hard problem: Suppose in the space A , for the problem \mathcal{A} , we suppose there is a polynomial-time algorithm that solves \mathcal{A} . That is, we can compute in polynomial time a result that satisfies \mathcal{A} in A . According to the definition of isomorphism, we can define a mapping function $h : A \rightarrow B$, where $h(a) = f(a)$. Since f is a bijective function, h is also a bijective function.

Now let us consider the problem \mathcal{B} in the space B . Given an input x' , we can map it back to the space A through the inverse mapping h^{-1} of the function h , and get the corresponding input $x = h^{-1}(x')$. We can then compute the result that satisfies \mathcal{A} in space A using a polynomial-time algorithm that solves problem \mathcal{A} in space A . Finally, we map the result back to the space B through the function h , and get the result that satisfies \mathcal{B} in the space B . The entire process can be completed in polynomial time.

Therefore, if \mathcal{A} is a hard problem, then \mathcal{B} is also a hard problem. \square

Lemma 17. *If space A and space B are isomorphic, space C and space D are isomorphic. Then space $A \times B$ and space $C \times D$ are also isomorphic.*

Proof. First, we know that A and B are isomorphic, then there is a bijection $f : A \rightarrow B$ that preserves the operations and inverse operations in A . Similarly, we know that C and D are isomorphic, and there exists a bijection $g : C \rightarrow D$ that preserves the operations and inverse operations in C . We can define a new mapping $h : A \times B \rightarrow C \times D$, mapping the elements (a, b) in $A \times B$ to the elements in $C \times D$ (c, d) , where $c = f(a)$ and $d = g(b)$.

Now we show that h is a bijective function.

1. **Mapping is injective.** Suppose there are two different elements $(a1, b1)$ and $(a2, b2)$ belonging to $A \times B$, and $h(a1, b1) = h(a2, b2)$. Then according to the definition of h , $f(a1) = f(a2)$ and $g(b1) = g(b2)$. Since f and g are both bijective functions, we can get $a1 = a2$ and $b1 = b2$. Therefore, h is injective.
2. **Mapping is surjective.** For any (c, d) belonging to $C \times D$, we can choose $a = f^{-1}(c)$ and $b = g^{-1}(d)$ to construct element (a, b) belongs to $A \times B$. By definition, $h(a, b) = (f(a), g(b)) = (c, d)$. Therefore, h is surjective.
3. **Mapping is homomorphic.** Assume $(a1, b1)$ and $(a2, b2)$ belong to $A \times B$, let us prove that $h((a1, b1) + (a2, b2)) = h(a1, b1) + h(a2, b2)$ and $h(-(a1, b1)) = -(h(a1, b1))$. From the definitions of vector addition and scalar multiplication and the properties of f and g we can get these two equations. Therefore, h holds operations and inverse operations.

In summary, we proved that there is a bijective function mapping the elements in $A \times B$ to the elements in $C \times D$, and this mapping preserves operations and inverse operations. Therefore, $A \times B$ is isomorphic to $C \times D$. \square

Lemma 18. *If space A , space B and space E are isomorphic, space C , space D and space F are isomorphic. Then space $A \times B \times E$ It is also isomorphic to the space $C \times D \times F$.*

Proof. According to Lemma 17, we know that if space A and space B are isomorphic, space C and space D are isomorphic. Then the space $A \times B$ and the space $C \times D$ are also isomorphic. That is, there is a mapping $h : A \times B \rightarrow C \times D$, such that the mapping is homomorphic and one to one. So we let the space

$X := A \times B$ and the space $Y := C \times D$, so the space X and the space Y are isomorphic. Combined with the Lemma 17, we have the space $A \times B \times E$ and the space $C \times D \times F$ which are also isomorphic. \square

Lemma 19 ([4], TH3.8). *Let m, n be integers, $\beta \geq \beta_\infty > 0$ be real numbers, and $q \geq \beta \cdot n^{\Omega(1)}$ be the modulus of an integer numbers with no more than $\text{poly}(n)$ integer divisors less than β_∞ , set $S = \{z \in \mathcal{M}_{m \times 1}(\mathbb{Z}) \setminus \{0\} \mid \|z\| \leq \beta \wedge \|z\|_\infty \leq \beta_\infty\}$. Then for $\gamma = \max\{1, \beta\beta_\infty/q\} \cdot O(\beta\sqrt{n})$, there is a difficult problem from n dimensional lattice efficient reduction of the S -collision-resolving search algorithm in $\text{SIS}(m, n, q)$ from SIVP_γ^n .*

Corollary 20 (When $n = 1$, the case of Lemma 19, we call it the *one-dimensional SIS problem*). *Let m be an integer, $\beta \geq \beta_\infty > 0$ be a real number, and $q \geq \beta$ be the modulus of an integer with no more than $\text{poly}(1)$ integer divisors less than β_∞ , the set $S = \{z \in \mathcal{M}_{m \times 1}(\mathbb{Z}) \setminus \{0\} \mid \|z\| \leq \beta \wedge \|z\|_\infty \leq \beta_\infty\}$. Then for $\gamma = \max\{1, \beta\beta_\infty/q\} \cdot O(\beta\sqrt{n})$, there is a difficult problem from n dimensional lattice efficient reduction of the S -collision-resolving search algorithm in $\text{SIS}(m, 1, q)$ from SIVP_γ^n .*

Definition 21 (One-dimensional SIS variant problem). *For $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{M}_{mn \times 1}(\mathbb{Z}_q)$, $\alpha_i^T \in \mathcal{M}_{n \times 1}(\mathbb{Z}_q)$, find $z = (z_1, z_2, \dots, z_n) \in \mathcal{M}_{1 \times mn}(\mathbb{Z}_q)$, $z_i \in \mathcal{M}_{1 \times n}(\mathbb{Z}_q)$, $\|z\| \leq \beta$, such that*

$$z\alpha = \sum_i z_i \alpha_i = (z_1, z_2, \dots, z_n)(\alpha_1, \alpha_2, \dots, \alpha_n)^T = 0.$$

Lemma 22. *If the SIS problem is hard, then the one-dimensional SIS variant problem is also hard.*

Proof. The correctness of Lemma 22 is obvious. We set $m' = mn$, so the one-dimensional SIS variant problem is transformed into the SIS problem. Therefore, m' satisfies the condition that the two are equivalent. \square

Theorem 23. *The Ring-SIS $_{|x=0}$ problem is equivalent in difficulty to the one-dimensional SIS variant problem in the classical lattice.*

$$sA = \mathcal{M}_{1 \times mn}(\mathbb{Z}_q) \times \mathcal{M}_{mn \times 1}(\mathbb{Z}_q) \mapsto \mathcal{M}_{1 \times 1}(\mathbb{Z}_q).$$

$$s_f A_f = (\mathbb{Z}_q^n[x]/\langle x^m + 1 \rangle) \times (\mathbb{Z}_q^n[x]/\langle x^m + 1 \rangle)|_{x=0} \mapsto \mathbb{Z}_q[x]/\langle x + 1 \rangle.$$

The first mapping (a certain row)

$$\begin{aligned} g : \mathbb{Z}_q^n[x]/\langle x^m + 1 \rangle &\rightarrow \mathcal{M}_{1 \times mn}(\mathbb{Z}_q) \\ a_0 + a_1x + \dots + a_{m-1}x^{m-1} &\mapsto (a_0, a_1, \dots, a_{m-1}). \end{aligned}$$

Second mapping (a certain column)

$$\begin{aligned} h : \mathbb{Z}_q^n[x]/\langle x^m + 1 \rangle &\rightarrow \mathcal{M}_{mn \times 1}(\mathbb{Z}_q) \\ b_0 + b_1x + \dots + b_{m-1}x^{m-1} &\mapsto (b_0, -b_{m-1}, \dots, -b_1). \end{aligned}$$

Then there is

$$\begin{aligned} (a_0 + a_1x + \dots + a_{m-1}x^{m-1})(b_0 + b_1x + \dots + b_{m-1}x^{m-1}) \bmod q \bmod (x^m + 1)|_{x=0} \\ = a_0b_0 - a_1b_{m-1} - \dots - a_{m-1}b \\ = g((a_0, a_1, \dots, a_{m-1})) \cdot h((b_0, -b_{m-1}, \dots, -b_1)). \end{aligned}$$

Therefore in the space $\mathcal{M}_{1 \times mn}(\mathbb{Z}_q) \times \mathcal{M}_{mn \times 1}(\mathbb{Z}_q)$ the inner product and the product defined in space $(\mathbb{Z}_q^n[x]/\langle x^m + 1 \rangle) \times (\mathbb{Z}_q^n[x]/\langle x^m + 1 \rangle)$ is homomorphic, and it is easy to prove that this operation is also a bijection.

According to the Lemma 17, it can be seen that the corresponding Ring-SIS $_{|x=0}$ on the ring is also difficult.

Theorem 24. *If the Ring-SIS $_{|x=0}$ problem is hard, then the Ring-SIS problem is also hard.*

Proof. For the Ring-SIS problem, if there is an algorithm \mathcal{W} that can solve the Ring-SIS problem, that is, find $f(x) \in \mathbb{Z}_q^{m_2}[x]$, such that

$$(f_1(x), \dots, f_m(x)) \cdot (g_1(x), \dots, g_m(x)) = \sum_i^m f_i(x)g_i(x) = 0 \quad (3)$$

This also means that the coefficient before x^0 in the equation(3) is also 0, then $f(x) \in \mathbb{Z}_q^{m_2}[x]$ is also an Ring-SIS $_{|x=0}$ problem solution

$$(f_1(x), \dots, f_m(x)) \cdot (g_1(x), \dots, g_m(x))|_{x=0} = \sum_i^m f_i(x)g_i(x)|_{x=0} = 0.$$

This is contradictory to the conclusion of Theorem 23, so the Ring-SIS problem is also difficult. \square

Theorem 25. *Problem Ring-SIS($m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m$) is "roughly the same" as difficult as problem SIS($m, n, q, \mathcal{D}_{\mathcal{L}, \alpha q}^m$).*

Proof. Due to the existence of the one-dimensional SIS problem collision-resolving oracle, the oracle can solve s_i for any form of $\sum_{i=1}^m s_i \alpha_i^1$. In order to be able to solve the SIS problem, we use Gauss's rule to convert the equation (4) into a ladder form and then use the one-dimensional SIS problem collision-resolving oracle to solve it. It is assumed that the SIS problem can be solved by asking $O(n)$ times. The dimensional SIS problem can be solved by colliding with the oracle.

$$As = \sum_{i=1}^m s_i \alpha_i = \begin{matrix} \sum_{i=1}^m s_i \alpha_i^{(1)} \\ \sum_{i=1}^m s_i \alpha_i^{(2)} \\ \sum_{i=1}^m s_i \alpha_i^{(3)} \\ \vdots \\ \sum_{i=1}^m s_i \alpha_i^{(n)} \end{matrix} \quad (4)$$

Among them, $A = (\alpha_1, \dots, \alpha_m)$, and $\alpha_i = (\alpha_i^{(1)}, \dots, \alpha_i^{(n)})$. The Ring-SIS problem also requires $O(n)$ times of querying Ring-SIS $_{|x=0}$ to solve the problem until it collides with the oracle.

$$\begin{array}{ccc} \text{One-dimensional SIS problem} & \xrightarrow{O(n) \text{ inquiries}} & \text{SIS problem} \\ \Downarrow & & \\ \text{Ring-SIS}_{|x=0} \text{ problem} & \xrightarrow{O(n) \text{ inquiries}} & \text{Ring-SIS problem} \end{array}$$

According to the form $F(x)G(x) = 0$, we have

$$\begin{aligned} F^T(x)G(x) &= \sum_l^m f^{(l)}(x)g^{(l)}(x) \\ &= \sum_l^m \left(a_0^{(l)}b_0^{(l)} + (-1) \sum_{j=1}^{n-1} a_j^{(l)}b_{n-j}^{(l)} \right) \\ &+ \sum_l^m \left(\sum_{i=1}^2 a_{i-1}^{(l)}b_{2-i}^{(l)} + (-1) \sum_{j=2}^{n-1} a_j^{(l)}b_{n+1-j}^{(l)} \right) x \\ &+ \sum_l^m \left(\sum_{i=1}^3 a_{i-1}^{(l)}b_{3-i}^{(l)} + (-1) \sum_{j=3}^{n-1} a_j^{(l)}b_{n+2-j}^{(l)} \right) x^2 \quad \text{mod } (x^n + 1) = 0. \\ &\vdots \\ &+ \sum_l^m \left(\sum_{i=1}^n a_{i-1}^{(l)}b_{n-i}^{(l)} \right) x^{n-1} \end{aligned} \quad (5)$$

We rewrite the equation (5) as

$$\begin{aligned}
\sum_l^m a_0^{(l)} b_0^{(l)} + (-1) \sum_{j=1}^{n-2} \left(\sum_l^m a_j^{(l)} b_{n-j}^{(l)} \right) &= 0 \\
\sum_i^2 \left(\sum_l^m a_{i-1}^{(l)} b_{2-i}^{(l)} \right) + (-1) \sum_{j=2}^{n-3} \left(\sum_l^m a_j^{(l)} b_{n+1-j}^{(l)} \right) &= 0 \\
\sum_i^3 \left(\sum_l^m a_{i-1}^{(l)} b_{3-i}^{(l)} \right) + (-1) \sum_{j=3}^{n-4} \left(\sum_l^m a_j^{(l)} b_{n+2-j}^{(l)} \right) &= 0 \\
&\vdots \\
\sum_i^n \left(\sum_l^m a_{i-1}^{(l)} b_{n-i}^{(l)} \right) &= 0
\end{aligned} \tag{6}$$

The equation (4) is consistent with the equation (6). We also need to convert it into a ladder form, and then use the Ring-SIS $_{|x=0}$ problem collision-resolving oracle to solve it. Therefore, the number of times that the Ring-SIS $_{|x=0}$ questions need to be asked to solve the Ring-SIS problem is the same as the number of one-dimensional SIS questions that need to be asked to solve the SIS problem, which can also be $O(n)$ times. Therefore, judging from the number of times the oracle is asked, the difficulty of the SIS problem is similar to that of the Ring-SIS problem. \square

Remark 26. *Although the proof of Theorem 25 may not be sufficient, there is one thing we are certain about. That is, if the one-dimensional SIS problem is difficult, then the Ring-SIS $_{|x=0}$ problem is also difficult, which implies that the difficulty level of the Ring-SIS problem is no lower than that of the Ring-SIS $_{|x=0}$ problem and no higher than that of the SIS problem, due to the following theorem.*

Theorem 27. *If there exists a oracle that can solve the SIS problem in polynomial time, then there also exists an efficient algorithm to solve the Ring-SIS problem.*

Proof. If there exists a collision-resolving oracle that can solve the SIS problem in polynomial time, i.e., find $s = (s_1, \dots, s_m)$, $s_i \in \mathbb{Z}_1$, such that

$$\sum_{i=1}^m \alpha_i s_i = 0,$$

we can construct an isomorphism mapping the vector $\alpha_i = (\alpha_i^1, \dots, \alpha_i^n)$ to $f_i(x) = \alpha_i^{(1)} + \alpha_i^{(2)}x + \dots + \alpha_i^{(n)}x^{n-1}$. Similarly, s_i is mapped to $g_i(x) = s_i$. Therefore, we have

$$\sum_{i=1}^m f_i g_i = 0.$$

Thus, $s = (s_1, \dots, s_m)$ is also a solution to the Ring-SIS problem. \square

4 Hardness of Ring-LWE with Small Uniform Errors

Lemma 28. *If there exists a collision-resolving oracle that can solve the Ring-LWE($m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}$) $_{|x=0}$ problem in polynomial time, then there also exists an efficient algorithm to solve the LWE($m, 1, q, \mathcal{D}_{\mathcal{L}, \alpha q}$) problem in polynomial time. Similarly, the reverse is also true.*

Proof. According to Theorem 23, assuming the existence of an oracle that solves the LWE($m, 1, q, \mathcal{D}_{\mathcal{L}, \alpha q}$) problem, i.e., finds $s \in \mathcal{M}_{mn \times 1}(\mathbb{Z}_q)$ such that $b = As + e$, where $A \in \mathcal{M}_{1 \times mn}(\mathbb{Z}_q)$ and $b, e \in \mathbb{Z}_q$. We can establish the following isomorphism mapping, namely

$$\begin{aligned}
b &= sA + e \\
\downarrow \\
b_f|_{x=0} &= (s_f \cdot a + e_f)|_{x=0}.
\end{aligned}$$

Here, $s_f, a \in \mathcal{R}_q$, $e_f = e + e_1x + \dots + e_{n-1}x^{n-1}$, $e_i \in \{1, 2, \dots, q-1\}$, $i \in \{1, \dots, n-1\}$. Clearly, we have established an equivalence between the $\text{LWE}(m, 1, q, \mathcal{D}_{\mathcal{L}, \alpha q})$ and $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q})|_{x=0}$ problems. By Lemma 16, we know that the existence of an oracle for $\text{LWE}(m, 1, q, \mathcal{D}_{\mathcal{L}, \alpha q})$ is equivalent to the existence of an efficient algorithm to solve $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q})|_{x=0}$. \square

Lemma 29. *If there exists an oracle that can solve the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem, then there must exist an efficient algorithm to solve the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)|_{x=0}$ problem.*

Proof. Let's assume that there exists an oracle \mathcal{W} that can solve the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem. In other words, it can find $s \in \mathcal{R}_q$ within polynomial time such that

$$b = as + e, \text{ where } a \in \mathcal{R}_q, e \in \mathcal{D}_{\mathcal{R}, \alpha q}.$$

Since $b = as + e$ is a polynomial, the same equation holds for the constant term of b , which is

$$b|_{x=0} = (as + e)|_{x=0}, \text{ where } a \in \mathcal{R}_q, e \in \mathcal{D}_{\mathcal{R}, \alpha q}.$$

Therefore, $s \in \mathcal{R}_q$ is also a solution to the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)|_{x=0}$ problem. \square

Lemma 30 ([4], Th 2.13). *Assuming that we can factorize q into $q = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ in $\text{poly}(n)$ polynomial time, let $0 < \alpha \leq 1/\omega_n$. If the $\text{LWE}(m, n, q, \mathcal{D}_{\mathbb{Z}, \alpha q}^m)$ problem is hard, where $m(n) = n^{O(1)}$, then $\text{LWE}(m', n, q, \mathcal{D}_{\mathbb{Z}, \alpha' q}^m)$ is also pseudorandom for any $m'(n) = n^{O(1)}$ and*

$$\alpha' \geq \max\{\alpha, \omega_n^{1+1/\ell} \cdot \alpha^{1/\ell}, \omega_n/p_1^{e_1}, \dots, \omega_n/p_k^{e_k}\},$$

where ℓ is an upper bound on p_i such that $p_i < \omega_n/\alpha'$.

Corollary 31. *Assuming that we can factorize q into $q = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ in polynomial time, let $0 < \alpha \leq 1/\omega$. If the $\text{LWE}(m, 1, q, \mathcal{D}_{\mathbb{Z}, \alpha q}^m)$ problem is hard, where m is any integer, then $\text{LWE}(m', 1, q, \mathcal{D}_{\mathbb{Z}, \alpha' q}^m)$ is also pseudorandom for any m' and*

$$\alpha' \geq \max\{\alpha, \omega^{1+1/\ell} \cdot \alpha^{1/\ell}, \omega/p_1^{e_1}, \dots, \omega/p_k^{e_k}\},$$

where ℓ is an upper bound on p_i such that $p_i < \omega/\alpha'$.

Lemma 32. *Assuming we can factorize $q = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ in polynomial time, where p_i are primes and e_i are positive integers. Let $0 < \alpha \leq 1/\omega_n$. If the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem is hard, where $m(n) = n^{O(1)}$, then $\text{Ring-LWE}(m', n, q, \mathcal{D}_{\mathcal{R}, \alpha' q}^m)$ is also pseudorandom for any $m'(n) = n^{O(1)}$, and*

$$\alpha' \geq \max\{\alpha, \omega_n^{1+1/\ell} \cdot \alpha^{1/\ell}, \omega_n/p_1^{e_1}, \dots, \omega_n/p_k^{e_k}\},$$

where ℓ is an upper bound on primes p_i such that $p_i < \omega_n/\alpha'$.

Proof. If there exists an oracle that solves the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem, then there must exist an algorithm to solve the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)|_{x=0}$ problem. According to Lemma 29, we know that the $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)|_{x=0}$ problem is equivalent to the $\text{LWE}(m, 1, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem. Therefore, there also exists an algorithm to solve the $\text{LWE}(m, 1, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem. Based on Corollary 31, we can conclude that there exists an algorithm to solve the $\text{LWE}(m', 1, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem.

Using the equivalence between $\text{Ring-LWE}(m', n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)|_{x=0}$ problem and $\text{LWE}(m', 1, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem again, we can obtain a solution to the $\text{Ring-LWE}(m', n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)|_{x=0}$ problem, and thus obtain an algorithm for solving $\text{Ring-LWE}(m', n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem using the conclusion of Lemma 29. \square

Theorem 33. *If there exists an oracle \mathcal{W} capable of solving $\text{Ring-LWE}(n, q, \mathcal{D}_{\mathcal{R}, \alpha q})$, then there also exists an algorithm \mathcal{W}' that can construct collision queries for $\text{Ring-LWE}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$, and thereby solve the $\text{Ring-SIS}(m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m)$ problem.*

Proof. For a Ring-LWE($n, q, \mathcal{D}_{\mathcal{R}, \alpha q}$) problem, given $(a_i, b_i) \in \mathcal{R}_q \times \mathcal{R}_q$, the goal is to find $s_i \in \mathcal{R}_q$ such that

$$b_i = a_i s_i + e_i.$$

Here, a_i and b_i are ring elements in \mathcal{R}_q , and s_i is the secret key while $e_i \in \chi$ is a small error term. If there exists an oracle \mathcal{W} capable of solving the Ring-LWE($n, q, \mathcal{D}_{\mathcal{R}, \alpha q}$) problem, then for a m -fold Ring-LWE($n, q, \mathcal{D}_{\mathcal{R}, \alpha q}$) problem, i.e., Ring-LWE($m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m$), we only need to query the oracle \mathcal{W} m times in order to obtain $s_i \in \mathcal{R}_q, i \in \{1, \dots, m\}$ such that

$$(b_1 = a_1 s_1 + e_1, \dots, b_m = a_m s_m + e_m). \quad (7)$$

as in Equation (7). Let $F(x) = (a_1, \dots, a_m)$, and consider $G(x) = (t_1, \dots, t_m)$ as a special solution to the Ring-SIS($m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m$) problem, such that

$$F^T(x)G(x) = a_1 t_1 + \dots + a_m t_m = 0,$$

and $a_i t_i = 0$ for $i \in \{1, \dots, m\}$.

Using the oracle machine \mathcal{W} again m times, we obtain $y_i = s_i + t_i$ for $i \in \{1, \dots, m\}$, such that

$$(b_1 = a_1(s_1 + t_1) + e_1, \dots, b_m = a_m(s_m + t_m) + e_m). \quad (8)$$

By combining the information from Equation (7) and Equation (8), we can obtain a solution to the Ring-SIS($m, n, q, \mathcal{D}_{\mathcal{R}, \alpha q}^m$) problem. \square

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Likewise. We transform it into

$$g^{(i,j,0)}(x) = b_0^{(i,j,0)} + b_1^{(i,j,0)}x^n + \dots + b_{n^k-1}^{(i,j,0)}x^{(n^k-1)n}.$$

We construct

$$G_j^{(0)}(x) := (g^{(1,j,0)}(x), g^{(2,j,0)}(x), \dots, g^{(m,j,0)}(x)), g^{(i,j,0)}(x) \in \mathbb{Z}_q[x]/\langle x^{n^k} + 1 \rangle, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.$$

We define $F^{(1)}(x) = ((F^{(0)}(x))^T G_1^{(0)}(x), (F^{(0)}(x))^T G_2^{(0)}(x), \dots, (F^{(0)}(x))^T G_m^{(0)}(x))$. At this point, $F^{(1)}(x)$ is also a Ring-SIS(m, n^c, q^c, β^c) problem. We extract a $\underline{F}^{(1)}(x)$ from $F^{(1)}(x)$, which is a Ring-SIS(m, n, q^c, β^c) problem, as stated in Equation (P2).

We let $\underline{F}^{(1)}(x) = (\underline{f}^{(1,1)}, \underline{f}^{(1,2)}, \dots, \underline{f}^{(1,m)})$, where $\underline{f}^{(1,i)} = \underline{a}_0^{(1,i)} + \underline{a}_1^{(1,i)}x + \dots + \underline{a}_{n-1}^{(1,i)}x^{n-1}$. Using the Ring-SIS(m, n, q, \mathcal{X}) oracle, we make m queries to $\underline{F}^{(0)}(x)$ and obtain

$$\begin{aligned} \underline{G}_j^{(1)}(x) &:= (\underline{g}^{(1,j,1)}(x), \underline{g}^{(2,j,1)}(x), \dots, \underline{g}^{(m,j,1)}(x)), \\ \underline{g}^{(i,j,0)}(x) &\in \mathbb{Z}_q[x]/\langle x^{n^k} + 1 \rangle, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}. \end{aligned}$$

Where,

$$\underline{g}^{(i,j,1)}(x) = b_0^{(i,j,1)} + b_1^{(i,j,1)}x + \dots + b_{n^k-1}^{(i,j,1)}x^{n^k-1}.$$

We could rewrite it as

$$g^{(i,j,1)}(x) = b_0^{(i,j,1)} + b_1^{(i,j,1)}x^n + \dots + b_{n^k-1}^{(i,j,1)}x^{(n^k-1)n}.$$

Construct a part of solution, that is

$$G_j^{(1)}(x) := (g^{(1,j,1)}(x), g^{(2,j,1)}(x), \dots, g^{(m,j,1)}(x)), g^{(i,j,1)}(x) \in \mathbb{Z}_q[x]/\langle x^{n^k} + 1 \rangle, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.$$

Suppose that $F^{(2)}(x) = ((F^{(1)}(x))^T G_1^{(1)}(x), (F^{(1)}(x))^T G_2^{(1)}(x), \dots, (F^{(1)}(x))^T G_m^{(1)}(x))$, and $F^{(2)}(x)$ is a Ring-SIS(m, n^c, q^c, β^c) problem. The solution to the recursive problem $F(x) = F^{(0)}(x) = \text{Ring-SIS}(m, n^c, q^c, \beta^c)$ is given by

$$G(x) = (\tilde{g}_1, \dots, \tilde{g}_m),$$

where

$$\tilde{g}_i = \left(\sum_{j_1=1}^m g_i^{j_1} \sum_{j_1=1}^m g_{j_1}^{j_2} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i, j_{n^c-n+1}}.$$

To verify that $G(x)$ is a solution to the problem $F(x) = F^{(0)}(x) = \text{Ring-SIS}(m, n^c, q^c, \beta^c)$, we only need to calculate $F^T(x)G(x) = 0$. For convenience, let's define

$$F(x) := (f^{(1)}(x), f^{(2)}(x), \dots, f^{(m)}(x)), f^{(i)}(x) \in \mathbb{Z}_q[x]/\langle x^{n^{k+1}} + 1 \rangle, i \in \{1, \dots, m\}.$$

where,

$$f^{(i)}(x) = \mathcal{F}_0^{(i)} + \dots + \mathcal{F}_{n^c-1}^{(i)}, \mathcal{F}_\lambda = a_0 x^\lambda + a_1 x^{\lambda+n} + \dots + a_{n^k-1} x^{\lambda+(n^k-1)n}, \lambda \in \{1, \dots, n\}.$$

Then we have

$$F^T(x)G(x) = \sum_{k=1}^m (\mathcal{F}_0^{(i)} + \dots + \mathcal{F}_{n^c-1}^{(i)}) \left(\sum_{j_1=1}^m g_i^{j_1} \sum_{j_1=1}^m g_{j_1}^{j_2} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i, j_{n^c-n+1}}.$$

Since $\sum_{i=1}^m (\mathcal{F}_0^{(i)} \sum_{j_1=1}^m g_i^{j_1}) = 0$, we have

$$\begin{aligned}
F^T(x)G(x) &= \sum_{i=1}^m (\mathcal{F}_0^{(i)} + \dots + \mathcal{F}_{n^c-1}^{(i)}) \left(\sum_{j_1=1}^m g_i^{j_1} \sum_{j_1=1}^m g_{j_1}^{j_2} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i,j_{n^c-n+1}} \\
&= \sum_{i=1}^m \left(\mathcal{F}_1^{(i)} \left(\sum_{j_1=1}^m g_i^{j_1} \right) + \dots + \mathcal{F}_{n^c-1}^{(i)} \left(\sum_{j_1=1}^m g_i^{j_1} \right) \right) \left(\sum_{j_1=1}^m g_{j_1}^{j_2} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i,j_{n^c-n+1}}.
\end{aligned}$$

Also, since

$$\sum_{i=1}^m \left(\mathcal{F}_1^{(i)} \sum_{j_1=1}^m \left(\sum_{j_1=1}^m g_i^{j_1} \right) \left(\sum_{j_1=1}^m g_{j_1}^{j_2} \right) \right) = 0,$$

we have

$$\begin{aligned}
F^T(x)G(x) &= \sum_{i=1}^m (\mathcal{F}_0^{(i)} + \dots + \mathcal{F}_{n^c-1}^{(i)}) \left(\sum_{j_1=1}^m g_i^{j_1} \sum_{j_1=1}^m g_{j_1}^{j_2} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i,j_{n^c-n+1}} \\
&= \sum_{i=1}^m \left(\mathcal{F}_1^{(i)} \left(\sum_{j_1=1}^m g_i^{j_1} \right) + \dots + \mathcal{F}_{n^c-1}^{(i)} \left(\sum_{j_1=1}^m g_i^{j_1} \right) \right) \left(\sum_{j_1=1}^m g_{j_1}^{j_2} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i,j_{n^c-n+1}} \\
&= \sum_{i=1}^m \left(\mathcal{F}_2^{(i)} \left(\sum_{j_1=1}^m g_i^{j_1} \right) \left(\sum_{j_1=1}^m g_{j_1}^{j_2} \right) + \dots + \mathcal{F}_{n^c-1}^{(i)} \left(\sum_{j_1=1}^m g_i^{j_1} \right) \left(\sum_{j_1=1}^m g_{j_1}^{j_2} \right) \right) \left(\sum_{j_2=1}^m g_{j_2}^{j_3} \dots \sum_{j_{n^c-n}=1}^m g_{j_{n^c-n}} \right) g_{i,j_{n^c-n+1}}.
\end{aligned}$$

If we continue this way, it is easy to prove that $G(x)$ is a solution to the Ring-SIS(m, n^c, q^c, β^c) problem. In this case, it will take a total of $m(n^c - n)/n + 1 = m(n^{c-1} - 1) + 1$ queries. \square