

# Further results on several classes of optimal ternary cyclic codes with minimum distance four

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## Abstract

Cyclic codes are a subclass of linear codes and have applications in consumer electronics, data storage systems, and communication systems as they have efficient encoding and decoding algorithms. In this paper, by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$  and using the multivariate method, we present three classes of optimal ternary cyclic codes in the case of  $m$  is odd and five classes of optimal ternary cyclic codes with explicit values  $e$ , respectively. In addition, two classes of optimal ternary cyclic codes  $\mathcal{C}_{(u,v)}$  are given.

**Keywords:** Cyclic code, Optimal code, Ternary code, Sphere packing bound.

## 1 Introduction

Let  $p$  be a prime and  $m$  be a positive integer. Let  $\mathbb{F}_p$  and  $\mathbb{F}_{p^m}$  denote the finite fields with  $p$  and  $p^m$  elements, respectively. An  $[n, k, d]$  linear code  $\mathcal{C}$  over the finite field  $\mathbb{F}_p$  is a  $k$ -dimensional subspace of  $\mathbb{F}_p^n$  with minimum Hamming distance  $d$ , and is called cyclic if any cyclic shift of a codeword is another codeword of  $\mathcal{C}$ . Assume  $\gcd(n, p) = 1$ . By identifying any vector  $(c_0, c_1, \dots, c_{n-1}) \in \mathbb{F}_p^n$  with  $c_0 + c_1x + c_2x^2 + \dots + c_{n-1}x^{n-1} \in \mathbb{F}_p[x]/(x^n - 1)$ , any cyclic code of length  $n$  over  $\mathbb{F}_p$  corresponds to an ideal of the polynomial residue class ring  $\mathbb{F}_p[x]/(x^n - 1)$ . It is well known that every ideal of  $\mathbb{F}_p[x]/(x^n - 1)$  is principal. The cyclic code can be expressed as  $\mathcal{C} = \langle g(x) \rangle$ , where  $g(x)$  is monic and has the least degree among all elements in  $\mathcal{C}$ . Then  $g(x)$  is called the generator polynomial of  $\mathcal{C}$  and  $h(x) = (x^n - 1)/g(x)$  is referred to as the parity-check polynomial of  $\mathcal{C}$ . Let  $A_i$  denote the number of codewords with Hamming weight  $i$  in a code  $\mathcal{C}$  of length  $n$  for  $1 \leq i \leq n$ . The weight enumerator of  $\mathcal{C}$  is defined by  $1 + A_1z + A_2z^2 + \dots + A_nz^n$ . Cyclic codes are an important subclass of linear codes and have been extensively studied [9]. Due to their wide applications in mathematics and engineering, such as cryptography [1] and sequence design [4].

Let  $\alpha$  be a generator of  $\mathbb{F}_{3^m}^* = \mathbb{F}_{3^m} \setminus \{0\}$  and  $m_i(x)$  be the minimal polynomial of  $\alpha^i$  over  $\mathbb{F}_3$ , where  $1 \leq i \leq 3^m - 1$ . Let  $u, v$  be two integers such that  $\alpha^u$  is not a Galois conjugate of  $\alpha^v$ , the cyclic code over  $\mathbb{F}_3$  with generator polynomial  $m_u(x)m_v(x)$  is denoted by  $\mathcal{C}_{(u,v)}$ . In recent years, many scholars had made much progress on optimal cyclic codes over finite fields with respect to the Sphere packing bound [9]. For  $(u, v) = (\frac{3^m+1}{2}, \frac{3^k+1}{2})$ , where  $m$  is odd,  $k$  is even, Zhou and Ding [27] gave a class of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$ .

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For  $(u, v) = (\frac{3^m+1}{2}, 2 \cdot 3^{\frac{m-1}{2}} + 1)$ , where  $m \geq 3$  is odd, Fan et al. [5] obtained a new class of optimal ternary cyclic codes and discussed the duals of them. For  $(u, v) = (\frac{3^m+1}{2}, 3^r + 2)$ , where  $m$  is odd,  $r$  is a Niho-type exponent with  $4r \equiv 1 \pmod{m}$ , Yan et al. [20] constructed a new class of optimal ternary cyclic codes and determined their duals. Then, Liu et al. [15] proposed two classes of new optimal ternary cyclic codes and determined the weight distribution of  $C_{(u,v)}^\perp$  for  $(u, v) = (2, \frac{3^m-1}{2} + 2(3^k+1))$ , where  $m$  is odd,  $\gcd(m, k) = 1$ . After that, for  $(u, v) = (3^m - 6, \frac{3^k+1}{2})$ , where  $m$  is even,  $k \in \{1, 3, m-1\}$ , Wang et al. [17] gave a class of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$ . On the other hand, when  $(u, v) = (1, e)$ , by using perfect nonlinear monomials, Carlet et al. [1] proposed several classes of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$ . In 2013, Ding and Hellesteth [3] obtained some optimal ternary cyclic codes with the same parameters by utilizing almost perfect nonlinear monomials  $x^e$  and some other monomials over  $\mathbb{F}_{3^m}$ . In addition, they also presented nine open problems. Later, by analyzing irreducible factors of certain polynomials with low degrees, an open problem for  $e = 2(3^{m-1} - 1)$  was solved by Li et al. [10], while the authors also presented several classes of optimal cyclic codes with parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  and  $[3^m - 1, 3^m - 2 - 2m, 5]$ . Through some subtle manipulation on solving certain equations, a conjecture for  $e = 2(1 + 3^h)$  was settled by Li et al. [12]. Remarkably, some optimal ternary codes given in [3] and [10] were generalized by Wang and Wu [18]. By solving some equations over  $\mathbb{F}_{3^m}$ , Liu et al. [14], Han and Yan [8] settled an open problem for  $e = 3^h + 5$  presented by [3], respectively. Then, Liu et al. [16] also did advance work on three open problems presented by [3]. Recently, Wang et al. [17] proposed a class of optimal ternary cyclic codes for  $e = \frac{3^m-1}{2} - k$ , where  $m$  is odd with  $m \not\equiv 0 \pmod{9}$ ,  $k \in \{7, 11, -19\}$ . Later, Zhao et al. [25] refined some optimal ternary cyclic codes in [18] and analyzed an open problem for  $e = \frac{3^{m-1}-1}{2} + 3^h + 1$  in [3]. For some advances about cyclic codes can refer to [2, 6, 7, 11, 19, 21, 22, 26] and the references therein.

In particular, Zha and Hu [23] presented six classes of optimal ternary cyclic codes with parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$  and using the multivariate method. Moreover, they left some conjectures about optimal ternary cyclic codes. In addition, They gave three classes of optimal ternary cyclic codes  $C_{(u,v)}$  and proved that  $C_{(\frac{3^m+1}{2}, \frac{3^m-1}{2}+v)}$  and  $C_{(1,v)}$  have the same optimality [24]. Motivated by their work, this paper aims to find more optimal ternary cyclic codes over  $\mathbb{F}_{3^m}$ . In this paper, by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$ , we obtain three classes of new optimal ternary cyclic codes  $C_{(1,e)}$ . Moreover, five new classes of optimal ternary cyclic codes  $C_{(1,e)}$  are constructed by using the multivariate method. In addition, we give two classes of optimal ternary cyclic codes  $C_{(u,v)}$ .

The remainder of the paper is organized as follows. Some preliminaries needed in the sequel are introduced in Section 2. In Section 3, we construct three classes of optimal ternary cyclic codes  $C_{(1,e)}$  and two classes of optimal ternary cyclic codes  $C_{(u,v)}$  with  $m$  is odd, respectively. Next, five classes of new optimal ternary cyclic codes  $C_{(1,e)}$  with explicit values  $e$  are proposed in Section 4. Finally, we give some concluding remarks in Section 5.

## 2 Preliminaries

In this section, we will introduce two useful results. The first one is about cyclotomic cosets. For a prime  $p$ , the  $p$ -cyclotomic coset modulo  $p^m - 1$  containing  $j$  is defined by

$$C_j = \{j \cdot p^s \pmod{p^m - 1} \mid s = 0, 1, \dots, m-1\}.$$

If  $e_1$  and  $e_2$  belong to the same  $p$ -cyclotomic coset, then the cyclic codes  $C_{(1,e_1)}$  and  $C_{(1,e_2)}$  are the same.

**Lemma 2.1.** ([13, Corollary 3.47]) *An irreducible polynomial over  $\mathbb{F}_{p^m}$  of degree  $n$  remains irreducible over  $\mathbb{F}_{p^{ml}}$  if and only if  $\gcd(l, n) = 1$ .*

**Lemma 2.2.** ([3, Lemma 2.1])  $|C_e| = m$  for any  $1 \leq e \leq p^m - 2$  with  $\gcd(e, p^m - 1) = 2$ .

As stated before, the ternary cyclic code with parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  is optimal with respect to the Sphere Packing bound. Ding and Helleseht presented the following determining rule of optimal ternary cyclic codes  $C_{(1,e)}$ .

**Theorem 2.1.** ([3, Theorem 4.1]) *Let  $e \notin C_1$  and  $|C_e| = m$ . The ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if the following conditions are satisfied:*

- C1:  $e$  is even;
- C2:  $(x + 1)^e + x^e + 1 = 0$  has a unique solution  $x = 1$  over  $\mathbb{F}_{3^m}$ ; and
- C3:  $(x + 1)^e - x^e - 1 = 0$  has a unique solution  $x = 0$  over  $\mathbb{F}_{3^m}$ .

Recently, Zha et al. [24] showed a link between the ternary cyclic codes  $C_{(\frac{3^m+1}{2}, \frac{3^m-1}{2}+e)}$  and  $C_{(1,e)}$  as follows.

**Theorem 2.2.** ([24, Theorem 3]) *Let  $m$  be odd,  $e$  be even with  $|C_e| = m$ . The ternary cyclic code  $C_{(\frac{3^m+1}{2}, \frac{3^m-1}{2}+e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$ .*

### 3 New optimal ternary cyclic codes in the case of $m$ is odd

In this section, let  $m$  be odd, by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$ , we present three classes of optimal ternary cyclic codes  $C_{(1,e)}$  and two classes of optimal ternary cyclic codes  $C_{(u,v)}$ , respectively.

**Lemma 3.1.** *Let  $m$  be odd,  $h$  be an integer with  $2h \equiv -1 \pmod{m}$ , i.e.,  $h = \frac{m-1}{2}$ . Then  $e \notin C_1$  and  $|C_e| = m$  if*

- (1)  $e = \frac{3^m-1}{2} + \frac{3^h+5}{2}$  and  $m \equiv 1 \pmod{4}$ ;
- (2)  $e = \frac{3^h+5}{2}$  and  $m \equiv 3 \pmod{4}$ .

**Proof.** It can be easily checked that  $e \notin C_1$ . Suppose there exists an integer  $1 \leq j \leq m - 1$  such that  $3^j \cdot e \equiv e \pmod{3^m - 1}$ , then  $(3^m - 1) \mid e(3^j - 1)$ . And we can get that  $(3^m - 1) \mid 2e(3^j - 1)$ , which means that  $(3^m - 1) \mid (3^h + 5)(3^j - 1)$ . Let  $t = \frac{m-1}{2}$ , then  $\gcd((3^h + 5)(3^j - 1), 3^m - 1) \leq \gcd(3^t + 5, 3^{2t+1} - 1) \cdot \gcd(3^j - 1, 3^m - 1) = \gcd(3^t + 5, 74) \cdot (3^{\gcd(m,j)} - 1) = 2(3^{\gcd(m,j)} - 1) < 3^m - 1$  since  $m$  is odd. This is a contradiction. Therefore,  $3^j \cdot e \not\equiv e \pmod{3^m - 1}$  for any  $1 \leq j \leq m - 1$ , which means that  $|C_e| = m$ .  $\square$

**Theorem 3.1.** *Let  $m$  be an odd prime,  $h$  be an integer with  $2h \equiv -1 \pmod{m}$ , i.e.,  $h = \frac{m-1}{2}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if one of the following conditions is met:*

- (1)  $e = \frac{3^m-1}{2} + \frac{3^h+5}{2}$ ,  $m \equiv 1 \pmod{4}$  and  $m \not\equiv 0 \pmod{17}$ ;
- (2)  $e = \frac{3^h+5}{2}$ ,  $m \equiv 3 \pmod{4}$ .

**Proof.** (1) Based on Lemma 3.1, the condition C1 in Theorem 2.1 is met. For any even integer  $e > 1$ , it can be easily checked that  $(x + 1)^e + x^e + 1 = 0$  has the only solution  $x = 1$  over  $\mathbb{F}_3$  and  $(x + 1)^e - x^e - 1 = 0$  has the only solution  $x = 0$  over  $\mathbb{F}_3$ . To prove the conditions C2 and C3 in Theorem 2.1 are satisfied, we need to show that there is no solution of equation

$$(x + 1)^e = \pm(x^e + 1)$$

in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ , which means that the equation

$$(x + 1)^{2e} - x^{2e} - 1 + x^e = 0 \quad (1)$$

has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . It is not difficult to verify that for any positive integer  $h$ ,

$$\begin{aligned} (x + 1)^{3^{h+5}} &= (x^{3^h} + 1)(x^3 + 1)(x + 1)^2 \\ &= x^{3^{h+5}} - x^{3^{h+4}} + x^{3^{h+3}} + x^{3^{h+2}} - x^{3^{h+1}} + x^{3^h} + x^5 - x^4 + x^3 + x^2 - x + 1. \end{aligned}$$

For  $e = \frac{3^m - 1}{2} + \frac{3^{h+5}}{2}$ , Eq. (1) can be reduced to

$$x((x^{\frac{3^m - 1}{2} + \frac{3^{h+1}}{2}} - x)^2 - (x^{3^h - 1} - 1)(x + 1)^3(x - 1)) = 0. \quad (2)$$

In the following, we will discuss the solutions of Eq. (2) in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Let  $h = \frac{m-1}{2}$ , then  $\gcd(m, h) = 1$  and  $\gcd(3^h - 1, 3^m - 1) = 2$ . Assume  $x^{\frac{3^h - 1}{2}} = \pm 1$ , then we have  $x^2 = 1$ , which follows from  $x^{3^h - 1} = 1$  and  $x^{3^m - 1} = 1$ , and hence we have  $x \in \mathbb{F}_3$ . Thus, the solutions of Eq. (2) in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  satisfy  $\theta := x^{\frac{3^h - 1}{2}} \notin \mathbb{F}_3$ . Then we can discuss the following two cases:  $x$  is a square and  $x$  is not a square over  $\mathbb{F}_{3^m}$ .

Case I:  $x$  is a square over  $\mathbb{F}_{3^m}$ . Since  $\theta \notin \mathbb{F}_3$ , it follows from Eq. (2) that

$$x^2(\theta - 1) - (\theta + 1)(x^4 - x^3 + x - 1) = 0,$$

which is equivalent to

$$-\theta(x^4 - x^3 - x^2 + x - 1) = x^4 - x^3 + x^2 + x - 1.$$

If  $x^4 - x^3 - x^2 + x - 1 = 0$ , then  $x^4 - x^3 + x^2 + x - 1 = 0$ , which implies that  $x = 0$ . It leads to a contradiction. Therefore,  $x^4 - x^3 - x^2 + x - 1 \neq 0$  and

$$\theta = -\frac{x^4 - x^3 + x^2 + x - 1}{x^4 - x^3 - x^2 + x - 1}.$$

Then

$$x^{3^h} = x\theta^2 = \frac{x^9 + x^8 + x^4 - x^3 + x^2 + x}{x^8 + x^7 - x^6 + x^5 + x + 1} := \frac{f(x)}{g(x)}, \quad (3)$$

where  $f(x) = x^9 + x^8 + x^4 - x^3 + x^2 + x$  and  $g(x) = x^8 + x^7 - x^6 + x^5 + x + 1$ . Recall that  $h = \frac{m-1}{2}$ . Taking the  $3^h$ -th power on both sides of Eq. (3), we get

$$x^{\frac{1}{3}} = \frac{f(x^{3^h})}{g(x^{3^h})} = \frac{f(x)^9 + f(x)^8 g(x) + f(x)^4 g(x)^5 - f(x)^3 g(x)^6 + f(x)^2 g(x)^7 + f(x) g(x)^8}{f(x)^8 g(x) + f(x)^7 g(x)^2 - f(x)^6 g(x)^3 + f(x)^5 g(x)^4 + f(x) g(x)^8 + g(x)^9}. \quad (4)$$

Plugging Eqs. (3) into (4) gives

$$\begin{aligned} & x(f(x)^8g(x) + f(x)^7g(x)^2 - f(x)^6g(x)^3 + f(x)^5g(x)^4 + f(x)g(x)^8 + g(x)^9)^3 \\ &= (f(x)^9 + f(x)^8g(x) + f(x)^4g(x)^5 - f(x)^3g(x)^6 + f(x)^2g(x)^7 + f(x)g(x)^8)^3. \end{aligned}$$

Thanks to the Magma computation, the above equation can be decomposed into the product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & x(x+1)(x-1)(x^7+x^4+x^3-1)(x^7+x^4-x^3-x-1)(x^7-x^4-x^3-1) \\ & (x^7+x^6-x^3-x-1)(x^7+x^6+x^4-x-1)(x^7+x^6+x^4-x^3-1) \\ & (x^9+x^8-x^6-x^4-x^2+x-1)(x^9-x^8+x^7+x^5+x^3-x-1) \\ & (x^{13}+x^9-x^7-x^6-x^4-x^3+x^2-1)(x^{13}+x^{11}+x^{10}-x^9-x^6-x^3+x^2-x-1) \\ & (x^{13}-x^{11}+x^{10}+x^9+x^7+x^6-x^4-1)(x^{13}+x^{12}+x^{10}-x^9-x^7+x^3-x^2-x-1) \\ & (x^{13}+x^{12}+x^{11}-x^{10}+x^6+x^4-x^3-x-1)(x^{13}+x^{12}-x^{11}+x^{10}+x^7+x^4-x^3-x^2-1) \\ & (x^{17}-x^{15}-x^{13}+x^{11}+x^{10}-x^9-x^8-x^6+x^5+x^4+x^3-x^2+1) \\ & (x^{17}-x^{15}+x^{14}+x^{13}+x^{12}-x^{11}-x^9-x^8+x^7+x^6-x^4-x^2+1) \\ & (x^{17}+x^{16}-x^{15}+x^{14}-x^{13}-x^{11}+x^{10}+x^9+x^7+x^6+x^5-x^3+x-1) \\ & (x^{17}+x^{16}-x^{15}+x^{14}-x^{13}-x^{12}-x^{11}+x^{10}-x^9+x^8-x^7-x^6-x^5-x^3-x+1) \\ & (x^{17}-x^{16}+x^{14}-x^{12}-x^{11}-x^{10}-x^8-x^7+x^6+x^4-x^3+x^2-x-1) \\ & (x^{17}-x^{16}-x^{14}-x^{12}-x^{11}-x^{10}+x^9-x^8+x^7-x^6-x^5-x^4+x^3-x^2+x+1). \end{aligned} \tag{5}$$

Since  $m \equiv 1 \pmod{4}$  is a prime, we can get that  $\gcd(m, 7) = 1$ , so the above six polynomials with degree 7 have no roots in  $\mathbb{F}_{3^m}$  by Lemma 2.1. Since  $m$  is a prime, we can get that  $\gcd(m, 9) = 1$ , so the above two polynomials with degree 9 have no roots in  $\mathbb{F}_{3^m}$  by Lemma 2.1. Since  $m \not\equiv 0 \pmod{17}$  is a prime, we can get that  $\gcd(m, 17) = 1$ , so the above six polynomials with degree 17 have no roots in  $\mathbb{F}_{3^m}$  by Lemma 2.1. When  $m = 13$ , we can check that the above six polynomials with degree 13 have no roots in  $\mathbb{F}_{3^{13}}$  by Magma. In conclusion, we know that Eq. (5) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Therefore, we can prove that Eq. (2) has no solution over  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

Case II:  $x$  is not a square over  $\mathbb{F}_{3^m}$ . Since  $\theta \notin \mathbb{F}_3$ , it follows from Eq. (2) that

$$x^2(\theta + 1) - (\theta - 1)(x^4 - x^3 + x - 1) = 0,$$

which can be written as

$$\theta(x^4 - x^3 - x^2 + x - 1) = x^4 - x^3 + x^2 + x - 1.$$

Similarly, it can be easily checked that  $x^4 - x^3 - x^2 + x - 1 \neq 0$  and

$$\theta = \frac{x^4 - x^3 + x^2 + x - 1}{x^4 - x^3 - x^2 + x - 1}.$$

Then

$$x^{3^h} = x\theta^2 = \frac{x^9 + x^8 + x^4 - x^3 + x^2 + x}{x^8 + x^7 - x^6 + x^5 + x + 1}.$$

Similar to Case I, we can prove that Eq. (2) has no solution over  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

In conclusion, we can get that the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m-1, 3^m-1-2m, 4]$  if  $e = \frac{3^m-1}{2} + \frac{3^h+5}{2}$ ,  $m \equiv 1 \pmod{4}$  and  $m \not\equiv 0 \pmod{17}$ .

(2) For  $e = \frac{3^h+5}{2}$ , The proof is similar to (1) and we omit it here.  $\square$

**Example 1.** Let  $m = 3$  and  $h = 1$ , then  $e = 4$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 + x^5 - x^4 - x^3 + x^2 + x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

**Example 2.** Let  $m = 5$  and  $h = 2$ , then  $e = 121 + 7 = 128$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^5}^*$  with  $\alpha^5 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[242, 232, 4]$  and generator polynomial  $x^{10} - x^8 - x^7 - x^6 + x^4 - x^2 + x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[242, 10, 144]$  and weight enumerator

$$1 + 2420x^{144} + 12100x^{153} + 34364x^{162} + 7744x^{171} + 2420x^{180}.$$

**Remark 1.** Note that for  $e = \frac{3^h+5}{2}$ ,  $2h \equiv -1 \pmod{m}$  and  $m \equiv 3 \pmod{4}$ , Theorem 3.1 is a special case of Open Problem 7.9 [3].

Based on Theorem 2.2, we have the following corollary.

**Corollary 3.1.** Let  $u = \frac{3^m+1}{2}$ ,  $h = \frac{m-1}{2}$  and  $m$  be an odd prime. Then the ternary cyclic code  $C_{(u,v)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if one of the following conditions is met:

- (1)  $v = \frac{3^h+5}{2}$ ,  $m \equiv 1 \pmod{4}$  and  $m \not\equiv 0 \pmod{17}$ ;
- (2)  $v = \frac{3^m-1}{2} + \frac{3^h+5}{2}$ ,  $m \equiv 3 \pmod{4}$ .

**Theorem 3.2.** Let  $m$  be odd,  $e$  be even,  $h$  be an integer satisfying that  $1 \leq h \leq m-1$  and  $e(3^h-1) \equiv \frac{3^m+1}{2} \pmod{3^m-1}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if the equation  $x^e + x - 1 = 0$  has no solution  $x$  in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  satisfying that  $(\chi(x), \chi(x+1)) = (-1, 1)$  or  $(\chi(x), \chi(x+1)) = (-1, -1)$ .

**Proof.** Since  $e$  is even, we can get  $e \notin C_1$ . Assume there exists an integer  $1 \leq j \leq m-1$  such that  $3^j \cdot e \equiv e \pmod{3^m-1}$ , then  $(3^m-1) \mid e(3^j-1)$ . And we can get that  $(3^m-1) \mid e(3^h-1)(3^j-1)$ , i.e.,  $3^m-1 \mid \frac{3^m+1}{2} \cdot (3^j-1)$ . Moreover, we have  $\gcd(3^m-1, \frac{3^m+1}{2} \cdot (3^j-1)) \leq \gcd(3^m-1, \frac{3^m+1}{2}) \cdot \gcd(3^m-1, 3^j-1) = 2(3^{\gcd(m,j)}-1) < 3^m-1$  since  $m$  is odd. This leads to a contradiction. Therefore,  $3^j \cdot e \not\equiv e \pmod{3^m-1}$  for any  $1 \leq j \leq m-1$ , which means that  $|C_e| = m$ . Thus, the condition C1 in Theorem 2.1 is satisfied. For any even integer  $e > 1$ , it can be easily checked that  $(x+1)^e + x^e + 1 = 0$  has the only solution  $x = 1$  over  $\mathbb{F}_3$  and  $(x+1)^e - x^e - 1 = 0$  has the only solution  $x = 0$  over  $\mathbb{F}_3$ . To prove the conditions C2 and C3 in Theorem 2.1 are satisfied, we need to show that there is no solution of equation

$$(x+1)^e = \pm(x^e+1)$$

in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ , which means that the equation

$$(x+1)^{e(3^h-1)} = (x^e+1)^{3^h-1}$$

has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Plugging  $\chi(x) = x^{\frac{3^m-1}{2}}$  into above equation, we have

$$\chi(x+1)(x^{e+1} + x^e + x + 1) = \chi(x)x^{e+1} + 1, \quad (6)$$

since  $e(3^h - 1) \equiv \frac{3^m+1}{2} \pmod{3^m-1}$ . For  $x \in \mathbb{F}_{3^m}^*$ ,  $\chi(x) = 1$  if  $x$  is a square. Otherwise,  $\chi(x) = -1$ . Next, we can divide the solutions of Eq. (6) into the following four cases.

Case I:  $\chi(x) = \chi(x+1) = 1$ . In this case, Eq. (6) turns to  $x(x^{e-1} + 1) = 0$ , which implies that  $x = 0$  or  $x^{e-1} = -1$ . Recall that  $m$  is odd and  $e$  is even, then  $-1$  is not a square in  $\mathbb{F}_{3^m}$  and  $e-1$  is odd. It implies that  $x^{e-1} = -1$  holds only if  $\chi(x) = -1$ . Therefore, there is no solution of Eq. (6) in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

Case II:  $\chi(x) = 1, \chi(x+1) = -1$ . In this case, Eq. (6) turns to  $(x-1)(x^e - 1) = 0$ , which implies that  $x = 1$  or  $x^e = 1$ . Note that  $m$  is odd and  $e$  is even. If  $x^e - 1 = 0$ , then  $x = \pm 1$ . Therefore, there is no solution of Eq. (6) in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

Case III:  $\chi(x) = \chi(x+1) = -1$ . In this case, Eq. (6) turns to

$$x^e + x - 1 = 0.$$

which implies that there is no solution of Eq. (6) in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  satisfying that  $\chi(x) = \chi(x+1) = -1$ .

Case IV:  $\chi(x) = -1, \chi(x+1) = 1$ . In this case, Eq. (6) becomes

$$x^{e+1} - x^e - x = 0.$$

Let  $y = \frac{1}{x}$ , then

$$y^e + y - 1 = 0,$$

where  $\chi(y) = \chi(y+1) = -1$  and  $y \in \mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Similarly, there is no solution of Eq. (6) in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

In summary, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if the equation  $x^e + x - 1 = 0$  has no solution  $x$  in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  satisfying that  $(\chi(x), \chi(x+1)) = (-1, 1)$  or  $(\chi(x), \chi(x+1)) = (-1, -1)$ .  $\square$

**Example 3.** Let  $m = 3$  and  $h = 2$ , then  $e = 18$ . The equation  $x^e + x - 1 = 0$  becomes  $x^{18} + x - 1 = 0$ . Let  $\omega$  is a primitive element of  $\mathbb{F}_{3^3}$ . The equation  $x^{18} + x - 1 = 0$  can be decomposed into the product of some irreducible factors over  $\mathbb{F}_{3^3}$  as

$$(x + \omega^{17})(x + \omega^{23})(x + \omega^{25})(x^2 + x - 1)(x^{13} + x^{12} + x^{11} - x^8 - x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - 1) = 0.$$

We can verify that  $\chi(-\omega^{17}) = \chi(-\omega^{23}) = \chi(-\omega^{25}) = 1$  by Magma. Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 + x^5 - x^3 - x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

**Lemma 3.2.** Let  $m$  be an odd prime and  $e = \frac{3^{m-1}-1}{2} + 3^h - 1$ , where  $h$  is an integer with  $0 \leq h \leq m-1$ . Then  $e \notin C_1$  and  $|C_e| = m$  in the following two conditions:

- (1)  $2h = 1 \pmod{m}$ , i.e.,  $h = \frac{m+1}{2}$ ;
- (2)  $2h = -1 \pmod{m}$ , i.e.,  $h = \frac{m-1}{2}$ .

**Proof.** It is easy to see that  $e \notin C_1$  since  $e$  is even. So, we only need to prove that  $|C_e| = m$ . It is well known that  $|C_e| \mid m$ , then we have  $|C_e| = 1$  or  $|C_e| = m$ , since  $m$  is an odd prime. If  $|C_e| = 1$ , then  $3(\frac{3^{m-1}-1}{2} + 3^h - 1) \equiv \frac{3^{m-1}-1}{2} + 3^h - 1 \pmod{3^m - 1}$ . So we can get that  $(3^m - 1) \mid 2(\frac{3^{m-1}-1}{2} + 3^h - 1)$ . Since  $2(\frac{3^{m-1}-1}{2} + 3^h - 1) = 3^{m-1} + 2 \cdot 3^h - 3$  and  $3^{m-1} + 2 \cdot 3^h - 3 \leq 3^m - 1$ , we obtain that  $3^{m-1} + 2 \cdot 3^h - 3 = 3^m - 1$ , then  $3^{m-1} = 3^h - 1$ . This is in contradiction with the assumption that  $0 \leq h \leq m - 1$ . Consequently, we can conclude that  $|C_e| = m$ .  $\square$

**Theorem 3.3.** *Let  $m$  be an odd prime and  $e = \frac{3^{m-1}-1}{2} + 3^h - 1$ , where  $h$  is an integer with  $0 \leq h \leq m - 1$ . If  $2h \equiv \pm 1 \pmod{m}$ , i.e.,  $h = \frac{m \pm 1}{2}$ , then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$ .*

**Proof.** According to Lemma 3.2, the condition C1 in Theorem 2.1 is met. It can be easily checked that  $(x + 1)^e + x^e + 1 = 0$  has the only solution  $x = 1$  over  $\mathbb{F}_3$  and  $(x + 1)^e - x^e - 1 = 0$  has the only solution  $x = 0$  over  $\mathbb{F}_3$ . To prove the conditions C2 and C3 in Theorem 2.1 are satisfied, we need to show that there is no solution of equation

$$(x + 1)^e = \pm(x^e + 1)$$

in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ , which means that

$$(x + 1)^{6e} = x^{6e} + 1 - x^{3e} \tag{7}$$

has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ .

Assume that  $x \in \mathbb{F}_{3^m} \setminus \mathbb{F}_3$  is a solution of Eq. (7). Then we can discuss the following two cases:  $x$  is a square and  $x$  is not a square over  $\mathbb{F}_{3^m}$ .

Case I:  $x$  is a square over  $\mathbb{F}_{3^m}$ . It can be verified that  $x^{6e} = x^{2 \cdot 3^{h+1} - 8}$  and  $x^{3e} = x^{\frac{3^m-1}{2} + 3^{h+1} - 4} = x^{3^{h+1} - 4}$ . Let  $x^{3^{h+1}} := \theta$ , we can get

$$\begin{aligned} & (x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)\theta^2 \\ & + (x^{12} - x^{11} + x^{10} - x^9 - x^7 + x^6 - x^5 + x^4)\theta \\ & - x^{16} + x^{15} - x^{14} + x^{13} - x^{12} + x^{11} - x^{10} + x^9 = 0. \end{aligned} \tag{8}$$

First, we suppose that  $x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1 \neq 0$ . Otherwise, we have  $x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)(x^2 + x - 1)(x^2 - x - 1) = 0$ . Since  $m$  is an odd prime and  $\gcd(m, 2) = 1$ , it then follows from Lemma 2.1 that the above three factors of degree 2 are irreducible polynomials over  $\mathbb{F}_{3^m}$ , which implies that  $x = 1$ . This is contrary to the assumption that  $x \in \mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Then Eq. (8) becomes

$$((x^2 + x - 1)\theta + x^5(x^2 - x - 1))((x^2 - x - 1)\theta - x^4(x^2 + x - 1)) = 0.$$

Hence, we can get  $\theta = \frac{-x^5(x^2 - x - 1)}{x^2 + x - 1}$  or  $\theta = \frac{x^4(x^2 + x - 1)}{x^2 - x - 1}$ .

Case I.1:  $\theta = \frac{-x^5(x^2 - x - 1)}{x^2 + x - 1}$ . In this case, we have

$$\theta = x^{3^{h+1}} = -\frac{x^5(x^2 - x - 1)}{x^2 + x - 1} := \frac{f(x)}{g(x)}, \tag{9}$$

where  $f(x) = -x^5(x^2 - x - 1) = -x^7 + x^6 + x^5$  and  $g(x) = x^2 + x - 1$ .

When  $2h \equiv 1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq (9), we obtain

$$x^{27} = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{-f(x)^7 + f(x)^6g(x) + f(x)^5g(x)^2}{f(x)^2g(x)^5 + f(x)g(x)^6 - g(x)^7} := \frac{F(x)}{G(x)},$$

where  $F(x) = -f(x)^7 + f(x)^6g(x) + f(x)^5g(x)^2$  and  $G(x) = f(x)^2g(x)^5 + f(x)g(x)^6 - g(x)^7$ , we calculate

$$x^{27}G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$x^{25}(x+1)^{25}(x-1) = 0.$$

This is contrary to the assumption that  $x \in \mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

When  $2h \equiv -1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq (9), we obtain

$$x^3 = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{-f(x)^7 + f(x)^6g(x) + f(x)^5g(x)^2}{f(x)^2g(x)^5 + f(x)g(x)^6 - g(x)^7} := \frac{F(x)}{G(x)},$$

where  $F(x) = -f(x)^7 + f(x)^6g(x) + f(x)^5g(x)^2$  and  $G(x) = f(x)^2g(x)^5 + f(x)g(x)^6 - g(x)^7$ , we calculate

$$x^3G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & x^3(x+1)^3(x-1)(x^7+x^4-x^3-x^2-1)(x^7+x^5+x^4-x^3-1)(x^7+x^6+x^4+x^2-x+1) \\ & (x^7+x^6+x^5+x^3+x^2-x+1)(x^7-x^6+x^5+x^3+x+1)(x^7-x^6+x^5+x^4+x^2+x+1) = 0. \end{aligned}$$

When  $m = 7$ , we can check that the above six polynomials with degree 7 have no roots in  $\mathbb{F}_{3^7}$  by Magma. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

Case I.2:  $\theta = \frac{x^4(x^2+x-1)}{x^2-x-1}$ . In this case, we have

$$\theta = x^{3^{h+1}} = \frac{x^4(x^2+x-1)}{x^2-x-1} := \frac{f(x)}{g(x)}, \quad (10)$$

where  $f(x) = x^4(x^2+x-1) = x^6+x^5-x^4$  and  $g(x) = x^2-x-1$ .

When  $2h \equiv 1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq. (10), we obtain

$$x^{27} = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{f(x)^6 + f(x)^5g(x) - f(x)^4g(x)^2}{f(x)^2g(x)^4 - f(x)g(x)^5 - g(x)^6} := \frac{F(x)}{G(x)},$$

where  $F(x) = f(x)^6 + f(x)^5g(x) - f(x)^4g(x)^2$  and  $G(x) = f(x)^2g(x)^4 - f(x)g(x)^5 - g(x)^6$ , we calculate

$$x^{27}G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$x^{16}(x+1)^{25}(x^3+x^2-x+1)(x^3-x^2+x+1) = 0.$$

When  $m = 3$ , we can check that the above two polynomials with degree 3 have no roots in  $\mathbb{F}_{3^3}$  by Magma. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

When  $2h \equiv -1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq. (10), we obtain

$$x^3 = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{f(x)^6 + f(x)^5g(x) - f(x)^4g(x)^2}{f(x)^2g(x)^4 - f(x)g(x)^5 - g(x)^6} := \frac{F(x)}{G(x)},$$

where  $F(x) = f(x)^6 + f(x)^5g(x) - f(x)^4g(x)^2$  and  $G(x) = f(x)^2g(x)^4 - f(x)g(x)^5 - g(x)^6$ , we calculate

$$x^3G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & x^3(x+1)^3(x^{15} + x^{14} + x^{13} + x^{12} - x^{11} + x^9 - x^7 + x^6 + x^4 - x^3 + x - 1) \\ & (x^{15} - x^{14} + x^{12} - x^{11} - x^9 + x^8 - x^6 + x^4 - x^3 - x^2 - x - 1) = 0. \end{aligned} \quad (11)$$

When  $m = 3$ , we can check that the above two polynomials with degree 15 are all decomposed as the product of three irreducible polynomials with degree 5 over  $\mathbb{F}_{3^3}$  by Magma. Furthermore, we know that Eq. (11) has no solution in  $\mathbb{F}_{3^3} \setminus \mathbb{F}_3$ , which follows from Lemma 2.1. When  $m = 5$ , we can verify that the above two polynomials with degree 15, are decomposed into five irreducible polynomials with degree 3 over  $\mathbb{F}_{3^5}$  by Magma. Furthermore, we know that Eq. (11) has no solution in  $\mathbb{F}_{3^5} \setminus \mathbb{F}_3$  by Lemma 2.1. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

Case II:  $x$  is not a square in  $\mathbb{F}_{3^m}$ . It can be verified that  $x^{6e} = x^{2 \cdot 3^{h+1} - 8}$  and  $x^{3e} = x^{\frac{3^m - 1}{2} + 3^{h+1} - 4} = -x^{3^{h+1} - 4}$ . Let  $\theta = x^{3^{h+1}}$ , we can get

$$\begin{aligned} & (x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)\theta^2 \\ & - (x^{12} - x^{11} + x^{10} - x^9 - x^8 - x^7 + x^6 - x^5 + x^4)\theta \\ & - x^{16} + x^{15} - x^{14} + x^{13} - x^{12} + x^{11} - x^{10} + x^9 = 0. \end{aligned} \quad (12)$$

From the proof of Case I, we have  $x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1 \neq 0$ . Then Eq. (12) becomes

$$((x^2 + 1)\theta + x^5(x - 1))((x - 1)\theta - x^4(x^2 + 1)) = 0,$$

which implies that  $\theta = \frac{-x^5(x-1)}{x^2+1}$  or  $\theta = \frac{x^4(x^2+1)}{x-1}$ .

Case II.1:  $\theta = \frac{-x^5(x-1)}{x^2+1}$ . In this case, we have

$$\theta = x^{3^{h+1}} = -\frac{x^5(x-1)}{x^2+1} := \frac{f(x)}{g(x)}, \quad (13)$$

where  $f(x) = -x^5(x-1) = -x^6 + x^5$  and  $g(x) = x^2 + 1$ .

When  $2h \equiv 1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq. (13), we obtain

$$x^{27} = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{-f(x)^6 + f(x)^5g(x)}{f(x)^2g(x)^4 + g(x)^6} := \frac{F(x)}{G(x)},$$

where  $F(x) = -f(x)^6 + f(x)^5g(x)$  and  $G(x) = f(x)^2g(x)^4 + g(x)^6$ , we calculate

$$x^{27}G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$x^{25}(x+1)^{16}(x^3+x^2-1)(x^3-x^2+1) = 0.$$

When  $m = 3$ , we can check that the above two polynomials with degree 3 have no roots in  $\mathbb{F}_{3^3}$  by Magma. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

When  $2h \equiv -1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq. (13), we obtain

$$x^3 = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{-f(x)^6 + f(x)^5g(x)}{f(x)^2g(x)^4 + g(x)^6} := \frac{F(x)}{G(x)},$$

where  $F(x) = -f(x)^6 + f(x)^5g(x)$  and  $G(x) = f(x)^2g(x)^4 + g(x)^6$ , we calculate

$$x^3G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & x^3(x+1)^3(x^{15} + x^{14} - x^{13} - x^{12} - x^9 - x^8 + x^7 + x^6 - x^5 + x + 1) \\ & (x^{15} - x^{14} - x^{13} + x^{12} + x^{11} - x^7 - x^6 - x^5 - x^4 + x^3 + x^2 - x + 1) = 0. \end{aligned} \quad (14)$$

When  $m = 3$ , we can check that the above two polynomials with degree 15 are all decomposed as the product of three irreducible polynomials with degree 5 over  $\mathbb{F}_{3^3}$  by Magma. Furthermore, we know that Eq. (14) has no solution in  $\mathbb{F}_{3^3} \setminus \mathbb{F}_3$ , which follows from Lemma 2.1. When  $m = 5$ , we can verify that the above two polynomials with degree 15 can be factored into five irreducible polynomials with degree 3 over  $\mathbb{F}_{3^5}$  by Magma. Furthermore, we know that Eq. (14) has no solution in  $\mathbb{F}_{3^5} \setminus \mathbb{F}_3$  by Lemma 2.1. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

Case II.2:  $\theta = \frac{x^4(x^2+1)}{x-1}$ . In this case, we have

$$\theta = x^{3^{h+1}} = \frac{x^4(x^2+1)}{x-1} := \frac{f(x)}{g(x)}, \quad (15)$$

where  $f(x) = x^4(x^2+1) = x^6 + x^4$  and  $g(x) = x-1$ .

When  $2h \equiv 1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq. (15), we obtain

$$x^{27} = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{f(x)^6 + f(x)^4g(x)^2}{f(x)g(x)^5 - g(x)^6} := \frac{F(x)}{G(x)},$$

where  $F(x) = f(x)^6 + f(x)^4g(x)^2$  and  $G(x) = f(x)g(x)^5 - g(x)^6$ , we calculate

$$x^{27}G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$x^{16}(x+1)^{16}(x^3-x^2+1)(x^3-x-1) = 0.$$

When  $m = 3$ , we can check that the above two polynomials with degree 3 have no roots in  $\mathbb{F}_{3^3}$  by Magma. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

When  $2h \equiv -1 \pmod{m}$  and taking the  $3^{h+1}$ -th power on both sides of Eq. (15), we obtain

$$x^3 = \left(\frac{f(x)}{g(x)}\right)^{3^{h+1}} = \frac{f(x)^6 + f(x)^4g(x)^2}{f(x)g(x)^5 - g(x)^6} := \frac{F(x)}{G(x)},$$

where  $F(x) = f(x)^6 + f(x)^4g(x)^2$  and  $G(x) = f(x)g(x)^5 - g(x)^6$ , we calculate

$$x^3G(x) - F(x) = 0.$$

Then the left-hand side of the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & x^3(x+1)^3(x^{15} + x^{14} - x^{10} + x^9 + x^8 - x^7 - x^6 - x^3 - x^2 + x + 1) \\ & (x^{15} - x^{14} + x^{13} + x^{12} - x^{11} - x^{10} - x^9 - x^8 + x^4 + x^3 - x^2 - x + 1) = 0. \end{aligned} \quad (16)$$

When  $m = 3$ , we can check that the above two polynomials with degree 15 are all decomposed as the product of three irreducible polynomials with degree 5 over  $\mathbb{F}_{3^3}$  by Magma. Furthermore, we know that Eq. (16) has no solution in  $\mathbb{F}_{3^3} \setminus \mathbb{F}_3$ , which follows from Lemma 2.1. When  $m = 5$ , we can verify that the above two polynomials with degree 15, can be factored into five irreducible polynomials with degree 3 over  $\mathbb{F}_{3^5}$  by Magma. Furthermore, we know that Eq. (16) has no solution in  $\mathbb{F}_{3^5} \setminus \mathbb{F}_3$  by Lemma 2.1. Therefore, we conclude that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$  if  $m$  is an odd prime.

To summarize, we know that Eq. (7) has no solution in  $\mathbb{F}_{3^m} \setminus \mathbb{F}_3$ . Hence, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if  $2h \equiv \pm 1 \pmod{m}$ .  $\square$

**Example 4.** Let  $m = 3$  and  $h = \frac{m+1}{2} = 2$ , then  $e = 12$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 + x^5 - x^4 - x^3 + x^2 + x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

**Example 5.** Let  $m = 3$  and  $h = \frac{m-1}{2} = 1$ , then  $e = 6$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 + x^5 - x^3 - x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

Based on Theorem 2.2, we have the following corollary.

**Corollary 3.2.** *Let  $m \geq 3$  be an odd prime. Let  $u = \frac{3^m+1}{2}$ ,  $v = 2 \cdot 3^{m-1} + 3^h - 2$ , where  $h$  is an integer with  $0 \leq h \leq m - 1$ . If  $2h \equiv \pm 1 \pmod{m}$ , i.e.,  $h = \frac{m \pm 1}{2}$ , then the ternary cyclic code  $C_{(u,v)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$ .*

## 4 New optimal ternary cyclic codes $C_{(1,e)}$ with explicit values $e$

In this section, by using the multivariate method, we propose five classes of optimal ternary cyclic codes  $C_{(1,e)}$  with explicit values  $e$ .

First, we give the following lemma needed later, which is an immediate consequence of Lemma 2.1 in [13].

**Lemma 4.1.** *Let  $f(x)$  be an irreducible polynomial over  $\mathbb{F}_q$  of degree  $n$  and  $k$  be a positive integer, where  $q$  is a prime power. Then  $f(x) = 0$  has no solution in  $\mathbb{F}_{q^k}$  if and only if  $k \not\equiv 0 \pmod{n}$ .*

**Theorem 4.1.** *Let  $m, e$  be two positive integers satisfying  $m > 1$  and  $7e \equiv 4 \pmod{3^m - 1}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m$  is odd and  $m \not\equiv 0 \pmod{9}$ .*

**Proof.** From the condition  $7e \equiv 4 \pmod{3^m - 1}$ , we can get that  $e$  is even and  $m \not\equiv 0 \pmod{6}$ , which means that  $e \notin C_1$  and  $\gcd(7, 3^m - 1) = 1$ . Therefore, we have  $\gcd(e, 3^m - 1) = \gcd(7e, 3^m - 1) = \gcd(4, 3^m - 1) = 2$  since  $m$  is odd. It follows from Lemma 2.2 that  $|C_e| = m$ . Hence, the condition C1 of Theorem 2.1 is satisfied. For any  $x \in \mathbb{F}_{3^m}$ , there exists  $\alpha, \beta \in \mathbb{F}_{3^m}$  such that  $x + 1 = \alpha^7, x = \beta^7$  and

$$\alpha^7 - \beta^7 = 1. \quad (17)$$

Next, we show that the conditions C2 and C3 of Theorem 2.1 are satisfied, respectively.

First of all, the equation  $(x + 1)^e + x^e + 1 = 0$  can be written as

$$\alpha^4 + \beta^4 = -1. \quad (18)$$

Eliminating  $\alpha$  from Eqs. (17) and (18) gives

$$(-\beta^4 - 1)^7 = (\beta^7 + 1)^4.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & (\beta - 1)^4(\beta^8 + \beta^6 - \beta^5 - \beta^3 - \beta^2 + \beta - 1)(\beta^8 - \beta^7 + \beta^6 - \beta^4 + \beta^2 - \beta + 1) \\ & (\beta^8 - \beta^7 + \beta^6 + \beta^5 + \beta^3 - \beta^2 - 1) = 0. \end{aligned} \quad (19)$$

By Lemma 4.1, we can get that three irreducible polynomials of degree 8 in Eq. (19) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{8}$ , which means that  $\beta = 1$ . Hence, we can get that  $x = 1$  is a unique solution of the equation  $(x + 1)^e + x^e + 1 = 0$  if and only if  $m \not\equiv 0 \pmod{8}$ .

Secondly, the equation  $(x + 1)^e - x^e - 1 = 0$  can be simplified as

$$\alpha^4 - \beta^4 = 1. \quad (20)$$

Combining Eqs. (17) and (20) leads to

$$(\beta^4 + 1)^7 = (\beta^7 + 1)^4.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\beta^4(\beta^2 + 1)(\beta^9 + \beta^8 - \beta^7 - \beta^6 - \beta^3 - \beta^2 + \beta - 1)(\beta^9 - \beta^8 + \beta^7 + \beta^6 + \beta^3 + \beta^2 - \beta - 1) = 0. \quad (21)$$

By Lemma 4.1, we can get that the irreducible polynomial of degree 2 in Eq. (21) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{2}$  and two irreducible polynomials of degree 9 in Eq. (21) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{9}$ , which means that  $\beta = 0$ . Hence, we can get that  $x = 0$  is a unique solution of the equation  $(x + 1)^e - x^e - 1 = 0$  if and only if  $m \not\equiv 0 \pmod{2}$  and  $m \not\equiv 0 \pmod{9}$ .

To sum up, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m$  is odd and  $m \not\equiv 0 \pmod{9}$ .  $\square$

Note that the conditions in Theorem 4.1 hold in the following cases: (i)  $e = \frac{5 \cdot 3^m - 1}{7}$ , when  $m \equiv 1 \pmod{6}$ ; (ii)  $e = \frac{2 \cdot 3^m + 2}{7}$ , when  $m \equiv 3 \pmod{6}$  and  $m \not\equiv 0 \pmod{9}$ ; (iii)  $e = \frac{6 \cdot 3^m - 2}{7}$ , when  $m \equiv 5 \pmod{6}$ .

**Example 6.** Let  $m = 3$ , then  $e = 8$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 - x^5 + x^4 + x^3 - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

**Theorem 4.2.** Let  $m, e$  be two positive integers satisfying  $m > 1$  and  $5e \equiv 3^m - 3 \pmod{3^m - 1}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m \not\equiv 0 \pmod{6}$ .

**Proof.** Since  $5e \equiv 3^m - 3 \pmod{3^m - 1}$ , then  $e$  is even and  $e \notin C_1$ . Suppose there exists an integer  $1 \leq j \leq m - 1$  such that  $3^j \cdot e \equiv e \pmod{3^m - 1}$ , then  $(3^m - 1) \mid e(3^j - 1)$ . And we can get that  $(3^m - 1) \mid 5e(3^j - 1)$ , i.e.,  $(3^m - 1) \mid (3^m - 3)(3^j - 1)$ . Furthermore, we have  $\gcd(3^m - 1, (3^m - 3)(3^j - 1)) \leq \gcd(3^m - 1, 3^m - 3) \cdot \gcd(3^m - 1, 3^j - 1) = 2(3^{\gcd(m,j)} - 1) < 3^m - 1$ , which leads to a contradiction. Therefore,  $3^j \cdot e \not\equiv e \pmod{3^m - 1}$  for any  $1 \leq j \leq m - 1$ , which means that  $|C_e| = m$ . Hence, the condition C1 of Theorem 2.1 is satisfied. It can be checked that  $m \not\equiv 0 \pmod{4}$  since  $5e \equiv 3^m - 3 \pmod{3^m - 1}$ , then we have  $\gcd(5, 3^m - 1) = 1$ . For any  $x \in \mathbb{F}_{3^m}$ , there exists  $\alpha, \beta \in \mathbb{F}_{3^m}$  such that  $x + 1 = \alpha^5, x = \beta^5$  and

$$\alpha^5 - \beta^5 = 1. \quad (22)$$

Next, we show that the conditions C2 and C3 of Theorem 2.1 are satisfied, respectively.

First, the equation  $(x + 1)^e + x^e + 1 = 0$  can be written as  $\alpha^{-2} + \beta^{-2} = -1$ . It can be verified that  $\alpha \neq 0, \beta \neq 0$  and  $(\beta^2 + 1)(\beta^{-2} + 1) \neq 0$ . Otherwise, if  $\beta^{-2} = -1$ , then  $\alpha^{-2} = 0$ , which contradicts to the assumption that  $\alpha \in \mathbb{F}_{3^m}$ . If  $\beta^2 = -1$ , then  $\beta^{-2} = -1$  and  $\alpha^{-2} = 0$ , a contradiction. Hence, we have

$$\alpha^2 = -\frac{1}{\beta^{-2} + 1} = -\frac{\beta^2}{\beta^2 + 1}. \quad (23)$$

Plugging Eqs. (23) into (22) turns to

$$(-\beta^2)^5 = (\beta^5 + 1)^2(\beta^2 + 1)^5.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$(\beta - 1)^2(\beta^6 + \beta^5 + \beta^3 + \beta + 1)(\beta^6 - \beta^5 + \beta^4 + \beta^2 + \beta - 1)(\beta^6 - \beta^5 - \beta^4 - \beta^2 + \beta - 1) = 0. \quad (24)$$

By Lemma 4.1, we can get that three irreducible polynomials of degree 6 in Eq. (24) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{6}$ , which means that  $\beta = 1$ . Hence, we can get that  $x = 1$  is a unique solution of the equation  $(x + 1)^e + x^e + 1 = 0$  if and only if  $m \not\equiv 0 \pmod{6}$ .

Secondly, we calculate the solutions of equation  $(x + 1)^e - x^e - 1 = 0$ . Obviously,  $x = 0$  is a solution of this equation. If  $x \neq 0$ , this equation can be simplified as  $\alpha^{-2} - \beta^{-2} = 1$ . Similarly, we can get  $(\beta^2 + 1)(\beta^{-2} + 1) \neq 0$ . Hence, we have

$$\alpha^2 = \frac{1}{\beta^{-2} + 1} = \frac{\beta^2}{\beta^2 + 1}. \quad (25)$$

Eliminating  $\alpha$  from Eqs. (22) and (25), we can get

$$(\beta^2)^5 = (\beta^5 + 1)^2(\beta^2 + 1)^5.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\beta^{20} - \beta^{18} + \beta^{16} - \beta^{15} + \beta^{14} + \beta^{13} - \beta^{12} - \beta^{11} + \beta^{10} - \beta^9 - \beta^8 + \beta^7 + \beta^6 - \beta^5 + \beta^4 - \beta^2 + 1 = 0. \quad (26)$$

By Lemma 4.1, we can get that Eq. (26) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{20}$ . And since  $m \not\equiv 0 \pmod{4}$ , then Eq. (26) has no solution in  $\mathbb{F}_{3^m}$ . Hence, we can get that  $x = 0$  is a unique solution of the equation  $(x + 1)^e - x^e - 1 = 0$ .

In conclusion, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m \not\equiv 0 \pmod{6}$ .  $\square$

Note that the conditions in Theorem 4.2 hold in the following cases: (i)  $e = \frac{3^m - 3}{5}$ , when  $m \equiv 1 \pmod{4}$ ; (ii)  $e = \frac{4 \cdot 3^m - 6}{5}$ , when  $m \equiv 2 \pmod{4}$  and  $m \not\equiv 0 \pmod{6}$ ; (iii)  $e = \frac{2 \cdot 3^m - 4}{5}$ , when  $m \equiv 3 \pmod{4}$ .

**Example 7.** Let  $m = 2$ , then  $e = 6$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^2}^*$  with  $\alpha^2 - \alpha - 1$ . Then the code  $C_{(1,e)}$  has parameters  $[8, 4, 4]$  and generator polynomial  $x^4 - x^3 - x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[8, 4, 4]$  and weight enumerator

$$1 + 20x^4 + 32x^5 + 8x^6 + 16x^7 + 4x^8.$$

**Example 8.** Let  $m = 6$ , then  $e = 582$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^6}^*$  with  $\alpha^6 - \alpha^4 + \alpha^2 - \alpha - 1$ . Then the code  $C_{(1,e)}$  has parameters  $[728, 716, 3]$  and generator polynomial  $x^{12} - x^{11} + x^{10} + x^8 + x^6 + x^2 + x - 1$ . Clearly, this is an almost optimal ternary cyclic code.

**Theorem 4.3.** *Let  $m, e$  be two positive integers satisfying  $m > 1$  and  $7e \equiv 3^m - 3 \pmod{3^m - 1}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m \not\equiv 0 \pmod{3}$ ,  $m \not\equiv 0 \pmod{4}$  and  $m \not\equiv 0 \pmod{22}$ .*

**Proof.** Since  $7e \equiv 3^m - 3 \pmod{3^m - 1}$ , it can be verified that  $e$  is even and  $e \notin C_1$ . Suppose that there exists an integer  $1 \leq j \leq m - 1$  such that  $3^j \cdot e \equiv e \pmod{3^m - 1}$ , then  $(3^m - 1) \mid e(3^j - 1)$ . And we can get that  $(3^m - 1) \mid 7e(3^j - 1)$ , i.e.,  $(3^m - 1) \mid (3^m - 3)(3^j - 1)$ . Moreover, we have  $\gcd(3^m - 1, (3^m - 3)(3^j - 1)) \leq \gcd(3^m - 1, 3^m - 3) \cdot \gcd(3^m - 1, 3^j - 1) = 2(3^{\gcd(m,j)} - 1) < 3^m - 1$ . This leads to a contradiction. Therefore,  $3^j \cdot e \not\equiv e \pmod{3^m - 1}$  for any  $1 \leq j \leq m - 1$ , which means that  $|C_e| = m$ . Hence, the condition C1 of Theorem 2.1 is met. It can be checked that

$m \not\equiv 0 \pmod{6}$  since  $7e \equiv 3^m - 3 \pmod{3^m - 1}$ , then  $\gcd(7, 3^m - 1) = 1$ . For any  $x \in \mathbb{F}_{3^m}$ , there exists  $\alpha, \beta \in \mathbb{F}_{3^m}$  such that  $x + 1 = \alpha^7$ ,  $x = \beta^7$  and

$$\alpha^7 - \beta^7 = 1. \quad (27)$$

Next, we show that conditions C2 and C3 of Theorem 2.1 are satisfied, respectively.

At first, the equation  $(x + 1)^e + x^e + 1 = 0$  can be written as  $\alpha^{-2} + \beta^{-2} = -1$ . Similar to the proof of Theorem 4.2, we can get  $(\beta^2 + 1)(\beta^{-2} + 1) \neq 0$  and

$$\alpha^2 = -\frac{1}{\beta^{-2} + 1} = -\frac{\beta^2}{\beta^2 + 1}. \quad (28)$$

Combining Eqs. (27) and (28) gives

$$(-\beta^2)^7 = (\beta^7 + 1)^2(\beta^2 + 1)^7.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$(\beta - 1)^{10}(\beta^3 - \beta - 1)(\beta^3 + \beta^2 - 1)(\beta^4 - \beta - 1)(\beta^4 + \beta^3 - 1)(\beta^4 - \beta^3 + \beta^2 - \beta + 1) = 0. \quad (29)$$

By Lemma 4.1, we can get that two irreducible polynomials of degree 3 in Eq. (29) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{3}$  and three irreducible polynomials of degree 4 in Eq. (29) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{4}$ , which means that  $\beta = 1$ . Hence, we can get that  $x = 1$  is a unique solution of the equation  $(x + 1)^e + x^e + 1 = 0$  if and only if  $m \not\equiv 0 \pmod{3}$  and  $m \not\equiv 0 \pmod{4}$ .

Secondly, we calculate the solutions of equation  $(x + 1)^e - x^e - 1 = 0$ . Obviously,  $x = 0$  is a solution of this equation. If  $x \neq 0$ , this equation can be written as  $\alpha^{-2} - \beta^{-2} = 1$ . Similarly, we can get  $(\beta^2 + 1)(\beta^{-2} + 1) \neq 0$  and

$$\alpha^2 = \frac{1}{\beta^{-2} + 1} = \frac{\beta^2}{\beta^2 + 1}. \quad (30)$$

Eliminating  $\alpha$  from Eqs. (27) and (30) turns to

$$(\beta^2)^7 = (\beta^7 + 1)^2(\beta^2 + 1)^7.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} &(\beta^3 + \beta^2 + \beta - 1)(\beta^3 - \beta^2 - \beta - 1) \\ &(\beta^{22} - \beta^{20} + \beta^{19} + \beta^{16} + \beta^{14} - \beta^{13} - \beta^{12} - \beta^{10} - \beta^9 + \beta^8 + \beta^6 + \beta^3 - \beta^2 + 1) = 0. \end{aligned} \quad (31)$$

By Lemma 4.1, we can get that two irreducible polynomials of degree 3 in Eq. (31) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{3}$  and the irreducible polynomial of degree 22 in Eq. (31) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{22}$ . Hence, we can get that  $x = 0$  is a unique solution of the equation  $(x + 1)^e - x^e - 1 = 0$  if and only if  $m \not\equiv 0 \pmod{3}$  and  $m \not\equiv 0 \pmod{22}$ .

In summary, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m \not\equiv 0 \pmod{3}$ ,  $m \not\equiv 0 \pmod{4}$  and  $m \not\equiv 0 \pmod{22}$ .  $\square$

Note that the conditions in Theorem 4.3 hold in the following cases: (i)  $e = \frac{3^m - 3}{7}$ , when  $m \equiv 1 \pmod{6}$ ; (ii)  $e = \frac{2 \cdot 3^m - 4}{7}$ , when  $m \equiv 2 \pmod{6}$ ,  $m \not\equiv 0 \pmod{4}$  and  $m \not\equiv 0 \pmod{22}$ ; (iii)  $e = \frac{3^{m+1} - 5}{7}$ , when  $m \equiv 4 \pmod{6}$ ,  $m \not\equiv 0 \pmod{4}$  and  $m \not\equiv 0 \pmod{22}$ ; (iv)  $e = \frac{4 \cdot 3^m - 6}{7}$ , when  $m \equiv 5 \pmod{6}$ .

**Example 9.** Let  $m = 2$ , then  $e = 2$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^2}^*$  with  $\alpha^2 - \alpha - 1$ . Then the code  $C_{(1,e)}$  has parameters  $[8, 4, 4]$  and generator polynomial  $x^4 - x^3 - x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[8, 4, 4]$  and weight enumerator

$$1 + 20x^4 + 32x^5 + 8x^6 + 16x^7 + 4x^8.$$

**Example 10.** Let  $m = 4$ , then  $e = 34$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^4}^*$  with  $\alpha^4 - \alpha^3 - 1$ . Then the code  $C_{(1,e)}$  has parameters  $[80, 72, 3]$  and generator polynomial  $x^8 - x^7 + x^6 - x^4 - x^3 - x^2 - x - 1$ . We can verify this is an almost optimal ternary cyclic code by the collection of the tables of best linear codes known maintained by Markus Grassl at <http://www.codetables.de/>.

**Theorem 4.4.** Let  $m, e$  be two positive integers satisfying  $m > 1$ , and  $5e \equiv 3^m - 5 \pmod{3^m - 1}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  and if and only if  $m$  is odd.

**Proof.** From the condition  $5e \equiv 3^m - 5 \pmod{3^m - 1}$ , we can get that  $e$  is even and  $m \not\equiv 0 \pmod{4}$ , then  $e \notin C_1$  and  $\gcd(5, 3^m - 1) = 1$ . Since  $m$  is odd, then  $\gcd(e, 3^m - 1) = \gcd(5e, 3^m - 1) = \gcd(3^m - 5, 3^m - 1) = 2$ . Therefore, we have  $|C_e| = m$ , which follows from Lemma 2.2. Hence, The condition C1 of Theorem 2.1 is satisfied. It can be checked that  $m \not\equiv 0 \pmod{4}$  since  $5e \equiv 3^m - 5 \pmod{3^m - 1}$ , then  $\gcd(5, 3^m - 1) = 1$ . For any  $x \in \mathbb{F}_{3^m}$ , there exists  $\alpha, \beta \in \mathbb{F}_{3^m}$  such that  $x + 1 = \alpha^5, x = \beta^5$  and

$$\alpha^5 - \beta^5 = 1. \quad (32)$$

Next, we show that the conditions C2 and C3 of Theorem 2.1 are satisfied, respectively.

First, the equation  $(x+1)^e + x^e + 1 = 0$  can be written as  $\alpha^{-4} + \beta^{-4} = -1$ . It can be verified that  $\alpha \neq 0, \beta \neq 0$  and  $(\beta^4 + 1)(\beta^{-4} + 1) \neq 0$ . Otherwise, if  $\beta^{-4} = -1$ , then  $\alpha^{-4} = 0$ , which contradicts to the assumption that  $\alpha \in \mathbb{F}_{3^m}$ . If  $\beta^4 = -1$ , then  $\beta^{-4} = -1$  and  $\alpha^{-4} = 0$ , a contradiction. Hence, we have

$$\alpha^4 = -\frac{1}{\beta^{-4} + 1} = -\frac{\beta^4}{\beta^4 + 1}. \quad (33)$$

Eliminating  $\alpha$  from Eqs. (32) and (33), we can get

$$(-\beta^4)^5 = (\beta^5 + 1)^4(\beta^4 + 1)^5.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} &(\beta - 1)^{10}(\beta^{10} - \beta^8 - \beta^7 - \beta^6 - \beta^5 - \beta^4 - \beta^3 - \beta^2 + 1) \\ &(\beta^{10} - \beta^8 - \beta^7 - \beta^5 - \beta^4 - \beta^3 + \beta + 1)(\beta^{10} + \beta^9 - \beta^7 - \beta^6 - \beta^5 - \beta^3 - \beta^2 + 1) = 0. \end{aligned} \quad (34)$$

By Lemma 4.1, we can get that three irreducible polynomials of degree 10 in Eq. (34) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{10}$ , which means that  $\beta = 1$ . Hence, we can get that  $x = 1$  is a unique solution of the equation  $(x + 1)^e + x^e + 1 = 0$  if and only if  $m \not\equiv 0 \pmod{10}$ .

Secondly, we calculate the solutions of equation  $(x + 1)^e - x^e - 1 = 0$ . Obviously,  $x = 0$  is a solution of this equation. If  $x \neq 0$ , this equation can be simplified as  $\alpha^{-4} - \beta^{-4} = 1$ . Similarly, we can get  $(\beta^4 + 1)(\beta^{-4} + 1) \neq 0$  and

$$\alpha^4 = \frac{1}{\beta^{-4} + 1} = \frac{\beta^4}{\beta^4 + 1}. \quad (35)$$

Plugging Eq. (35) into (32) leads to

$$(\beta^4)^5 = (\beta^5 + 1)^4(\beta^4 + 1)^5.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & (\beta^2 + 1)(\beta^6 - \beta^5 + \beta^4 - \beta^3 + \beta^2 - \beta + 1) \\ & (\beta^{32} + \beta^{31} - \beta^{30} - \beta^{29} + \beta^{27} + \beta^{26} + \beta^{25} - \beta^{24} - \beta^{23} - \beta^{21} + \beta^{20} - \beta^{19} + \beta^{18} - \beta^{17} \\ & - \beta^{16} - \beta^{15} + \beta^{14} - \beta^{13} + \beta^{12} - \beta^{11} - \beta^9 - \beta^8 + \beta^7 + \beta^6 + \beta^5 - \beta^3 - \beta^2 + \beta + 1) = 0. \end{aligned} \quad (36)$$

By Lemma 4.1, we can get that the irreducible polynomial of degree 2 in Eq. (36) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{2}$ , the irreducible polynomial of degree 6 in Eq. (36) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{6}$  and the irreducible polynomial of degree 32 in Eq. (36) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{32}$ . Hence, we can get that  $x = 0$  is a unique solution of the equation  $(x + 1)^e - x^e - 1 = 0$  if and only if  $m \not\equiv 0 \pmod{2}$ .

To sum up, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m$  is odd.  $\square$

Note that the conditions in Theorem 4.4 hold in the following cases: (i)  $e = \frac{2 \cdot 3^m - 6}{5}$ , when  $m \equiv 1 \pmod{4}$ ; (ii)  $e = \frac{4 \cdot 3^m - 8}{5}$ , when  $m \equiv 3 \pmod{4}$ .

**Example 11.** Let  $m = 3$ , then  $e = 20$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 - x^5 + x^4 + x^3 - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

**Theorem 4.5.** Let  $m, e$  be two positive integers satisfying  $m > 1$  and  $7e \equiv 3^m - 5 \pmod{3^m - 1}$ . Then the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m$  is odd,  $m \not\equiv 0 \pmod{5}$  and  $m \not\equiv 0 \pmod{9}$ .

**Proof.** From the condition  $7e \equiv 3^m - 5 \pmod{3^m - 1}$ , we can get that  $e$  is even and  $m \not\equiv 0 \pmod{6}$ , which implies that  $e \notin C_1$  and  $\gcd(7, 3^m - 1) = 1$ . Therefore, we have  $\gcd(e, 3^m - 1) = \gcd(7e, 3^m - 1) = \gcd(3^m - 5, 3^m - 1) = 2$  since  $m$  is odd. Consequently, we can get  $|C_e| = m$ , which follows from Lemma 2.2. Hence, the condition C1 of Theorem 2.1 is met. For any  $x \in \mathbb{F}_{3^m}$ , there exists  $\alpha, \beta \in \mathbb{F}_{3^m}$  such that  $x + 1 = \alpha^7, x = \beta^7$  and

$$\alpha^7 - \beta^7 = 1. \quad (37)$$

Next, we show that the conditions C2 and C3 of Theorem 2.1 are satisfied, respectively.

At first, the equation  $(x + 1)^e + x^e + 1 = 0$  can be written as  $\alpha^{-4} + \beta^{-4} = -1$ . Similar to the proof of Theorem 4.4, we can get  $(\beta^4 + 1)(\beta^{-4} + 1) \neq 0$  and

$$\alpha^4 = -\frac{1}{\beta^{-4} + 1} = -\frac{\beta^4}{\beta^4 + 1}. \quad (38)$$

Drugging Eq. (38) into (37) yields

$$(-\beta^4)^7 = (\beta^7 + 1)^4(\beta^4 + 1)^7.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & (\beta - 1)^2(\beta^9 - \beta^8 - \beta^7 + \beta^5 - \beta^4 + \beta^3 - 1)(\beta^9 - \beta^6 + \beta^3 + \beta - 1)(\beta^9 - \beta^8 - \beta^6 + \beta^3 - 1) \\ & (\beta^9 + \beta^8 - \beta^4 + \beta^3 + \beta^2 - 1)(\beta^9 - \beta^6 + \beta^5 - \beta^4 + \beta^2 + \beta - 1)(\beta^9 - \beta^7 - \beta^6 + \beta^5 - \beta - 1) = 0. \end{aligned} \quad (39)$$

By Lemma 4.1, we can get that six irreducible polynomials of degree 9 in Eq. (39) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{9}$ , which means that  $\beta = 1$ . Hence, we can get that  $x = 1$  is a unique solution of the equation  $(x + 1)^e + x^e + 1 = 0$  if and only if  $m \not\equiv 0 \pmod{9}$ .

Secondly, we calculate the solutions of equation  $(x + 1)^e - x^e - 1 = 0$ . It is clear that  $x = 0$  is a solution of this equation. If  $x \neq 0$ , this equation is equivalent to  $\alpha^{-4} - \beta^{-4} = 1$ . Similarly, we can get  $(\beta^4 + 1)(\beta^{-4} + 1) \neq 0$  and

$$\alpha^4 = \frac{1}{\beta^{-4} + 1} = \frac{\beta^4}{\beta^4 + 1}. \quad (40)$$

Substituting Eqs. (40) into (37) leads to

$$(\beta^4)^7 = (\beta^7 + 1)^4(\beta^4 + 1)^7.$$

Then the above equation can be decomposed into a product of some irreducible factors over  $\mathbb{F}_3$  as

$$\begin{aligned} & (\beta^2 + 1)(\beta^5 + \beta^4 + \beta^3 + \beta^2 - \beta + 1)(\beta^5 - \beta^4 + \beta^3 + \beta^2 + \beta + 1) \\ & (\beta^8 + \beta^7 - \beta^6 + \beta^5 + \beta^3 - \beta^2 + \beta + 1)(\beta^{36} - \beta^{35} - \beta^{33} - \beta^{32} - \beta^{31} - \beta^{29} + \beta^{27} - \beta^{26} - \beta^{25} \\ & - \beta^{24} + \beta^{23} + \beta^{19} + \beta^{18} + \beta^{17} + \beta^{13} - \beta^{12} - \beta^{11} - \beta^{10} + \beta^9 - \beta^7 - \beta^5 - \beta^4 - \beta^3 - \beta + 1) = 0. \end{aligned} \quad (41)$$

By Lemma 4.1, we can get that the irreducible polynomial of degree 2 in Eq. (41) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{2}$ , two irreducible polynomials of degree 5 in Eq. (41) have no solutions in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{5}$ , the irreducible polynomial of degree 8 in Eq. (41) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{8}$  and the irreducible polynomial of degree 36 in Eq. (41) has no solution in  $\mathbb{F}_{3^m}$  if and only if  $m \not\equiv 0 \pmod{36}$ . Hence, we can get that  $x = 0$  is a unique solution of the equation  $(x + 1)^e - x^e - 1 = 0$  if and only if  $m \not\equiv 0 \pmod{2}$  and  $m \not\equiv 0 \pmod{5}$ .

In conclusion, the ternary cyclic code  $C_{(1,e)}$  has parameters  $[3^m - 1, 3^m - 1 - 2m, 4]$  if and only if  $m$  is odd,  $m \not\equiv 0 \pmod{5}$  and  $m \not\equiv 0 \pmod{9}$ .  $\square$

Note that the conditions in Theorem 4.5 hold in the following cases: (i)  $e = \frac{2 \cdot 3^m - 6}{7}$ , when  $m \equiv 1 \pmod{6}$  and  $m \not\equiv 0 \pmod{5}$ ; (ii)  $e = \frac{5 \cdot 3^m - 9}{7}$ , when  $m \equiv 3 \pmod{6}$ ,  $m \not\equiv 0 \pmod{5}$  and  $m \not\equiv 0 \pmod{9}$ ; (iii)  $e = \frac{3^m - 5}{7}$ , when  $m \equiv 5 \pmod{6}$  and  $m \not\equiv 0 \pmod{5}$ .

**Example 12.** Let  $m = 3$ , then  $e = 18$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^3}^*$  with  $\alpha^3 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[26, 20, 4]$  and generator polynomial  $x^6 + x^5 - x^3 - x - 1$ .

The dual of  $C_{(1,e)}$  is a ternary cyclic code with parameters  $[26, 6, 15]$  and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$

**Example 13.** Let  $m = 5$ , then  $e = 34$ . Let  $\alpha$  be the generator of  $\mathbb{F}_{3^5}^*$  with  $\alpha^5 - \alpha + 1$ . Then the code  $C_{(1,e)}$  has parameters  $[242, 232, 3]$  and generator polynomial  $x^{10} + x^7 - x^3 - x - 1$ . We can verify this is an almost optimal ternary cyclic code by the collection of the tables of best linear codes known maintained by Markus Grassl at <http://www.codetables.de/>.

**Remark 2.** Note that Theorems 4.2, 4.3, 4.4 can solve the remaining problems of [23]. Furthermore, we have found that the necessary and sufficient conditions of Theorem 4.2 in [23] should be  $m > 1$  and  $m \not\equiv 0 \pmod{5}$ . And the necessary and sufficient conditions of Theorem 4.3 in [23] should be  $m > 2$ ,  $m \not\equiv 0 \pmod{5}$  and  $m \not\equiv 0 \pmod{9}$ .

## 5 Concluding remarks

This paper mainly investigated the construction of optimal ternary cyclic codes over  $\mathbb{F}_{3^m}$ . On the one hand, by analyzing the solutions of certain equations over  $\mathbb{F}_{3^m}$ , we obtained three classes of optimal ternary cyclic codes  $C_{(1,e)}$  in the case of  $m$  is odd. On the other hand, we proposed five new classes of optimal ternary cyclic codes  $C_{(1,e)}$  with explicit values  $e$  by using the multivariate method. As a byproduct, two classes of optimal ternary cyclic codes  $C_{(u,v)}$  were given. It is a continuation and generalization of some previous works in [23, 24]. We summarized all known optimal ternary cyclic codes  $C_{(1,e)}$  and  $C_{(u,v)}$  over  $\mathbb{F}_{3^m}$  in Tables 1 and 2, respectively. Finding more optimal ternary cyclic codes over  $\mathbb{F}_{3^m}$  would be interesting.

**Table 1**

Known optimal ternary cyclic codes  $C_{(1,e)}$  over  $\mathbb{F}_{3^m}$

$e$	Conditions	Reference
$\frac{3^h+1}{2}$	$h$ is odd, $\gcd(m, h) = 1$	[1]
$3^h + 1$	$\frac{m}{\gcd(m,h)}$ is odd	[1]
$\frac{3^m-3}{2}$	$m \geq 5$ is odd	[3]
$\frac{3^{m+1}+1}{4} + \frac{3^m-1}{2}$	$m \geq 3$ is odd	[3]
$(3^{\frac{m+1}{4}} - 1)(3^{\frac{m+1}{2}} + 1)$	$m \equiv 3 \pmod{4}$	[3]
$3^{\frac{m+1}{2}-1}$ or $\frac{3^{m+1}-1}{8}$	$m \equiv 3 \pmod{4}$	[3]
$\frac{3^{\frac{m+1}{2}-1}}{2} + \frac{3^m-1}{2}$	$m \equiv 1 \pmod{4}$	[3]
$\frac{3^{m+1}-1}{8} + \frac{3^m-1}{2}$	$m \equiv 1 \pmod{4}$	[3]
$3^h - 1$	$\gcd(h, m) = \gcd(3^h - 2, 3^m - 1) = 1$	[3]
$\frac{3^h-1}{2}$	$m$ is odd, $h$ is even	[3]
$2(3^{m-1} - 1)$ or $5(3^{m-1} - 1)$	$\gcd(h, m) = \gcd(h - 1, m) = 1$	[3]
$\frac{3^m-1}{2} + r$	$m$ is odd, $m \not\equiv 0 \pmod{3}$	[10]
$\frac{3^m-1}{2} - r$	$r = 7, m$ is odd or $r = 10, m \equiv 2 \pmod{4}$	[10]
$2(3^h + 1)$	$r = 2, m \equiv 2 \pmod{4}; r = 5, m$ is odd	[10]
$e \equiv \frac{3^m-1}{2} + 3^h + 1 \pmod{3^m - 1}$	$r = 7; r = 11, m \not\equiv 0 \pmod{9}; r = -19, m \not\equiv 0 \pmod{9}$	[17]
$e \equiv \frac{3^m-1}{2} + 3^h - 1 \pmod{3^m - 1}$	$m$ is odd	[12]
$3^h + 5$	$m$ is even, $\frac{m}{\gcd(m,h)}$ is odd	[18]
$3^h + 13$	see [18]	[18]
$\frac{3^{m-1}-1}{2} + 3^h + 1$	$m \equiv 0 \pmod{4}, m \geq 4, h = \frac{m}{2}$	[8]
$\frac{3^{m-1}-1}{2} + 3^h$	$m \equiv 2 \pmod{4}, m \geq 6, h = \frac{m+2}{2}$	[8]
$e(3^s + a) \equiv 3^t + a \pmod{3^m - 1}$	$2h \equiv 1 \pmod{m}, m \geq 5$ is odd, $\gcd(m, 3) = 1$	[8]
	$2h \equiv -1 \pmod{m}, m \geq 5$ is odd prime	[16]
	$2h \equiv -1 \pmod{m}, m \geq 7$ is odd prime	[16]
	$2h \equiv 1 \pmod{m}, m \geq 5$ is odd, $\gcd(m, 3) = 1$	[8]
	$2h \equiv \pm 1 \pmod{m}, m \geq 5$ is odd prime	[16]
	see [25]	[25]
	$m$ is even, $\gcd(m, h + 1) = \gcd(3^{h+1} - 2, 3^m - 1) = 1$	[25]
	$a = 1, \gcd(m, t - s) = \gcd(m, t + s) = 1$ or	
	$a = -1, \gcd(m, t) = \gcd(m, t - s) = 1$	[18, 25]

Continue on the next page

The continuation of Table 1

$e$	Conditions	Reference
$3^{\frac{m+1}{2}+5}$	$m \equiv 1 \pmod{4}, m \not\equiv 0 \pmod{3}$	[23]
$3^{\frac{m+1}{2}+5} + 3^{\frac{m-1}{2}}$	$m \equiv 3 \pmod{4}, m \not\equiv 0 \pmod{3}$	[23]
$3^h+7$	$m$ is odd, $h$ is even	[23]
$3^{\frac{h+7}{2}} + 3^{\frac{m-1}{2}}$	$m$ is odd, $h$ is odd	[23]
$e(3^h+1) \equiv 3^{\frac{m+1}{2}} \pmod{3^m-1}$	$m$ is odd, $e$ is even	[23]
$5e \equiv 2 \pmod{3^m-1}$	$m \not\equiv 0 \pmod{3}$	[23]
$5e \equiv 4 \pmod{3^m-1}$	$m > 2, m \not\equiv 0 \pmod{3}, m \not\equiv 0 \pmod{5}$	[23]
$7e \equiv 2 \pmod{3^m-1}$	$m \not\equiv 0 \pmod{5}, \gcd(m, 6) = 1$ or $m \equiv 3 \pmod{6}$	[23]
$3^{\frac{m-1}{2}+5} + 3^{\frac{m-1}{2}}$	$m$ is odd prime, $m \equiv 1 \pmod{4}, m \not\equiv 0 \pmod{17}$	Theorem 3.1
$3^{\frac{m-1}{2}+5}$	$m$ is odd prime, $m \equiv 3 \pmod{4}$	Theorem 3.2
$e(3^h-1) \equiv 3^{\frac{m+1}{2}} \pmod{3^m-1}$	see Theorem 3.2	Theorem 3.2
$3^{\frac{m-1}{2}-1} + 3^h - 1$	$2h \equiv \pm 1 \pmod{m}, m$ is odd prime	Theorem 3.3
$7e \equiv 4 \pmod{3^m-1}$	$m > 1$ is odd, $m \not\equiv 0 \pmod{9}$	Theorem 4.1
$5e \equiv 3^m - 3 \pmod{3^m-1}$	$m > 1, m \not\equiv 0 \pmod{6}$	Theorem 4.2
$7e \equiv 3^m - 3 \pmod{3^m-1}$	$m > 1, m \not\equiv 0 \pmod{3}, m \not\equiv 0 \pmod{4}, m \not\equiv 0 \pmod{22}$	Theorem 4.3
$5e \equiv 3^m - 5 \pmod{3^m-1}$	$m > 1$ is odd	Theorem 4.4
$7e \equiv 3^m - 5 \pmod{3^m-1}$	$m > 1$ is odd, $m \not\equiv 0 \pmod{5}, m \not\equiv 0 \pmod{9}$	Theorem 4.5

Table 2

Known optimal ternary cyclic codes  $C_{(u,v)}$  over  $\mathbb{F}_{3^m}$ 

$u$	$v$	Conditions	Reference
$3^m+1$	$3^s+1$	$m$ is odd, $s$ is even, $\gcd(m, s) = 1$	[27]
$3^m+1$	$2 \cdot 3^l + 1$	$m = 2l + 1$	[5]
$3^m+1$	$3^r + 2$	$m \geq 3$ is odd, $4r \equiv 1 \pmod{m}, 9 \nmid m$	[20]
$3^m - 6$	$3^k+1$	see [17]	[17]
2	$3^{\frac{k-1}{2}} + 2(3^k - 1)$	$m$ is odd, $\gcd(m, k) = \gcd(3^k - 2, 3^m - 1) = 1$	[15]
2	$3^{\frac{k-1}{2}} + 2(3^k + 1)$	$m$ is odd, $\gcd(m, k) = 1$	[15]
$3^k+1$	$3^l+1$	$\frac{l}{\gcd(m,l)}$ is odd, $\frac{m}{\gcd(m,l)}$ is even, $\gcd(m, k \pm l) = 1$	[24]
$2^i$	$3^{\frac{m-1}{2}} + 2^i \cdot e$	$m$ is odd, $C_{(1,e)}$ is optimal ternary cyclic code	[24]
$u$	$3^{\frac{m-1}{2}} + (3^k - 1)u$	$m$ is odd, $k = \frac{m+1}{2}, \gcd(u, 3^m - 1) = 2$	[24]
$3^m+1$	$3^{\frac{m-1}{2}} + e$	$m$ is odd, $C_{(1,e)}$ is optimal ternary cyclic code	[24]
$3^m+1$	$3^{\frac{m-1}{2}+5}$	$m$ is odd prime, $m \equiv 1 \pmod{4}, m \not\equiv 0 \pmod{17}$	Corollary 3.1
$3^m+1$	$3^{\frac{m-1}{2}} + 3^{\frac{m-1}{2}+5}$	$m$ is odd prime, $m \equiv 3 \pmod{4}$	Corollary 3.1
$3^m+1$	$2 \cdot 3^{m-1} + 3^h - 2$	$2h \equiv \pm 1 \pmod{m}, m \geq 3$ is odd prime	Corollary 3.2

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